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SOLUTION OF A GENERAL LINEAR DIFFERENCE EQUATION

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Abstract

A matrix solution and a determinantal solution are obtained for a general linear recurrence relation.

A few years back, Brown [1] gave the solution of a three-term linear difference equation

$$a_0(n)u_n + a_1(n)u_{n-1} + a_2(n)u_{n-2} = 0, \qquad n \geqslant 2,$$

with $a_0(n) \neq 0$ for all $n \geqslant 2$, in terms of certain determinants. In a recent paper, Singh [2] has given the solution for an $(r+1)$-term homogeneous linear difference equation in terms of certain lower Hessenberg determinants, and has also given a matrix solution.

In this note, we obtain a matrix solution as well as a determinantal solution of the non-homogeneous linear difference equation

$$a_0(n)u_n + a_1(n)u_{n-1} + \cdots + a_r(n)u_{n-r} = v(n), \qquad n > r, \tag{1}$$

with $a_0(n) \neq 0$ for all $n \geqslant r$. Besides, we establish that the particular solutions appearing in the solution obtained in [2] constitute a linearly independent set under certain conditions.

Following Singh [2], we shall be using the notations:

$a_t(n) = 0$ if t is a negative integer or a positive integer $> r$.

$p_k = \prod_{h=r}^{k} a_0(h)$, empty products being taken to be 1.

$A_k = [a_{i-j}(kr + i - 1)]$; $i, j = 1, 2, \ldots, r$; $k = 1, 2, \cdots$.

$B_k = [a_{r-(j-i)}(kr + i - 1)]$; $i, j = 1, 2, \ldots, r$; $k = 1, 2, \cdots$.

$N = [n/r]$, where $[x]$ denotes the integral part of x; $N' = n - Nr + 1$.

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$A_{(n)} = [a_{i-j}(Nr + i - 1)]$; $i, j = 1, 2, \ldots, N'$.

$B_{(n)} = [a_{r-(j-i)}(Nr + i - 1)]$; $i = 1, 2, \ldots, N'$; $j = 1, 2, \ldots, r$.

$U_{(r,n)} = [u_r, u_{r+1}, \ldots, u_n]^T$; $U_{(kr,(k+1)r-1)} \equiv U_k$.

$D_m^n(r, s) = |a_{i-j+1}(m + i - 1)|$; $i = 1, 2, \ldots, n - m + 1$;

$j = 2 - s, 2, 3, \ldots, n - m + 1$, for $n \geqslant m + 1$;

$D_m^m(r, s) = a_s(m)$; $D_{n+1}^n(r, 1) = 1$ and $D_r^n(r, s) = D_s^n$.

$V_{(r,n)} = [v(r), v(r + 1), \ldots, v(n)]^T$; $V_{(kr,(k+1)r-1)} \equiv V_k$.

$W_{(s,n)} = [a_s(r), a_{s+1}(r + 1), \ldots, a_{s+n-r}(n)]^T$.

$\tilde{B}_k = A_k^{-1}B_k$, $\tilde{V}_k = A_k^{-1}V_k$; $k = 1, 2, \cdots$.

We shall first obtain a matrix solution of (1). If we put $n = kr, kr + 1, \ldots, (k + 1)r - 1$, in (1), we get the matrix reduction formula

$$A_k U_k = V_k - B_k U_{k-1}, \qquad k \geqslant 1,$$

whence we can easily see that

$$U_k = \left\{(-1)^k \prod_{s=k}^{1} \tilde{B}_s\right\} U_0 + \tilde{V}_k + \prod_{t=k-1}^{1} (-1)^{k+t}\left(\prod_{s=k}^{t+1} \tilde{B}_s\right)\tilde{V}_t, \qquad k \geqslant 2, \tag{2}$$
$$U_1 = \tilde{V}_1 - \tilde{B}_1 U_0.$$

Therefore, if $n = kr + t$, $0 \leqslant t \leqslant r - 1$, the general solution of the linear difference equation (1) is given by u_{kr+t} = the $(t + 1)$-th element of the matrix U_k, where U_k is defined by (2).

We now obtain the solution of (1) in terms of the determinants D_s^n and $D_m^n(r, 1)$. Let $n = r, r + 1, \ldots, n$ in (1). Then we get

$$A_{(r,n)}U_{(r,n)} = (V_1 - B_1 U_0)_n, \qquad r \leqslant n \leqslant 2r - 1$$
$$= \begin{bmatrix} V_1 - B_1 U_0 \\ V_{(2r,n)} \end{bmatrix}, \qquad n \geqslant 2r, \tag{3}$$

where $(V_1 - B_1 U_0)_n$ denotes the $(n - r + 1)$-vector obtained by taking the first $(n - r + 1)$ components of $V_1 - B_1 U_0$, and $A_{(r,n)}$ = the $(n - r + 1)$-th leading principal submatrix of A_1, if $r \leqslant n \leqslant 2r - 1$,

$$= \left[\begin{array}{cccccc} A_1 & O_r & \cdots & O_r & O_r & O_{r,N'} \\ B_2 & A_2 & \cdots & O_r & O_r & O_{r,N'} \\ O_r & B_3 & \cdots & O_r & O_r & O_{r,N'} \\ \hdashline O_r & O_r & \cdots & B_{N-1} & A_{N-1} & O_{r,N'} \\ O_{N',r} & O_{N',r} & \cdots & O_{N',r} & B_{(n)} & A_{(n)} \end{array}\right], \quad \text{if } n \geqslant 2r,$$

$O_{r,s}$ denoting the null matrix of dimension $r \times s$, with $O_{r,r} \equiv O_r$.

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Applying Cramer's rule to the linear non-homogeneous system (3), we get in particular

$$|A_{(r,n)}|u_n = C_{(r,n)}, \tag{4}$$

where

$$\begin{aligned} C_{(r,n)} &= |A^*_{(r,n)}(V_1 - B_1U_0)_n|, \qquad r \leqslant n \leqslant 2r-1, \\ &= \begin{vmatrix} & V_1 - B_1U_0 \\ A^*_{(r,n)} & \\ & V_{(2r,n)} \end{vmatrix}, \qquad n \geqslant 2r, \end{aligned}$$

$A^*_{(r,n)}$ denoting the matrix obtained from $A_{(r,n)}$ by omitting the last column.

Writing $C_{(r,n)}$ as a sum of $(r+1)$ determinants, we obtain

$$C_{(r,n)} = |A^*_{(r,n)}V_{(r,n)}| - \sum_{s=r}^{1} |A^*_{(r,n)}W_{(s,n)}|u_{r-s}.$$

Now bringing the last column in all these $(r+1)$ determinants to the leading position while retaining the order of the rest of the columns in them, we find that

$$C_{(r,n)} = (-1)^{n-r+1}\left\{\sum_{s=r}^{1} D_s^n u_{r-s} - D_{(r,n)}\right\}. \tag{5}$$

Here $D_{(r,n)} \equiv |V_{(r,n)}A^*_{(r,n)}|$ is obtainable from D_s^n by replacing its first column by $V_{(r,n)}$, so that, expanding along this column, we get

$$D_{(r,n)} = \sum_{s=r}^{n} (-1)^{s-r} v(s)p_{s-1}D^n_{s+1}(r, 1). \tag{6}$$

Further

$$|A_{(r,n)}| = p_n. \tag{7}$$

Therefore, from equations (4), (5) and (7), the general solution of (1) is given by

$$u_n = (-1)^{n-r+1}p_n^{-1}\left\{\sum_{s=r}^{1} D_s^n u_{r-s} - D_{(r,n)}\right\}, \qquad n \geqslant r, \tag{8}$$

where $D_{(r,n)}$ is given by (6). Taking $v(n) = 0$, $n \geqslant r$, in (6) and (8), we get the following solution for the homogeneous case obtained by Singh [2]:

$$u_n = (-1)^{n-r+1}p_n^{-1}\sum_{s=r}^{1} D_s^n u_{r-s}, \qquad n \geqslant r.$$

We now make certain observations regarding the homogeneous case just referred to. In this case, the r particular solutions $(-1)^{n-r+1}p_n^{-1}D_s^n$, $s = r, r-1, \ldots, 1$ (corresponding, respectively, to the initial conditions $u_{r-k} = \delta_s^k$,

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$k = r, r-1, \ldots, 1$, where δ_j^i is the Kronecker delta) form a linearly independent set of solutions if

$$a_r(h) \neq 0, \qquad r \leqslant h \leqslant 2r-1. \tag{9}$$

This can be seen by observing that a non-trivial linear relation for $n \geqslant r$ between the determinants D_s^n (as functions of the variable $n \geqslant r$) implies the existence of a homogeneous system of r equations in r undetermined coefficients with the determinant

$$E_r^r \equiv |D_j^{r+i-1}| = 0, \qquad i, j = 1, 2, \ldots, r.$$

But, we have [2]

$$E_r^r = \left\{ \prod_{n=r}^{2r-2} p_n \right\} q_{2r-1},$$

where $q_k = \prod_{h=r}^{k} a_r(h)$. Since $p_n \neq 0$ for $n \geqslant r$ by assumption, $E_r^r \neq 0$ if (9) holds.

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