Equation (2) becomes

$$y = 2Cx^2 + \big(y(1) - y(0) - 2C\big)x + y(0),$$

which is (5).

The final step is to check that no spurious solutions have been introduced [4, p. 26]. Substituting (5) into (1) gives an identity, which shows that there are none.

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## 107.32 A new inductive proof of the AM - GM inequality

In what follows, we denote by $A_n$ and $G_n$ the arithmetic and geometric means of $n$ non-negative real numbers $a_1, a_2, \ldots, a_n$ $(n \geqslant 1)$, that is,

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \text{ and } G_n = \sqrt[n]{a_1 a_2 \ldots a_n}.$$

Then the famous arithmetic mean - geometric mean inequality (see, e.g., [1, Subsection 2.1] and [2, Section 5]) states that

$$A_n \geqslant G_n, \tag{1}$$

where equality holds if, and only if, $a_1 = a_2 = \ldots = a_n$.

Several proofs of the arithmetic mean-geometric mean inequality are known in the literature (see, e.g., [3]). It was used as the inductive hypothesis in [4, 5, 6, 7, 8].

Notice that in the well known "forward-backward induction proof" of the inequality (1), established in 1821 by A. L. Cauchy [9, p. 457] (also see [10, Chapter 4]), Cauchy showed that $A_{2^n} \geqslant G_{2^n}$ $(n = 1, 2, \ldots)$, and then that the hypothesis $A_n \geqslant G_n$ implies that $A_{n-1} \geqslant G_{n-1}$.

The aim of this short Note is to provide a new inductive proof of the arithmetic mean - geometric mean inequality. This proof is based on the following lemma.

*Lemma*: The inequality

$$A_{2^n} \geqslant G_{2^n} \tag{2}$$

is true for all $n = 1, 2, \ldots$ . Moreover, the equality in (2) holds if, and only if, $a_1 = a_2 = \ldots = a_{2^n}$.

*Proof*: We proceed by induction on $n \geqslant 1$. The induction base is the inequality

$$\frac{a_1 + a_2}{2} \geqslant \sqrt{a_1 a_2}, \tag{3}$$

which follows immediately from the inequality $\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \geqslant 0$. Moreover, $A_2 = G_2$ if, and only if, $a_1 = a_2$.

Suppose that (2) is true for $n = k$ and that the equality in (2) holds if, and only if, $a_1 = a_2 = \ldots = a_{2^k}$. Using the inequality (2) with $n = k$ (the inductive hypothesis) with $\left(a_1 + a_2 + \ldots + a_{2^k}\right)/2^k$ and $\left(a_{2^k+1} + a_{2^k+2} + \ldots + a_{2^{k+1}}\right)/2^k$ instead of $a_1$ and $a_2$, respectively, and then applying the inequality (3), we find that

$$\begin{aligned} A_{2^{k+1}} &= \frac{a_1 + a_2 + \ldots + a_{2^{k+1}}}{2^{k+1}} \\ &= \frac{1}{2}\left(\frac{a_1 + a_2 + \ldots + a_{2^k}}{2^k} + \frac{a_{2^k+1} + a_{2^k+2} + \ldots + a_{2^{k+1}}}{2^k}\right) \\ &\geqslant \frac{1}{2}\left(\sqrt[2^k]{a_1 a_2 \ldots a_{2^k}} + \sqrt[2^k]{a_{2^k+1} a_{2^k+2} \ldots a_{2^{k+1}}}\right) \\ &\geqslant \sqrt[2^{k+1}]{a_1 a_2 \ldots a_{2^{k+1}}} = G_{2^{k+1}}. \end{aligned}$$

Hence, $A_{2^{k+1}} \geqslant G_{2^{k+1}}$ and clearly, the equality holds if, and only if, $a_1 = a_2 = \ldots = a_{2^{k+1}}$. This completes the inductive proof of Lemma.

*Proof of the inequality* (1): Applying the inequality (2) to $2^n$ non-negative real numbers $a_1, a_2, \ldots, a_n, \underbrace{G_n, \ldots, G_n}_{2^n - n}$ (where $G_n = \sqrt[n]{a_1 a_2 \ldots a_n}$, we find that

$$\frac{1}{2^n}\left(a_1 + a_2 + \ldots + \underbrace{G_n + \ldots + G_n}_{2^n - n}\right) \geqslant \sqrt[2^n]{a_1 a_2 \ldots a_n (G_n)^{2^n - n}} = G_n,$$

whence taking $a_1 + a_2 + \dots + a_n = nA_n$, we immediately obtain the inequality $A_n \geqslant G_n$. Finally, since by the Lemma, in the above inequality the equality holds if, and only if, $a_1 = a_2 = \dots = a_n = G_n$, it follows that $A_n = G_n$ is true if, and only if, $a_1 = a_2 = \dots = a_n$.

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## 107.33 On the antiderivatives of a monotone function and its inverse

As far as we know, computing antiderivatives of the inverse trigonometric functions and logarithmic function all are initial examples of integration by parts in calculus. In this Note we are motivated by the question that is it possible to compute the above mentioned antiderivatives without using integration by parts? The common property of these functions is their monotonicity. Based on this fact, we demonstrate a geometric argument to relate the antiderivatives of a monotone function $f$ and its inverse $f^{-1}$. Although our geometric implication essentially carries ideas