RING ISOMORPHISMS OF BANACH ALGEBRAS

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1. Introduction. In discussing an isomorphism between two Banach algebras, one will ordinarily tacitly assume that the mapping is linear (i.e., preserves the complex scalars as well as the ring operations). In general this cannot be avoided; for instance if the two Banach algebras are just the field of complex numbers, then the isomorphism is unrestricted, and could be given by any one of the myriads of discontinuous automorphisms of the complex numbers. A similar remark applies generally to the finite-dimensional case. But if the algebras are genuinely infinite-dimensional in an appropriate sense, interesting results become possible. The first such theorem was proved by Arnold **(1)**: if A and B are both algebras of all bounded operators on *infinite-dimensional* Banach spaces, then any ring isomorphism between A and B is automatically real-linear (or alternatively, it is either linear or conjugate linear relative to complex scalars). Kakutani and Mackey **(5)** used a similar argument in connection with their characterization of complex Hilbert space. Rickart **(7)** generalized Arnold's theorem to the case of primitive Banach algebras with minimal ideals.

In this paper we shall extend Rickart's result to any semi-simple Banach algebra, the precise theorem being as follows: *if ϕ is a ring isomorphism from one semi-simple Banach algebra A onto another, then A is a direct sum $A_1 \oplus A_2 \oplus A_3$ with A_1 finite-dimensional, ϕ linear on A_2, and ϕ conjugate linear on A_3.* Some of the preliminary lemmas (particularly Lemmas 7 and 9) may be of independent interest.

2. Elements with infinite spectrum. Let A be a Banach algebra,[1] x an element in A. We define the non-zero spectrum of x to consist of all scalars λ such that $-\lambda^{-1}x$ is not quasi-regular. We insert 0 in the spectrum of x unless A has a unit element and x is regular.

LEMMA 1. *If there exists a non-zero element z with $zx - \lambda z = 0$, then λ is in the spectrum of x.*

Proof. If not, suppose y is the quasi-inverse of $-\lambda^{-1}x$, so that

$$-\lambda^{-1}x + y - \lambda^{-1}xy = 0.$$

A left multiplication by z yields the contradiction $z = 0$.

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[1]All our Banach algebras will admit complex scalars. The results can probably be extended to the case where only real scalars are assumed, but it seemed preferable in the present paper to avoid the extra complications.

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LEMMA 2. *If a Banach algebra possesses an infinite set of orthogonal idempotents, then it has an element with an infinite spectrum.*

Proof. Denote the idempotents by e_i and write $x = \Sigma\lambda_i e_i$, where λ_i are distinct numbers satisfying $|\lambda_i| \, ||e_i|| < 2^{-i}$. Since $e_i x = \lambda_i e_i$, it follows from Lemma 1 that λ_i is in the spectrum of x.

LEMMA 3. *Let e be an idempotent in a Banach algebra A. Then the non-zero spectrum of an element of eAe is the same, whether computed in eAe or in A. The same is true*[2] *for the subalgebra* $(1 - e)\, A(1 - e)$.

Proof. We can cover both cases by using a symbol f for either e or $1 - e$. The problem comes to this: given an element fxf which has a quasi-inverse in A, prove that it already has a quasi-inverse in fAf. Let y denote the quasi-inverse in A, so that

$$fxf + y + fxfy = fxf + y + yfxf = 0.$$

On left and right multiplying by f, we see that fyf is likewise a quasi-inverse of fxf.

LEMMA 4. *Let A be a Banach algebra with unit element and radical R. Suppose that every element of A has only one number in its spectrum. Then R consists precisely of all elements with spectrum* 0, *and A/R is one-dimensional.*

Proof. Let N be the set of all elements with spectrum 0; of course $N \supset R$. We note that the elements of A having (two-sided) inverses are precisely those not in N. We are now going to prove that N is a right ideal, for which purpose we have two things to verify.

(1) If $x \in N$ and $y \in A$, we have to prove $xy \in N$. If not, then xy is regular, say with inverse z. Thus x has at any rate a right inverse, namely yz. But if yz is not also a left inverse of x, then the element yzx is an idempotent other than 0 and 1, and has 0 and 1 in its spectrum, contrary to hypothesis.

(2) If $x, y \in N$, we must show $x - y \in N$. If not, $x - y = u$ is regular. By what we have just shown, both xu^{-1} and yu^{-1} are in N. But then $xu^{-1} = 1 + yu^{-1}$ is regular, a contradiction.

We have thus shown that N is a right ideal. Since it consists of quasi-regular elements, it is part of the radical. Hence $N = R$, and it is immediate that A/R is one-dimensional.

LEMMA 5. *Let A be a commutative Banach algebra having exactly r regular maximal ideals. Then A contains r orthogonal* (*non-zero*) *idempotents.*

Proof. Let R be the radical of A. Then A/R is the direct sum of r copies of the complex numbers. It is known that the r orthogonal idempotents in A/R, which arise in this way, can be lifted to orthogonal idempotents in A (see for example **(2)**).

[2]The symbol 1 is used here formally, and does not indicate that we are assuming the presence of a unit element.

Our next lemma is an elementary purely algebraic one, recorded for the convenience of the reader.

LEMMA 6. *Let A be a ring with unit element and no nilpotent ideals. Suppose that $1 = e_1 + \ldots + e_n$ with the e's orthogonal idempotents such that each e_iAe_i is a division ring. Then A has the descending chain condition on right ideals (and so is the direct sum of a finite number of matrix rings over division rings).*

Proof. Since A has no nilpotent ideals, and e_iAe_i is a division ring, it is known (**4**, p. 13) that e_iA is a minimal right ideal. Moreover $A = e_1A + \ldots + e_nA$ is a direct sum decomposition of A into a finite number of minimal right ideals. It follows from the Jordan-Hölder theorem that A has the descending chain condition on right ideals.

LEMMA 7. *In any infinite-dimensional semi-simple Banach algebra there exists an element with an infinite spectrum.*

Proof. Suppose that A is a semi-simple Banach algebra, and that every element of A has a finite spectrum. We shall prove that A is finite-dimensional.

It cannot be the case that every element has 0 spectrum, for then A would be all radical. Select an element x with some non-zero spectrum, and let B be the closed subalgebra generated by x. One knows that the number of regular maximal ideals in B is the same as the number of non-zero numbers in the spectrum of x. It follows from Lemma 5 that B contains idempotents.

Let then e_1 be an idempotent in A. By Lemma 3, the Banach algebra e_1Ae_1 inherits the hypothesis that all its elements have finite spectrum. There may exist in e_1Ae_1 an idempotent e_2 other than 0 and e_1, and then a third one e_3 inside e_2Ae_2, etc.; but this cannot continue indefinitely, for we would find the infinite set $\{e_i - e_{i+1}\}$ of orthogonal idempotents, contrary to Lemma 2.

Thus we may assume that e_1Ae_1 contains no idempotents other than 0 and e_1. For any y in e_1Ae_1 we form the closed subalgebra C generated by y and e_1. If y has two or more numbers in its spectrum, then C has at least two maximal ideals (the presence of 0 in the spectrum is not treated specially here, since C has a unit element). Since this contradicts Lemma 5, it must be the case that every element in e_1Ae_1 has a one-element spectrum. It is known that e_1Ae_1, along with A, is semi-simple. It now follows from Lemma 4 that e_1Ae_1 is one-dimensional.

Let $e_1, \ldots, e_n$ be a maximal set of idempotents in A such that each e_iAe_i is one-dimensional. (That such a maximal set is finite follows from Lemma 2.) We write $e = e_1 + \ldots + e_n$ and turn our attention to $(1 - e)A(1 - e)$; according to Lemma 3, its elements all have finite spectrum. If $(1 - e)A(1 - e)$ is not all radical, the preceding argument shows that we can find in it an idempotent e_{n+1} with $e_{n+1}Ae_{n+1}$ one-dimensional. This contradicts the maximality of $e_1, \ldots, e_n$. Hence $(1 - e)A(1 - e)$ is all radical. But on the other hand it, like A, is semi-simple. Hence $(1 - e)A(1 - e) = 0$. We next observe that $(1 - e)A$ is a nilpotent right ideal, and so $(1 - e)A = 0$ and similarly

$A(1-e)=0$. In other words, e is a unit element for A. We are now ready to apply Lemma 6. In the light of the Gelfand-Mazur theorem that all Banach division algebras are one-dimensional, we conclude that A is finite-dimensional. This completes the proof of Lemma 7.

3. A remark on ideals. Our program is to study mappings which are isomorphisms purely in the ring-theoretic sense. It is therefore important for us to know that certain ideals (such as primitive ideals) which are defined with no reference to scalars, are *automatically* algebra ideals when they occur in an algebra. Actually, for later purposes we wish to consider more generally the admissibility of the ideal under general operators, where by an operator on a ring we mean an additive endomorphism that commutes with all left and right multiplications. The principal fact is given in the following lemma.

LEMMA 8. *Let A be any ring and M a regular maximal right ideal in A. Then M automatically admits any operator on A.*

Proof. Let the operator be denoted by θ and placed on the left. Then for $x \in M$ we have to prove $\theta x \in M$. Let e be a left unit modulo M, so that $ey - y \in M$ for all y in A. If θx is not in M, then the right ideal generated by θx and M must be all of A. In particular we have

$$e = \theta x(a+n) + m$$

where $a \in A$, $m \in M$, and n is an integer. On right multiplying by e and commuting θ past x we get

$$e^2 = x(\theta ae + \theta ne) + me \in M.$$

Now $e^2 - e \in M$; this tell us that e is in M, whence M is all of A, a contradiction.

From Lemma 8 we deduce that any primitive ideal is automatically operator-admissible. More generally, if I is any two-sided ideal such that A/I is semi-simple then I is admissible, for in that case I is an intersection of regular maximal right ideals.

4. The centroid. By the *centroid*[3] of a ring A we mean the set of all operators on A where, as above, an operator is an additive endomorphism commuting with all left and right multiplications. If A has a unit element, the centroid is easily seen to coincide with the ordinary center of A.

Suppose that A is an algebra over a field F. Then the elements of F form part of the centroid. We call A *central* if the elements of F in this way form all of the centroid.

LEMMA 9. *Any primitive Banach algebra is central.*

[3]This term (used by Artin in a Princeton seminar) seems better than earlier ones that have been used, such as "multiplication centralizer" in **(3)**.

Proof. Let M be a regular maximal right ideal such that A is faithfully represented by right multiplication on A/M. We propose to invoke the theory of the *eigenring* B of M, for which we refer the reader to (**3**, p. 236) and (**6**, Lemma 3). We summarize briefly: B is defined to be the set of all a in A with $aM \subset M$, M is a two-sided ideal in the subring B, B/M is a division ring and in fact coincides in a natural way with the ring of all endomorphisms of A/M commuting with all right multiplications on A/M.

Now in our case we have further that M is an algebra ideal (Lemma 8). Also, any regular maximal right ideal in a Banach algebra is closed. It follows further that B is closed and that B/M is a Banach division algebra which, by the Gelfand-Mazur theorem, is simply the complex numbers.

Let an element θ of the centroid be presented. By Lemma 8, θ sends M into itself and accordingly induces an additive endomorphism of A/M. This latter manifestly commutes with all right multiplications by elements of A. We are thus led to a certain element of B/M, i.e. to a complex number λ. The information we have is that for any a in A, $\theta a - \lambda a \in M$. Then for a further element x in A

$$\theta xa - \lambda xa = x(\theta a - \lambda a) \in M.$$

Thus right multiplication by $\theta a - \lambda a$ sends A into M, and induces the 0 map on A/M. Since the representation of A on A/M is faithful, we have $\theta a - \lambda a = 0$. The centroid of A therefore coincides with the complex numbers and A is central.

5. The primitive case. Next we need a lemma which is concerned with the construction of an entire function with desired properties.

LEMMA 10. *Let $\{\lambda_i\}$ be a sequence of distinct non-zero complex numbers. For each i let there be given a discontinuous automorphism σ_i of the complex numbers (the σ's need not be distinct). Then there exists an entire function f, vanishing at 0, such that the set $\{\sigma_i[f(\lambda_i)]\}$ is unbounded.*

Proof. The function f will be constructed as a sum Σg_n of polynomials. Assuming that $g_1, \ldots, g_{n-1}$ have been selected, we take

$$g_n = cz(z - \lambda_1) \ldots (z - \lambda_{n-1}).$$

The coefficient c is to be chosen so that

$$|g_n(z)| < 2^{-n} \quad \text{for} \quad |z| \leqslant n, \tag{1}$$

$$|\sigma_n[g_1(\lambda_n) + \ldots + g_n(\lambda_n)]| > n. \tag{2}$$

Of course (1) merely requires that c be suitably small. This having been arranged, we can achieve (2), for it is known that a discontinuous automorphism σ_n is unbounded on any open subset of the complex plane.

By (1), the sum Σg_n converges uniformly on any bounded set. Hence the sum f is an entire function. Since $g_i(\lambda_n) = 0$ for $i > n$, the terms from the

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$(n + 1)$-st on do not disturb (2) and we have $|\sigma_n[f(\lambda_n)]| > n$. Finally $f(0) = 0$ since each g_n vanishes at 0.

We can now dispose of the primitive case.

LEMMA 11. *Let A and B be infinite-dimensional primitive Banach algebras. Then any ring isomorphism from A onto B is automatically real-linear (and hence either linear or conjugate linear relative to complex scalars).*

Proof. The given isomorphism induces an isomorphism between the centroids of A and B (this is a general fact about isomorphisms between rings). By Lemma 10 the centroid of both A and B is just the complex numbers. So the isomorphism between the two centroids is describable as an automorphism σ of the complex numbers. Our problem is to prove that σ is continuous, and we suppose the contrary. By Lemma 7, A possesses an element x with an infinite spectrum. Let $\lambda_1, \lambda_2, \ldots$ be distinct non-zero numbers in the spectrum of x. Apply Lemma 10 (with all the σ_i's equal to σ). If f is the resulting entire function, then f may be applied to x to yield a well defined element y in A, and the spectrum of y contains $f(\lambda_1), f(\lambda_2), \ldots$. If we write y' for the image of y under the isomorphism, the spectrum of y' will contain the numbers $\sigma[f(\lambda_i)]$. But these numbers are unbounded, whereas in any Banach algebra the spectrum of every element is bounded. This contradiction shows that σ must be continuous, and completes the proof of Lemma 11.

6. The main theorem.

The next step in the discussion is to show that discontinuous automorphisms of the complex numbers can arise at only a finite number of primitive ideals. We first need a lemma somewhat analogous to Lemma 2.

LEMMA 12. *Let A be a Banach algebra, $\{M_i\}$ an infinite set of regular maximal two-sided ideals in A. Then we can find an element x in A and distinct non-zero complex numbers λ_i such that λ_i is in the spectrum of*[4] *$x(M_i)$.*

Proof. By the Chinese remainder theorem there exists an element y_i with $y_i(M_i) = 1$, $y_i(M_j) = 0$ for $j < i$. We proceed to choose numbers α_i and λ_i in succession. Having selected them up to $i - 1$, we take α_i satisfying

$$0 < \alpha_i < 2^{-i}||y_i||,$$

and such that $(\alpha_1 y_1 + \ldots + \alpha_i y_i)(M_i)$ has $\lambda_i \neq 0$ in its spectrum, where λ_i is any number different from $\lambda_1, \ldots, \lambda_{i-1}$. We define $x = \Sigma \alpha_i y_i$. Since the terms from y_{i+1} on map into 0 mod M_i, x has the desired property that $x(M_i)$ has λ_i in its spectrum.

Before stating the next lemma, we consider the following situation: ϕ is a ring isomorphism of a Banach algebra A onto a second one B, P is a primitive ideal in A, and $Q = \phi(B)$. Then (see the remark after Lemma 8) P is an algebra ideal in A. Moreover it is known that P is closed; indeed P is an intersection of regular maximal right ideals and the latter are closed. Thus A/P is a primitive

[4]The notation $x(M)$ denotes the image of x in the natural homomorphism from A onto A/M.

Banach algebra. The same is true of B/Q, and we observe that ϕ induces a ring isomorphism of A/P onto B/Q.

LEMMA 13. *Let A be a Banach algebra and ϕ a ring isomorphism of A onto a second Banach algebra B. Then there can exist only a finite number of primitive ideals P in A such that the isomorphism induced by ϕ on A/P is not real-linear.*

Proof. Suppose on the contrary that there are an infinite number of such primitive ideals and denote them by P_i. Then by Lemma 11 each A/P_i is finite-dimensional (and hence a total matrix algebra). In particular P_i is regular maximal. We now apply Lemma 12 and produce an element x in A and distinct non-zero numbers λ_i such that, for all i, $x(P_i)$ has λ_i in its spectrum. Write σ_i for the discontinuous automorphism of the complex numbers associated with the isomorphism of A/P_i. We apply Lemma 10, and write $y = f(x)$ with f the entire function given there. Then $y(P_i)$ has $f(\lambda_i)$ in its spectrum. Passing to the algebra B, we observe that the image of $\phi(y)$ in $B/\phi(P_i)$ has $\sigma_i[f(\lambda_i)]$ in its spectrum. Then further all these numbers $\sigma_i[f(\lambda_i)]$ lie in the spectrum of $\phi(y)$ itself. This contradicts the boundedness of the spectrum of $\phi(y)$.

We shall now state and prove the main theorem of the paper.

THEOREM. *Let A and B be semi-simple Banach algebras and let ϕ be a ring isomorphism of A onto B. Then we may write $A = A_1 \oplus A_2 \oplus A_3$ with A_1 finite-dimensional, ϕ linear on A_2, and ϕ conjugate linear on A_3.*

COROLLARY. *If A has no finite-dimensional ideals, then any ring isomorphism of A onto B is automatically real-linear.*

Proof. We first single out (by Lemma 13) the finite number of primitive ideals $P_1, \ldots, P_r$ in A such that the induced isomorphism on A/P_i is not real-linear. We shall prove that P_1 is a direct summand of A, or rather the equivalent statement that Q_1 is a direct summand of B, where $Q_i = \phi(P_i)$. By Lemma 11, each A/P_i is finite-dimensional, whence P_i is regular maximal. By the Chinese remainder theorem, there exists in A an element x with $x(P_1) = 1$, $x(P_2) = \ldots = x(P_r) = 0$. For any other primitive ideal P we have $||x(P)|| \leqslant ||x||$, and *a fortiori* the spectrum of $x(P)$ is bounded by $||x||$. Write σ for the discontinuous automorphism of the complex numbers attached to the isomorphism on A/P_1. There exists a complex number λ such that $|\sigma(\lambda)| > 2|\lambda|\ ||x||$. Write $y = \phi(\lambda x)/\sigma(\lambda)$. Then $y(Q_1) = 1$, $y(Q_2) = \ldots = y(Q_r) = 0$. Let $Q = \phi(P)$ be any other primitive ideal in B. Since the induced isomorphism from A/P onto B/Q is real-linear, it preserves the absolute value of the spectrum of any element. Since the spectrum of $x(P)$ is bounded by $||x||$ we compute that the spectrum of $y(Q)$ is bounded by $||x||\ |\lambda|/|\sigma(\lambda)| < \frac{1}{2}$. We apply to y the Cauchy integral, in the appropriate version for algebras that may lack a unit element:

$$e = \frac{1}{2\pi i}\int_C (-\lambda^{-1}y)'\lambda^{-1}d\lambda.$$

Here C may be taken to be a circle of radius $\frac{1}{2}$ about 1, and the prime denotes quasi-inverse. Then by known properties of this integral we have that $e(Q_1) = 1$, while $e(Q) = 0$ for every other primitive ideal in B (including of course $Q_2, \ldots, Q_r$). Because of the semi-simplicity of B it follows that e is a central idempotent, and indeed we have the desired direct sum decomposition $B = Q_1 \oplus eB$.

Transferring this back by the inverse of ϕ, we have likewise that P_1 is a direct summand of A. We may of course subject $P_2, \ldots, P_r$ to the same treatment. The result is to reduce our problem to the case where ϕ is already real-linear, and we accordingly make that assumption in the rest of the proof.

This last portion of the proof is purely algebraic, and is best understood by making use of the centroid. Multiplication by i is an operator on the ring A and thereby gives rise to a centroid element. Likewise we get an element of the centroid of B from multiplication by i. This latter may be transferred to the centroid of A, via the given isomorphism of A and B. We now have two centroid elements of A, both with square equal to -1. The vital thing is to know that they commute, for then their product will be a centroid element with square 1. Such an element splits A into a direct sum $A_2 \oplus A_3$ of ideals, on the first of which it is the identity, on the second the negative of the identity. This is the decomposition we are seeking.

So it remains only to convince ourselves that the centroid of a semi-simple ring is commutative. Here is a stronger result: if a ring A has no total left annihilator other than 0, then its centroid is commutative. Given x in A and θ_1, θ_2 in the centroid, we have to prove $\theta_1\theta_2 x = \theta_2\theta_1 x$. It is enough to prove that this holds after a right multiplication by y. By repeated use of the fact that the θ's commute with left and right multiplications we find:

$$[(\theta_1\theta_2)x]y = (\theta_1\theta_2)(xy) = \theta_1(x\cdot\theta_2 y) = \theta_1 x\cdot\theta_2 y,$$
$$[(\theta_2\theta_1)x]y = (\theta_2\theta_1)(xy) = \theta_2(\theta_1 x\cdot y) = \theta_1 x\cdot\theta_2 y.$$

With this the proof of the theorem is complete.

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