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Preduals and Nuclear Operators Associated with Bounded, p-Convex, p-Concave and Positive p-Summing Operators

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Abstract. We use Krivine's form of the Grothendieck inequality to renorm the space of bounded linear maps acting between Banach lattices. We construct preduals and describe the nuclear operators associated with these preduals for this renormed space of bounded operators as well as for the spaces of p-convex, p-concave and positive p-summing operators acting between Banach lattices and Banach spaces. The nuclear operators obtained are described in terms of factorizations through classical Banach spaces via positive operators.

1 Introduction

In studying Banach lattices, J. L. Krivine proved the following lattice form of Grothendieck's inequality [18]: If E and F are Banach lattices and $T\colon E \to F$ is a bounded linear operator, then

$$\Big\| \Big(\sum_{i=1}^{n} |Tx_i|^2 \Big)^{1/2} \Big\| \leq K_G \|T\| \Big\| \Big(\sum_{i=1}^{n} |x_i|^2 \Big)^{1/2} \Big\| \tag{1.1}$$

for any $x_1, \dots, x_n \in E$, where K_G denotes the universal Grothendieck constant and $\|T\|$ denotes the operator norm of T (see also [25]).

Of course, sense has to be made of the square of a member of a Banach lattice. Krivine developed a functional calculus in Banach lattices, which gives meaning to such squares (see §2 below) and which plays an important role in the study of the geometry of Banach lattices, *cf.* [7, 9, 18, 22, 25].

Let

$$L(E,F) = \{T\colon E \to F \mid T \text{ is linear}\}, \quad \mathcal{L}(E,F) = \{T \in L(E,F) \mid T \text{ is bounded}\}.$$

We use (1.1) together with (2.1) and (2.2) below to renorm $\mathcal{L}(E,F)$ in a natural way, to find a predual for this renormed space and to describe the nuclear operators pertaining to this predual in terms of a factorization result. This provides analogues, in the renormed setting, of the injective and the projective norms on $E \otimes F$ and gives a description of the associated nuclear operators.

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The methods employed are general enough so as to also yield preduals (see Theorem 5.2 below) and nuclear operators (see Theorem 7.4 below) associated with these preduals for the p-convex, p-concave and positive p-summing operators, as can be found in [1, 18].

It may be of some interest to note that these classes of operators are apparently less general than operator ideals in the sense of Pietsch (see [24, 26]), in that one-sided compositions by positive maps are required rather than bounded ones. Furthermore, some of the norms under consideration are not tensor norms (in the sense of Grothendieck), but are reasonable cross norms.

2 Preliminaries

We follow the terminology of [8, 10] with regard to normed tensor products. If X and Y are Banach spaces and α is a norm on $X \otimes Y$, we denote the normed space $(X \otimes Y, \alpha)$ by $X \otimes_\alpha Y$, its norm completion by $X \widetilde{\otimes}_\alpha Y$ and its continuous dual by $(X \otimes_\alpha Y)'$. A norm α on $X \otimes Y$ is called a *reasonable cross norm* (*cf.* [8, 10, 14]) if α satisfies the conditions:

(i) For $x \in X$ and $y \in Y$, $\alpha(x \otimes y) \leq \|x\| \, \|y\|$.
(ii) For $x' \in X'$ and $y' \in Y'$, $x' \otimes y' \in (X \otimes_\alpha Y)'$ and $\|x' \otimes y'\| \leq \|x'\| \, \|y'\|$.

It is well known that the inequalities in (i) and (ii) may be replaced by equality.

We denote the injective cross norm on $X \otimes Y$ by $\|\cdot\|_\epsilon$ or by ϵ and the projective cross norm on $X \otimes Y$ by $\|\cdot\|_\pi$ or by π (see [8–10, 14, 25, 26]).

For Riesz spaces, we follow the terminology of [30], and for Banach lattices, we follow [22, 23, 29, 30].

If E is a Banach lattice, $x_1, \dots, x_n \in E$ and $1 \leq p \leq \infty$, then by Krivine's functional calculus, $\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \in E$, where

$$\Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} = \sup\Big\{\sum_{i=1}^n a_i x_i \;\Big|\; a_i \in \mathbb{R}, \sum_{i=1}^n |a_i|^q \leq 1\Big\}$$

for $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} = \bigvee_{i=1}^n |x_i|$$

for $p = \infty$ (see [22, pp. 42–44] and [18]).

There is a connection between $\left\| \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \right\|$ and the elements of a suitable tensor product of ℓ^p and E, for which we resort to Chaney's M-norm [3]: If E is Banach lattice and X is a Banach space, then the M-norm on $X \otimes E$ is given by

$$\|u\|_M = \inf\Big\{ \Big\| \sum_{i=1}^n \|x_i\| \, |y_i| \Big\| \;\Big|\; u = \sum_{i=1}^n x_i \otimes y_i \Big\}$$

and also by

$$\Big\|\sum_{i=1}^{n} x_i \otimes y_i\Big\|_M = \Big\|\sup\Big\{\sum_{i=1}^{n} \langle x', x_i\rangle y_i \;\Big|\; x' \in X' \text{ and } \|x'\| \leq 1\Big\}\Big\|,$$

where $x_1, \dots, x_n \in X$ and $y_1, \dots, y_n \in E$ (see also [13, 29]). The M-norm is a reasonable cross norm on $X \otimes E$ (*cf.* [3, Theorem 1.4]) and is equal to Schaefer's m-norm on $X \otimes F$, (see [29, Ch. 4, §7] and [17, Ch. 4, §5]).

Recall from [8] that if X and Y are Banach spaces and α is a reasonable cross norm on $X \otimes Y$, then the *transpose* of α, denoted by ${}^t\alpha$ and defined on $Y \otimes X$ by

$${}^t\alpha\Big(\sum_{i=1}^{n} y_i \otimes x_i\Big) = \alpha\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big),$$

is a reasonable cross norm on $Y \otimes X$ (see also [6, 10, 14]).

The transpose of the M-norm, *i.e.*,

$$\|u\|_{{}^tM} = \inf\Big\{\Big\|\sum_{i=1}^{n} \|y_i\|\,|x_i|\Big\| \;\Big|\; u = \sum_{i=1}^{n} x_i \otimes y_i\Big\},$$

$$\Big\|\sum_{i=1}^{n} x_i \otimes y_i\Big\|_{{}^tM} = \Big\|\sup\Big\{\sum_{i=1}^{n} \langle y', y_i\rangle x_i \;\Big|\; y' \in X' \text{ and } \|y'\| \leq 1\Big\}\Big\|,$$

where $x_1, \dots, x_n \in E$ and $y_1, \dots, y_n \in X$, is Schaefer's l-norm.

Furthermore, if $E = L^p(\mu)$, where (Ω, Σ, μ) is a σ-finite measure space and $1 \leq p < \infty$, then the transpose of the M-norm is the Bochner norm on $L^p(\mu, X)$. Consequently, $L^p(\mu, X)$ is isometrically isomorphic to $L^p(\mu)\,\widetilde{\otimes}_{{}^tM}\, X$.

If E and F are Banach lattices. Then

$$E_+ \otimes F_+ = \Big\{\sum_{i=1}^{n} x_i \otimes y_i \;\Big|\; x_i \in E_+, y_i \in F_+, n \in \mathbb{N}\Big\}$$

is the *projective cone* in $E \otimes F$, where E_+ denotes the positive cone of E.

The norms M and ${}^tM$ have the property that $E\,\widetilde{\otimes}_M\, F$ and $E\,\widetilde{\otimes}_{{}^tM}\, F$, if equipped with the respective norm closures of the projective cone in $E \otimes F$, are Banach lattices [3, 20, 29].

The isometric isomorphism between $L^p(\mu, E)$ and $L^p(\mu)\,\widetilde{\otimes}_{{}^tM}\, E$, mentioned above, is a Riesz isometry (*i.e.*, the isometry preserves the vector lattice structure), provided that $L^p(\mu)\,\widetilde{\otimes}_{{}^tM}\, E$ is equipped with the closure of its projective cone.

Let e_i be the i-th standard unit vector in ℓ^p, *i.e.*, $e_i = (\delta_{ij})$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. If X is a Banach space and E is a Banach lattice, let

$$S(X) = \Big\{\sum_{i=1}^{n} e_i \otimes x_i \;\Big|\; x_i \in X \text{ and } n \in \mathbb{N}\Big\},$$

$$S_+(E) = \Big\{\sum_{i=1}^{n} e_i \otimes x_i \;\Big|\; x_i \in E_+ \text{ and } n \in \mathbb{N}\Big\}.$$

If X is a Banach space, $1 \leq p \leq \infty$, and α is a reasonable cross norm on $\Lambda^p \otimes X$, then $S(X)$ is dense in $\Lambda^p \widetilde{\otimes}_\alpha X$, and if X is a Banach lattice, then $cl_\alpha S_+(X) = cl_\alpha(\Lambda^p_+ \otimes X_+)$ (*cf.* [20, §7]).

It is shown in [20, Lemma 8.1.] that if $x_1, \dots, x_n \in E$, then

$$\Big\| \Big(\sum_{i=1}^n |x_i|^p \Big)^{1/p} \Big\| = \Big\| \sum_{i=1}^n e_i \otimes x_i \Big\|_{\ell^p \otimes_M E} \tag{2.1}$$

for $1 \leq p < \infty$, and

$$\Big\| \bigvee_{i=1}^n |x_i| \Big\| = \Big\| \sum_{i=1}^n e_i \otimes x_i \Big\|_{c_0 \otimes_M E}. \tag{2.2}$$

In terms of the M-tensor product, (1.1) can be stated as

$$\Big\| \sum_{i=1}^n e_i \otimes T x_i \Big\|_{\ell^2 \otimes_M F} \leq K_G \|T\| \Big\| \sum_{i=1}^n e_i \otimes x_i \Big\|_{\ell^2 \otimes_M E}. \tag{2.3}$$

We let $\Lambda^p = \ell^p$ for $1 \leq p < \infty$ and $\Lambda^\infty = c_0$ and denote the identity map on X by id_X for any Banach space X.

The p-convex and p-concave operators (*cf.* [22]) can also be described in terms of suitable tensor products equipped with the M-norm or its transpose ${}^tM$. Let E and F be Banach lattices, X and Y be Banach spaces and $1 \leq p \leq \infty$. Then it follows from [22, p. 45] that a linear operator $T\colon E \to Y$ is p-concave if and only if

$$\mathrm{id}_{\Lambda^p} \otimes T\colon \Lambda^p \otimes_M E \to \Lambda^p \otimes_{{}^tM} Y$$

is continuous, and a linear operator $T\colon X \to F$ is p-convex if and only if

$$\mathrm{id}_{\Lambda^p} \otimes T\colon \Lambda^p \otimes_{{}^tM} X \to \Lambda^p \otimes_M F$$

is continuous.

3 The Banach Space $\mathcal{G}_{(\gamma_p,\delta_p)}(X, Y)$

Let X, X_1, Y, Y_1 be Banach spaces and γ and δ reasonable cross norms on $X_1 \otimes Y_1$ and $X \otimes Y$, respectively. If $S \in \mathcal{L}(X_1, X)$, $R \in \mathcal{L}(Y_2, Y)$ and $S \otimes R \in \mathcal{L}(X_1 \otimes_\gamma Y_1, X \otimes_\delta Y)$, then since δ is a reasonable cross norm,

$$\|(S \otimes R)(x \otimes y)\| = \|Sx\| \, \|Ry\| \text{ for all } (x, y) \in X_1 \times Y_1,$$

from which we get

$$\|S \otimes R\| \geq \|S\| \, \|R\|.$$

In particular, if $R \in \mathcal{L}(Y_1, Y)$ and $\mathrm{id}_X \otimes R \in \mathcal{L}(X \otimes_\gamma Y_1, X \otimes_\delta Y)$, then

$$\|\mathrm{id}_X \otimes R\| \geq \|R\|. \tag{3.1}$$

Definition 3.1 Let X and Y be Banach spaces, γ and δ reasonable cross norms on $\Lambda^p \otimes X$ and $\Lambda^p \otimes Y$, respectively, and $1 \leq p \leq \infty$. Set

$$\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y) := \{T \in L(X,Y) \mid \mathrm{id}_{\Lambda^p} \otimes T \in \mathcal{L}(\Lambda^p \otimes_\gamma X, \Lambda^p \otimes_\delta Y)\}$$

and

$$\|T\|_{\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)} := \|\mathrm{id}_{\Lambda^p} \otimes T\| \text{ for all } T \in \mathcal{G}_{(\gamma_p,\delta_p)}(X,Y). \tag{3.2}$$

Proposition 3.2 *Let X and Y be Banach spaces, γ and δ reasonable cross norms on $\Lambda^p \otimes X$ and $\Lambda^p \otimes Y$, respectively, and $1 \leq p \leq \infty$. Then $\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)}$.*

Proof It is clear that $\|\cdot\|_{\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)}$ is a norm on $\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)$. To prove completeness, let (T_n) be a Cauchy sequence in $\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)$. Then $(\mathrm{id}_{\Lambda^p} \otimes T_n)$ is a Cauchy sequence in $\mathcal{L}(\Lambda^p \otimes_\gamma X, \Lambda^p \otimes_\delta Y)$. Hence, $(\mathrm{id}_{\Lambda^p} \otimes T_n)$ converges to some $S \in \mathcal{L}(\Lambda^p \otimes_\gamma X, \Lambda^p \widetilde{\otimes}_\delta Y)$. By (3.1) and (3.2), (T_n) is also a Cauchy sequence in $\mathcal{L}(X,Y)$ and hence converges in operator norm to some $T \in \mathcal{L}(X,Y)$. Consequently, if $x \in X$, then $S(e_k \otimes x) = \lim_{n\to\infty}(\mathrm{id}_{\Lambda^p} \otimes T_n)(e_k \otimes x) = \lim_{n\to\infty}(e_k \otimes T_n x) = e_k \otimes Tx = (\mathrm{id}_{\Lambda^p} \otimes T)(e_k \otimes x)$, from which we get $S(\Lambda^p \otimes_\gamma X) \subseteq (\Lambda^p \otimes_\delta Y)$ and $S = \mathrm{id}_{\Lambda^p} \otimes T$. But then $T \in \mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)$ and $T_n \to T$ in $\|\cdot\|_{\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)}$. ■

We renorm $\mathcal{L}(E,F)$.

Theorem 3.3 *If E and F are Banach lattices and $T\colon E \to F$, then $\mathcal{G}_{(M_2,M_2)}(E,F)$ is a Banach space which is linearly and topologically isomorphic to $\mathcal{L}(E,F)$. Moreover,*

$$\|T\| \leq \|T\|_{\mathcal{G}_{(M_2,M_2)}(E,F)} \leq K_G\|T\|.$$

Proof The proof follows from (2.3), (3.1) and Proposition 3.2. ■

The positive p-summing operators considered in [1] can also be described in terms of a suitable tensor product (see Example 3.4 below), for which we need the following notation: If E and F are Banach lattices, let

$$L_+(E,F) = \{T\colon E \longrightarrow F \mid T \text{ is linear and positive}\}.$$

We denote by $|\epsilon|$ the norm on $E \otimes F$ induced by the r-norm

$$\|T\|_r := \inf\{\|S\| \mid \pm T \leq S,\ S \in L_+(E',F)\}$$

defined on $\mathcal{L}^r(E',F) = L_+(E',F) - L_+(E',F)$. The $|\epsilon|$-norm is a reasonable cross norm on $E \otimes F$ and $E \widetilde{\otimes}_{|\epsilon|} F$ is a Banach lattice with positive cone the $|\epsilon|$-closure of the projective cone $E_+ \otimes F_+$ (*cf.* [20]).

If E is a Banach lattice, the norms $|\epsilon|$ and ${}^tM$ agree on $\ell^1 \otimes E$ (see [29, Ch. 4] or [20, Theorem 8.2]) and the norms ϵ and M agree on $c_0 \otimes E$ (see [3, Proposition 1.5] or [29, Ch.4]).

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Example 3.4

$\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)$	$CN_p(X,Y)$	$\mathfrak{P}_p(X,Y)$	$\mathcal{S}_p(X,Y)$	$\mathcal{L}^{\mathrm{pcav}}(X,Y)$	$\mathcal{L}^{\mathrm{pvex}}(X,Y)$
$\|\cdot\|_{\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y)}$	N_p	P_p	σ_p	$\|\cdot\|_{\mathrm{pcav}}$	$\|\cdot\|_{\mathrm{pvex}}$
Operator	Cohen p-Nuclear	p-Absolutely summing	Positive p-summing	p-concave	p-convex
X Banach	space	space	lattice	lattice	space
Y Banach	space	space	space	space	lattice
γ	ϵ	ϵ	$\lvert\epsilon\rvert$	M	${}^tM$
δ	π	${}^tM$	${}^tM$	${}^tM$	M
p	$1 < p < \infty$	$1 \le p < \infty$	$1 \le p < \infty$	$1 \le p \le \infty$	$1 \le p \le \infty$
Reference	[5]	[6, p. 127]	[1]	[18,22]	[18,22]

In the terminology of [29], the ∞-convex operators are known as *majorizing* operators and the 1-concave operators are known as *cone absolutely summing* operators. In the terminology of [1], the latter operators are known as *positive* 1-*summing* operators.

4 The Norm $\|\cdot\|_{(\gamma_p,\delta_q)}$

Let X and Y be Banach spaces, E and F Banach lattices and $\frac{1}{p}+\frac{1}{q}=1$.

Our aim is to construct a predual for $\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y')$, for special cases of γ_p and δ_p. We, therefore, consider suitable norms on $X\otimes Y$, and resort to vector-valued sequence spaces.

Denote the space of weakly p-summable sequences in X by $\ell_p^{\mathrm{weak}}(X)$ for $1 \le p \le \infty$. The following result, formulated by Grothendieck, relates $\ell_p^{\mathrm{weak}}(X)$ to $\mathcal{L}(\ell^q, X)$ (see [5] and [16, §19.4]).

Theorem 4.1 *Let X be a Banach space and let $\frac{1}{p}+\frac{1}{q}=1$.*

(i) *$\ell_p^{\mathrm{weak}}(X)$ is isometrically isomorphic to $\mathcal{L}(\ell^q, X)$ for $1 < p \le \infty$ and $\ell_1^{\mathrm{weak}}(X)$ is isometrically isomorphic to $\mathcal{L}(c_0, X)$.*

(ii) *$\ell_p^{\mathrm{weak}}(X')$ is isometrically isomorphic to $\mathcal{L}(X, \ell^p)$ for $1 \le p \le \infty$ and $(c_0)^{\mathrm{weak}}(X')$ is isometrically isomorphic to $\mathcal{L}(X, c_0)$.*

It is well known that

$$\Lambda_{p,c}^{\mathrm{weak}}(X) := \left\{ (x_i) \in \Lambda_p^{\mathrm{weak}}(X) \;\middle|\; \|(x_i)_{i=n}^{\infty}\|_{\Lambda_p^{\mathrm{weak}}(X)} \to 0 \text{ as } n \to \infty \right\}$$

is isometrically isomorphic to $\Lambda^p \widetilde{\otimes}_\epsilon X$ for $1 \le p \le \infty$. (For details, see [5, 6, 9, 11, 12, 14, 20, 21].)

To proceed with the task at hand, let

$$\mathcal{F}(X) := \{(x_1, \dots, x_n, 0, \dots) \mid x_i \in X \text{ and } n \in \mathbb{N}\},$$

$$\mathcal{F}_+(E) := \{(x_1, \dots, x_n, 0, \dots) \mid x_i \in E_+ \text{ and } n \in \mathbb{N}\}.$$

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We recall from [20] that $\mathcal{F}(E)$ is an Archimedean Riesz space with positive cone

$$\mathcal{F}_+(E) = \mathcal{F}(E) \cap (E^{\mathbb{N}})_+,$$

where $(E^{\mathbb{N}})_+$ is the cone in $E^{\mathbb{N}}$ generated by the pointwise order, *i.e.*,

$$(x_i) \leq (y_i) \Leftrightarrow x_i \leq y_i \text{ for all } i \in \mathbb{N}.$$

The map

$$\kappa\colon \sum_{k=1}^{n} e_k \otimes x_k \mapsto (x_1, x_2, \dots, x_n, 0, \dots),$$

provides an identification of $S(X)$ and $\mathcal{F}(X)$ and also yields a one-to-one correspondence between $S_+(E)$ and $\mathcal{F}_+(E)$ (*cf.* [20, §7]).

Let $1 \leq p \leq \infty$ and α a reasonable cross norm on $\Lambda^p \otimes X$. Set

$$\Lambda_\alpha^{p,c}(X) := \Big\{ (x_i) \in E^{\mathbb{N}} \Bigm| \sum_{i=1}^{\infty} e_i \otimes x_i \in \Lambda^p \mathbin{\widetilde{\otimes}_\alpha} X \Big\}$$

and

$$\|(x_i)\|_{\Lambda_\alpha^{p,c}(X)} := \Big\| \sum_{i=1}^{\infty} e_i \otimes x_i \Big\|_{\Lambda^p \widetilde{\otimes}_\alpha X}.$$

Hence by our discussion above,

$$\Lambda_\epsilon^{p,c}(X) = \Lambda_{p,c}^{\text{weak}}(X) \text{ for } 1 \leq p \leq \infty.$$

Theorem 4.2 *Let X be a Banach space, $1 \leq p \leq \infty$ and α a reasonable cross norm on $\Lambda^p \otimes X$. Then*

(i) *$\Lambda_\alpha^{p,c}(X)$ is a Banach space with respect to the norm $\|\cdot\|_{\Lambda_\alpha^{p,c}(X)}$.*

(ii) *$(x_i) \in \Lambda_\alpha^{p,c}(X)$ if and only if $\lim_{m\to\infty} \|(x_i)_{i=m}^{\infty}\|_{\Lambda_\alpha^{p,c}(X)} = 0$.*

(iii) *$\mathcal{F}(X) \subseteq \Lambda_\alpha^{p,c}(X)$ and $cl_\alpha \mathcal{F}(X) = \Lambda_\alpha^{p,c}(X)$.*

(iv) *$\Lambda^p \mathbin{\widetilde{\otimes}_\alpha} X$ is isometrically isomorphic to $\Lambda_\alpha^{p,c}(X)$; the isometry is given by the continuous extension of κ to $\Lambda^p \mathbin{\widetilde{\otimes}_\alpha} X$.*

If E is a Banach lattice, then the isometric isomorphism in (iv) *yields a one-to-one correspondence between $cl_\alpha(\Lambda_+^p \otimes E_+)$ and*

$$\Lambda_\alpha^{p,c}(E)_+ := \{(x_i) \in (E^{\mathbb{N}})_+ \bigm| \sum_{i=1}^{\infty} e_i \otimes x_i \in \Lambda^p \mathbin{\widetilde{\otimes}_\alpha} E\}.$$

Proof If $0 = \|(x_i)\|_{\Lambda_\alpha^{p,c}(X)} = \|\sum_{i=1}^{\infty} e_i \otimes x_i\|_{\Lambda^p \widetilde{\otimes}_\alpha X}$, then since $\|\cdot\|_\epsilon \leq \|\cdot\|_\alpha$, it follows from Theorem 4.1 and $(\sum_{i=1}^{\infty} e_i \otimes x_i)e_j = x_j$, that $x_j = 0$ for all $j \in \mathbb{N}$. It is then easy to verify that $\|\cdot\|_{\Lambda_\alpha^{p,c}(X)}$ is a norm and

$$(x_i) \in \Lambda_\alpha^{p,c}(X) \text{ if and only if } \lim_{m\to\infty} \|(x_i)_{i=m}^{\infty}\|_{\Lambda_\alpha^{p,c}(X)} = 0,$$

i.e., (ii) holds.

To prove (iii), note that if $x_1, \dots, x_n \in X$, then $\sum_{i=1}^n e_i \otimes x_i \in \Lambda^p \otimes_\alpha X$ and so $\mathcal{F}(X) \subseteq \Lambda_\alpha^{p,c}(X)$. Also, by (ii), we get that $\Lambda_\alpha^{p,c}(X)$ is contained in the norm completion, denoted by H, of $(\mathcal{F}(X), \|\cdot\|_{\Lambda_\alpha^{p,c}(X)})$. But H is isometrically isomorphic to $\Lambda^p \widetilde{\otimes}_\alpha X$; the isomorphism is given by the continuous extension of κ, denoted by $\widetilde{\kappa}\colon \Lambda^p \widetilde{\otimes}_\alpha X \to H$. We claim that $H \subseteq \Lambda_\alpha^{p,c}(X)$. If $z \in H$, then there exists $\sum_{i=1}^\infty e_i \otimes x_i \in \Lambda^p \widetilde{\otimes}_\alpha X$ such that $z = \widetilde{\kappa}\big(\sum_{i=1}^\infty e_i \otimes x_i\big)$. Then $(x_i) \in \Lambda_\alpha^{p,c}(X)$ and

$$(x_i) = \widetilde{\kappa}\Big(\sum_{i=1}^\infty e_i \otimes x_i\Big) = z.$$

It follows from what has just been shown that $\Lambda_\alpha^{p,c}(X)$ is norm complete and that $\Lambda_\alpha^{p,c}(X)$ is isometrically isomorphic to $\Lambda^p \widetilde{\otimes}_\alpha X$. Thus, (i) and (iv) hold.

The statement about the order holds due to the fact that κ yields a one-to-one correspondence between $S_+(E)$ and $\mathcal{F}_+(E)$, as mentioned above, and $cl_\alpha S_+(E) = cl_\alpha(\Lambda_+^p \otimes E_+)$, as mentioned in §2. ■

Where convenient, we shall identify $\Lambda^p(X) \widetilde{\otimes}_\alpha X$ and $\Lambda_\alpha^{p,c}(X)$ and denote

$$\|(x_i)\|_{\Lambda_\alpha^{p,c}(X)} \ \text{ by } \ \|(x_i)\|_{\Lambda^p \widetilde{\otimes}_\alpha X}.$$

Also, if the space of all absolutely p-summable sequences (x_n) in X is denoted by $\ell_p^{\text{strong}}(X)$, then $\ell_p^{\text{strong}}(X)$ is isometrically isomorphic to $\ell^p \widetilde{\otimes}_{t_M} X$. (For details of $\ell_p^{\text{strong}}(X)$, see [9].) Hence,

$$\ell_p^{\text{strong}}(X) = \Lambda_{t_M}^{p,c}(X) \text{ for } 1 \le p < \infty.$$

Using the notation convention made above, *i.e.*, by denoting $\|(x_i)\|_{\Lambda_\alpha^{p,c}(X)}$ by $\|(x_i)\|_{\Lambda^p \widetilde{\otimes}_\alpha X}$, and considering $1 \le p \le \infty$, the Chevet–Saphar norms g_p and d_p are given by

$$g_p(u) = \inf\Big\{ \|(x_i)\|_{\Lambda^p \otimes_{t_M} X} \|(y_i)\|_{\Lambda^q \otimes_\epsilon Y} \ \Big|\ u = \sum_{i=1}^n x_i \otimes y_i \Big\},$$

$$d_p(u) = \inf\Big\{ \|(x_i)\|_{\Lambda^p \otimes_\epsilon X} \|(y_i)\|_{\Lambda^q \otimes_{t_M} Y} \ \Big|\ u = \sum_{i=1}^n x_i \otimes y_i \Big\}$$

for all $u \in X \otimes Y$, and the Cohen norms w_p are given by

$$w_p(u) = \inf\Big\{ \|(x_i)\|_{\Lambda^p \otimes_\epsilon X} \|(y_i)\|_{\Lambda^q \otimes_\epsilon Y} \ \Big|\ u = \sum_{i=1}^n x_i \otimes y_i \Big\}$$

for all $u \in X \otimes Y$ (see [4–6, 19, 26–28]).

The norm w_2 is Grothendieck's important *Hilbertian* norm on $X \otimes Y$ [6, 14, 19, 25, 26]. It is well known that these norms are reasonable cross norms on $X \otimes Y$ (in fact, these norms are tensor norms in the sense of Grothendieck, see [5, 6, 12, 26]).

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For $1 \le p \le \infty$ and E a Banach lattice, denote the space of positive weakly p-summable sequences in E by $\ell_p^{|\text{weak}|}(E)$ (see [2, 20, 21] for details). It is a Banach lattice with the ordering induced by the pointwise ordering on $E^{\mathbb{N}}$. Furthermore, $\Lambda^p \widetilde{\otimes}_{|\epsilon|} E$ is Riesz and isometrically isomorphic to $\Lambda_{p,c}^{|\text{weak}|}(E)$, where

$$\Lambda_{p,c}^{|\text{weak}|}(E) := \left\{ (x_i) \in \Lambda_p^{|\text{weak}|}(E) \;\middle|\; \|(x_i)_{i=n}^{\infty}\|_{\Lambda_p^{|\text{weak}|}(E)} \to 0 \text{ as } n \to \infty \right\}.$$

Hence,

$$\Lambda_{|\epsilon|}^{p,c}(E) = \Lambda_{p,c}^{|\text{weak}|}(E) \text{ for } 1 \le p \le \infty.$$

The ${}^tM$-norm has the following description, which is a Chevet–Saphar–Cohen norm look-alike:

$$\|u\|_{{}^tM} = \inf\Big\{ \|(x_i)\|_{\ell^1 \otimes_{|\epsilon|} X} \|(y_i)\|_{c_0 \otimes_\epsilon Y} \;\Big|\; u = \sum_{i=1}^n x_i \otimes y_i \Big\}$$

for all $u \in E \otimes Y$ (*cf.* [21]). However, ${}^tM$ is not a tensor norm, as was noted by Pisier [6, 25].

Definition 4.3 Let X and Y be Banach spaces, γ and δ reasonable cross norms on $\Lambda^p \otimes X$ and $\Lambda^q \otimes Y$, respectively, $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Define

$$\|u\|_{(\gamma_p,\delta_q)} := \inf\Big\{ \|(x_i)\|_{\Lambda^p \otimes_\gamma X} \|(y_i)\|_{\Lambda^q \otimes_\delta Y} \;\Big|\; u = \sum_{i=1}^n x_i \otimes y_i \Big\} \text{ for all } u \in X \otimes Y.$$

The convention is used to denote $(\gamma_1, \delta_\infty)$ by (γ_1, δ_0) and $(\gamma_\infty, \delta_0)$ by (γ_0, δ_1).

In general, it seems difficult to verify the triangle inequality for $\|\cdot\|_{(\gamma_p,\delta_q)}$. However, it follows easily from the definition of $\|\cdot\|_{(\gamma_p,\delta_q)}$ that

- $w_p(u) \le \|u\|_{(\gamma_p,\delta_q)}$ for all $u \in X \otimes Y$;
- if $u \in X \otimes Y$, then $\|u\|_{(\gamma_p,\delta_q)} = 0$ if and only if $u = 0$;
- if $\lambda \in \mathbb{R}$, then $\|\lambda u\|_{(\gamma_p,\delta_q)} = |\lambda| \, \|u\|_{(\gamma_p,\delta_q)}$ for all $u \in X \otimes Y$;
- if $x \in X$ and $y \in Y$, then $\|x \otimes y\|_{(\gamma_p,\delta_q)} \le \|x\| \, \|y\|$;
- if $x' \in X'$ and $y' \in Y'$, then $x' \otimes y' \in (X \otimes_{(\gamma_p,\delta_q)} Y)'$; because by Cohen's result that w_p is a reasonable cross norm, we get that $x' \otimes y' \in (X \otimes_{w_p} Y)'$, hence, the inequality $w_p(u) \le \|u\|_{(\gamma_p,\delta_q)}$ implies $x' \otimes y' \in (X \otimes_{(\gamma_p,\delta_q)} Y)'$ and $\|x' \otimes y'\|_{(\gamma_p,\delta_q)'} \le \|x' \otimes y'\|_{w_p'} \le \|x'\| \, \|y'\|$.

For a number of interesting special cases we provide verification of the triangle inequality for $\|\cdot\|_{(\gamma_p,\delta_q)}$.

Theorem 4.4 *Let E and F Banach lattices, X and Y Banach spaces, $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \le p \le \infty$. Then*

(i) $\|\cdot\|_{(M_2,M_2)}$ *and* $\|\cdot\|_{(|\epsilon|_p,M_q)}$ *are reasonable cross norms on* $E \otimes F$.
(ii) $\|\cdot\|_{({}^tM_p,M_q)}$ *is a reasonable cross norm on* $X \otimes F$.

(iii) $\|\cdot\|_{(M_p,{}^tM_q)}$ *and* $\|\cdot\|_{(|\epsilon|_p,{}^tM_q)}$ *are reasonable cross norms on* $E\otimes Y$.

Proof (i) Consider $\|\cdot\|_{(|\epsilon|_p,M_q)}$. By the remark preceding Example 3.4,

$$\|\cdot\|_{(|\epsilon|_1,M_0)}=\|\cdot\|_{(|\epsilon|_1,\epsilon_0)}=\|\cdot\|_{{}^tM},$$

and since the latter is a norm, we may assume that $p\neq 1$.

By our remarks preceding the theorem, we only need to verify the triangle inequality. Let $u_1,u_2\in E\otimes F$ and let $\epsilon>0$ be given. For $i=1,2$ there are representations

$$u_i=\sum_{j=1}^{n_i}\bar{x}_j^{(i)}\otimes\bar{y}_j^{(i)}$$

such that

$$\|(\bar{x}_j^{(i)})\|_{\Lambda^p\otimes_{|\epsilon|}E}\|(\bar{y}_j^{(i)})\|_{\Lambda^q\otimes_M F}\leq\|u_i\|_{(|\epsilon|_p,M_q)}+\epsilon.$$

Let

$$x_j^{(i)}=\Big[\|(\bar{y}_j^{(i)})\|_{\Lambda^q\otimes_M F}\Big]^{1/p}\Big[\Big[\|(\bar{x}_j^{(i)})\|_{\Lambda^p\otimes_{|\epsilon|}E}\Big]^{1/q}\Big]^{-1}\bar{x}_j^{(i)}$$

$$y_j^{(i)}=\Big[\|(\bar{x}_j^{(i)})\|_{\Lambda^p\otimes_{|\epsilon|}E}\Big]^{1/q}\Big[\Big[\|(\bar{y}_j^{(i)})\|_{\Lambda^q\otimes_M F}\Big]^{1/p}\Big]^{-1}\bar{y}_j^{(i)}.$$

Then

$$\|(x_j^{(i)})\|_{\Lambda^p\otimes_{|\epsilon|}E}=\Big[\|(\bar{x}_j^{(i)})\|_{\Lambda^p\otimes_{|\epsilon|}E}\Big]^{1/p}\Big[\|(\bar{y}_j^{(i)})_{\Lambda^q\otimes_M F}\Big]^{1/p}\leq\big(\|u_i\|_{(|\epsilon|_p,M_q)}+\epsilon\big)^{1/p},$$

$$\|(y_j^{(i)})\|_{\Lambda^q\otimes_M F}\leq\big(\|u_i\|_{(|\epsilon|_p,M_q)}+\epsilon\big)^{1/q}$$

and

$$u_1+u_2=\sum_{i=1}^{2}\sum_{j=1}^{n_i}\bar{x}_j^{(i)}\otimes\bar{y}_j^{(i)}=\sum_{i=1}^{2}\sum_{j=1}^{n_i}x_j^{(i)}\otimes y_j^{(i)}.$$

If $B(E')_+=\{x'\in E'\mid\|x'\|\leq 1\}$, then

$$\begin{aligned}
&\|(x_j^{(i)})_{i,j}\|_{\Lambda^p\otimes_{|\epsilon|}E}\\
&=\sup\Big\{\Big\|\big(\langle|x_j^{(i)}|,x'\rangle\big)_{i,j}\Big\|_{\Lambda^p}\ \Big|\ x'\in B(E')_+\Big\}\qquad\text{(cf. [20, §7])}\\
&=\sup_{x'\in B(E')_+}\Big\|\Big(\langle|x_1^{(1)}|,x'\rangle,\langle|x_2^{(1)}|,x'\rangle,\dots,\langle|x_{n_1}^{(1)}|,x'\rangle,\langle|x_1^{(2)}|,x'\rangle,\\
&\qquad\qquad\langle|x_2^{(2)}|,x'\rangle,\dots,\langle|x_{n_2}^{(2)}|,x'\rangle,0,0,\dots\Big)\Big\|_{\Lambda^p}\\
&=\sup_{x'\in B(E')_+}\Big\|\Big(\big\|\big(\langle|x_j^{(1)}|,x'\rangle\big)_{j=1}^{n_1}\big\|_{\Lambda^p},\big\|\big(\langle|x_j^{(2)}|,x'\rangle\big)_{j=1}^{n_2}\big\|_{\Lambda^p},0,0,\dots\Big)\Big\|_{\Lambda^p}\\
&\leq\Big\|\Big(\big\|(x_j^{(1)})_{j=1}^{n_1}\big\|_{\Lambda^p\otimes_{|\epsilon|}E},\big\|(x_j^{(2)})_{j=1}^{n_2}\big\|_{\Lambda^p\otimes_{|\epsilon|}Y},0,0,\dots\Big)\Big\|_{\Lambda_p}\\
&\leq\big(\|u_1\|_{(|\epsilon|_p,M_q)}+\|u_2\|_{(|\epsilon|_p,M_q)}+2\epsilon\big)^{1/p}.
\end{aligned}$$

Hence,

$$\big\| (x_j^{(i)})_{i,j} \big\|_{\Lambda^p \otimes_{|\epsilon|} E} \leq \big(\|u_1\|_{(|\epsilon|_p, M_q)} + \|u_2\|_{(|\epsilon|_p, M_q)} + 2\epsilon \big)^{1/p}.$$

Also,

$$\begin{aligned}
\|(y_j^{(i)})_{i,j}\|_{\Lambda^q \otimes_M F} &= \big\| (y_1^{(1)}, y_2^{(1)}, \dots, y_{n_1}^{(1)}, y_1^{(2)}, \dots, y_{n_2}^{(2)}, 0, 0, \dots) \big\| \\
&= \Big\| \big(|y_1^{(1)}|^q + |y_2^{(1)}|^q + \dots + |y_{n_1}^{(1)}|^q + |y_1^{(2)}|^q + |y_2^{(2)}|^q + \dots + |y_{n_2}^{(2)}|^q \big)^{1/q} \Big\| \\
&\leq \Big\| \Big(\sum_{i=1}^{n_1} |y_i^{(1)}|^q \Big)^{1/q} \Big\| + \Big\| \Big(\sum_{i=1}^{n_2} |y_i^{(2)}|^q \Big)^{1/q} \Big\| \\
&= \big\| (y_j^{(1)})_{j=1}^{n_1} \big\|_{\Lambda^q \otimes_M F} + \big\| (y_j^{(2)})_{j=1}^{n_2} \big\|_{\Lambda^q \otimes_M F} \\
&\leq \big(\|u_1\|_{(|\epsilon|_p, M_q)} + \|u_2\|_{(|\epsilon|_p, M_q)} + 2\epsilon \big)^{1/q}.
\end{aligned}$$

All the other proofs follow in a similar manner. ∎

5 A Predual for $\mathcal{G}_{(\gamma_p, \delta_p)}(X, Y')$

It is well known (see [3, 5, 6, 15, 29]) that if X and Y are Banach spaces and E and F are Banach lattices, then

(i) $(X \otimes_{w_p} Y)' = CN_p(X, Y')$ for $1 < p < \infty$;
(ii) $(X \otimes_{d_p} Y)' = \mathfrak{P}_p(X, Y')$ for $1 \leq p < \infty$;
(iii) $(E \otimes_{(|\epsilon|_1, \epsilon_0)} Y)' = \mathcal{L}^{1\mathrm{cav}}(E, Y')$.

Let $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p \leq \infty$.

The projective norm has the property that $X \widetilde{\otimes}_\pi Y \hookrightarrow X'' \widetilde{\otimes}_\pi Y$ is an isometry for any Banach spaces X and Y (*cf.* [26, p. 25]), so $c_0 \widetilde{\otimes}_\pi Y \hookrightarrow \ell^\infty \widetilde{\otimes}_\pi Y$ is an isometry. Due to the fact that ℓ^p has the metric approximation property, the nuclear operators are isometrically embedded in the integral operators. Consequently, $\Lambda^p \widetilde{\otimes}_\pi Y' \hookrightarrow (\Lambda^q \otimes_\epsilon Y)'$ is an isometry (into).

It is also known (*cf.* [3]) that $\Lambda^p \widetilde{\otimes}_{tM} Y' \hookrightarrow (\Lambda^q \otimes_{tM} Y)'$ is an isometry (into) and $\Lambda^p \widetilde{\otimes}_M F' \hookrightarrow (\Lambda^q \otimes_M F)'$ is a Riesz isometry (into), provided that $\Lambda^p \widetilde{\otimes}_M F'$ is equipped with the closure of the projective cone. If Y is a Banach lattice, the former isometry is also a Riesz isometry [3, 17, 29].

Proposition 5.1 *Let X and Y be Banach spaces, $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p \leq \infty$. Suppose*

(i) *γ and δ are reasonable cross norms on $\Lambda^p \otimes X$ and $\Lambda^p \otimes Y'$, respectively;*
(ii) *there exists a reasonable cross norm θ on $\Lambda^q \otimes Y$ such that $\Lambda^p \widetilde{\otimes}_\delta Y' \hookrightarrow \big(\Lambda^q \otimes_\theta Y\big)'$ is an isometry;*
(iii) *$\|\cdot\|_{(\gamma_p, \theta_q)}$ is a reasonable cross norm on $X \otimes Y$.*

Then $(X \otimes_{(\gamma_p, \theta_q)} Y)' = \mathcal{G}_{(\gamma_p, \delta_p)}(X, Y')$.

Proof Let $T \in \mathcal{G}_{(\gamma_p,\delta_p)}(X, Y')$ and define $f_T \colon X \otimes Y \to \mathbb{R}$ by

$$f_T\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big) = \sum_{i=1}^{n} \langle y_i, Tx_i \rangle.$$

Note that $(Tx_i) \in \Lambda^p \otimes_\delta Y' \hookrightarrow (\Lambda^q \otimes_\theta Y)'$ and $(y_i) \in \Lambda^q \otimes_\theta Y$; hence,

$$\sum_{i=1}^{n} \langle y_i, Tx_i \rangle = \big\langle (y_i)_{i=1}^n, (Tx_i)_{i=1}^n \big\rangle.$$

Thus,

$$\begin{aligned}
\Big| f_T\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big) \Big| &= \Big| \sum_{i=1}^{n} \langle y_i, Tx_i \rangle \Big| \\
&= \big| \big\langle (y_i)_{i=1}^n, (Tx_i)_{i=1}^n \big\rangle \big| \\
&\leq \big\| (Tx_i)_{i=1}^n \big\|_{\Lambda^p \otimes_\delta Y'} \big\| (y_i)_{i=1}^n \big\|_{\Lambda^q \otimes_\theta Y} \\
&\leq \|T\|_{\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y')} \|(x_i)_{i=1}^n\|_{\Lambda^p \otimes_\gamma X} \|(y_i)_{i=1}^n\|_{\Lambda^q \otimes_\theta Y}.
\end{aligned}$$

Consequently, if $u \in X \otimes_{(\gamma_p,\theta_q)} Y$, then $|f_T(u)| \leq \|T\|_{\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y')} \|u\|_{(\gamma_p,\theta_q)}$, *i.e.*,

$$\|f_T\| \leq \|T\|_{\mathcal{G}_{(\gamma_p,\delta_p)}(X,Y')}.$$

Conversely, let $f \in (X \otimes_{(\gamma_p,\theta_q)} Y)'$. Define $T_f \colon X \to Y'$ by $\langle y, T_f x \rangle = f(x \otimes y)$. Then

$$\begin{aligned}
\|f\| &= \sup\big\{ |\langle u, f \rangle| \;\big|\; \|u\|_{(\gamma_p,\theta_q)} \leq 1 \big\} \\
&= \sup\Big\{ \Big| \sum_{i=1}^{n} \langle y_i, T_f x_i \rangle \Big| \;\Big| \\
&\qquad\qquad \inf\Big\{ \big\| (x_i)_{i=1}^n \big\|_{\Lambda^p \otimes_\gamma X} \big\| (y_i)_{i=1}^n \big\|_{\Lambda^q \otimes_\theta Y} \;\Big|\; u = \sum_{i=1}^{n} x_i \otimes y_i \Big\} \leq 1 \Big\} \\
&\geq \sup\Big\{ \Big| \sum_{i=1}^{n} \langle y_i, T_f x_i \rangle \Big| \;\Big|\; \big\| (x_i)_{i=1}^n \big\|_{\Lambda^p \otimes_\gamma X} \leq 1 \text{ and } \big\| (y_i)_{i=1}^n \big\|_{\Lambda^q \otimes_\theta Y} \leq 1 \Big\} \\
&= \sup\Big\{ \sup\Big\{ \Big| \sum_{i=1}^{n} \langle y_i, T_f x_i \rangle \Big| \;\Big|\; \big\| (y_i)_{i=1}^n \big\|_{\Lambda^q \otimes_\theta Y} \leq 1 \Big\} \;\Big|\; \big\| (x_i)_{i=1}^n \big\|_{\Lambda^p \otimes_\gamma X} \leq 1 \Big\} \\
&= \sup\Big\{ \big\| (T_f x_i)_{i=1}^n \big\|_{\Lambda^p \otimes_\delta Y'} \;\Big|\; \big\| (x_i)_{i=1}^n \big\|_{\Lambda^p \otimes_\gamma X} \leq 1 \Big\},
\end{aligned}$$

because $T_f x_i \in Y'$ for $i = 1 \dots n$, and by assumption, $\Lambda^p \widetilde{\otimes}_\delta Y' \hookrightarrow (\Lambda^q \otimes_\theta Y)'$ is an isometry; thus

$$\begin{aligned}\big\| (T_f x_i)_{i=1}^n \big\|_{\Lambda^p \otimes_\delta Y'} &= \big\| (T_f x_i)_{i=1}^n \big\|_{(\Lambda^q \otimes_\theta Y)'} \\ &= \sup\Big\{ \Big| \sum_{i=1}^n \langle y_i, T_f x_i \rangle \Big| \;\Big|\; \|(y_i)_{i=1}^n\|_{\Lambda^q \otimes_\theta Y} \le 1 \Big\}.\end{aligned}$$

Hence,

$$\sup\Big\{ \big\| (T_f x_i)_{i=1}^n \big\|_{\Lambda^p \otimes_\delta Y'} \;\Big|\; \big\| (x_i)_{i=1}^n \big\|_{\Lambda^p \otimes_\gamma X} \le 1 \Big\} = \|T_f\|_{\mathcal{G}_{(\gamma_p, \delta_p)}(X, Y')}.$$

Thus, $\|f\| \ge \|T_f\|_{\mathcal{G}_{(\gamma_p, \delta_p)}(X, Y')}$. ■

We can now construct preduals for the renormed space of bounded maps, the positive p-summing, the p-concave and the p-convex operators. The cases $p = \infty$ in (iii) and $p = 1$ in (iv) below are well known:

Theorem 5.2 *If X and Y are Banach spaces, E and F Banach lattices and $\frac{1}{p} + \frac{1}{q} = 1$, then*

(i) $(E \otimes_{(M_2, M_2)} F)' = \mathcal{G}_{(M_2, M_2)}(E, F')$;
(ii) $(E \otimes_{(|\epsilon|_p, {}^t M_q)} Y)' = \mathcal{S}_p(E, Y')$ *for* $1 \le p < \infty$;
(iii) $(E \otimes_{(M_p, {}^t M_q)} Y)' = \mathcal{L}^{\mathrm{pvex}}(E, Y')$ *for* $1 \le p \le \infty$;
(iv) $(X \otimes_{({}^t M_p, M_q)} F)' = \mathcal{L}^{\mathrm{pcav}}(X, F')$ *for* $1 \le p \le \infty$.

Proof (i) The M-norm is a reasonable cross norm on $\ell^2 \otimes E$ and on $\ell^2 \otimes F'$. As mentioned above, the M-norm is also self-dual in the sense that $\ell^2 \widetilde{\otimes}_M F' \hookrightarrow (\ell^2 \otimes_M F)'$ is a Riesz isometry (into). It was shown in Theorem 4.4 that $\| \cdot \|_{(M_2, M_2)}$ is a norm on $E \otimes F$. It therefore follows from Proposition 5.1 that $(E \otimes_{(M_2, M_2)} F)' = \mathcal{G}_{(M_2, M_2)}(E, F')$.

The remaining proofs follow by using analogous arguments. ■

Krivine's version (1.1) of Grothendieck's inequality translates via Theorem 3.3 and Theorem 5.2(a) into the following.

Corollary 5.3 *Let E and F be Banach lattices. Then*

$$\frac{1}{K_G} \|u\|_\pi \le \|u\|_{(M_2, M_2)} \le \|u\|_\pi \text{ for all } u \in E \otimes F.$$

6 Representing the Elements of $X \mathbin{\widetilde{\otimes}}_{(\gamma_p,\delta_q)} Y$

Definition 6.1 Let X and Y be Banach spaces, $\frac{1}{p}+\frac{1}{q}=1$, γ a reasonable cross norm on $\Lambda^p \otimes X$, δ a reasonable cross norm on $\Lambda^q \otimes Y$, $\|\cdot\|_{(\gamma_p,\delta_q)}$ a reasonable cross norm on $X \otimes Y$ and $1 \le p \le \infty$. Then $\|\cdot\|_{(\gamma_p,\delta_q)}$ is said to satisfy property (REP) on $X \otimes Y$ provided that $u \in X \mathbin{\widetilde{\otimes}}_{(\gamma_p,\delta_p)} Y$ if and only if $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, where $(x_i) \in \Lambda^p \mathbin{\widetilde{\otimes}}_\gamma X$ and $(y_i) \in \Lambda^q \mathbin{\widetilde{\otimes}}_\delta Y$, and

$$\|u\|_{(\gamma_p,\delta_q)} = \inf \|(x_i)\|_{\Lambda^p \widetilde{\otimes}_\gamma X'} \|(y_i)\|_{\Lambda^q \widetilde{\otimes}_\delta Y},$$

where the inf is taken over all such representations of u.

It was shown in [12] that the Cohen norms w_p have property (REP) and it was shown in [20] that the norms M and ${}^tM$ have property (REP). Before we consider other norms with property (REP), we recall some facts about Krivine's sequence space.

Let E be a Banach lattice and $\widetilde{E(\ell^p)}$ the space of all sequences (x_i) of elements of E for which

$$\|(x_i)\|_{\widetilde{E(\ell^p)}} := \sup_n \Big\| \Big(\sum_{i=1}^{n} |x_i|^p \Big)^{1/p} \Big\| \in \mathbb{R} \quad \text{if } 1 \le p < \infty$$

or

$$\|(x_i)\|_{\widetilde{E(\ell^\infty)}} := \sup_n \Big\| \Big(\bigvee_{i=1}^{n} |x_i| \Big) \Big\| \in \mathbb{R} \quad \text{if } p = \infty.$$

Denote by $E(\ell^p)$ the closed subspace of $\widetilde{E(\ell^p)}$ spanned by the sequences (x_i) which are eventually zero, and denote $E(\ell^\infty)$ by $E(c_0)$. It was shown in [20, §8] that $E(\Lambda^p)$ is a Banach lattice and the map κ, as defined in §4, can be extended to a Riesz isometry from the Banach lattice $\Lambda^p \mathbin{\widetilde{\otimes}}_M E$ onto $E(\Lambda^p)$. In the notation of Theorem 4.2, we have

$$\Lambda_M^{p,c}(E) = E(\Lambda^p) \text{ for } 1 \le p \le \infty.$$

By adapting the notation, the method used in [12] to prove that w_p has (REP), can also be used to prove the following.

Theorem 6.2 *Let E and F be Banach lattices, X and Y Banach spaces, $\frac{1}{p}+\frac{1}{q}=1$ and $1 \le p \le \infty$. Then*

(i) *the reasonable cross norms $\|\cdot\|_{(M_2,M_2)}$ and $\|\cdot\|_{(|\epsilon|_p,M_q)}$ on $E \otimes F$ have property (REP);*
(ii) *the reasonable cross norm $\|\cdot\|_{({}^tM_p,M_q)}$ on $X \otimes F$ has property (REP);*
(iii) *the reasonable cross norms $\|\cdot\|_{(M_p,{}^tM_q)}$ and $\|\cdot\|_{(|\epsilon|_p,{}^tM_q)}$ on $E \otimes Y$ have property (REP).*

Proof The proofs are based on [12] (see also [21, Theorem 4.1]). We keep the first part of the proof general up to the point where we need specific information about the norms γ and δ.

Suppose $(a_i) \in \Lambda^p \widetilde{\otimes}_\gamma X$ and $(y_i) \in \Lambda^q \widetilde{\otimes}_\delta Y$. Let $u_r = \sum_{i=1}^r a_i \otimes y_i$. Then for each $r \in \mathbb{N}$,

$$\|u_r\|_{(\gamma_p,\delta_q)} \le \|(a_i)_{i=1}^r\|_{\Lambda^p\otimes_\gamma X} \|(y_i)_{i=1}^r\|_{\Lambda^q\otimes_\delta Y} \le \|(a_i)\|_{\Lambda^p\widetilde{\otimes}_\gamma X} \|(y_i)\|_{\Lambda^q\widetilde{\otimes}_\delta Y}.$$

It then follows easily that (u_r) converges to $u := \sum_{i=1}^{\infty} a_i \otimes y_i$ in (γ_p, δ_q)-norm and

$$\|u\|_{(\gamma_p,\delta_q)} \le \|(a_i)\|_{\ell^1\widetilde{\otimes}_\gamma X}\|(y_i)\|_{\Lambda^q\widetilde{\otimes}_\delta Y}. \tag{6.1}$$

Conversely, let $u \in X\widetilde{\otimes}_{(\gamma_p,\delta_q)} Y$ and let $\epsilon > 0$ be given. Then there exists a sequence (u_n) in $X \otimes Y$ such that

$$\|u - u_i\|_{(\gamma_p,\delta_q)} < (1/2)^{2i+1}\epsilon$$

for $i = 0, 1, 2, \dots$. Then

$$\|u_{i+1} - u_i\|_{(\gamma_p,\delta_q)} \le \|u_{i+1} - u\|_{(\gamma_p,\delta_q)} + \|u - u_i\|_{(\gamma_p,\delta_q)} < (1/4)^i\epsilon$$

for $i = 0, 1, 2, \dots$. Hence $u_{i+1} - u_i$ has a representation

$$u_{i+1} - u_i = \sum_{n=1}^{n_{i+1}} a_n^{(i+1)} \otimes y_n^{(i+1)},$$

with $a_n^{(i+1)} \in X$, $y_n^{(i+1)} \in Y$,

$$\left\| \left(a_n^{(i+1)}\right)_{n=1}^{n_{i+1}} \right\|_{\Lambda^p\otimes_\gamma X} \left\| \left(y_n^{(i+1)}\right)_{n=1}^{n_{i+1}} \right\|_{\Lambda^q\otimes_\delta X} < (1/4)^i\epsilon,$$

where

$$\left\| \left(a_n^{(i+1)}\right)_{n=1}^{n_{i+1}} \right\|_{\Lambda^p\otimes_\gamma X} \le \left[(1/4)^i\epsilon\right]^{1/2},$$
$$\left\| \left(y_n^{(i+1)}\right)_{n=1}^{n_{i+1}} \right\|_{\Lambda^q\otimes_\delta Y} \le \left[(1/4)^i\epsilon\right]^{1/2}.$$

Since

$$\|u_0\|_{(\gamma_p,\delta_q)} \le \|u\|_{(\gamma_p,\delta_q)} + \|u - u_0\|_{(\gamma_p,\delta_q)} < \|u\|_{(\gamma_p,\delta_q)} + \epsilon/2,$$

there is a representation $u_0 = \sum_{n=1}^{n_0} a_n^{(0)} \otimes y_n^{(0)}$ with $a_n^{(0)} \in X$, $y_n^{(0)} \in Y$,

$$\left\| \left(a_n^{(0)}\right)_{n=1}^{n_0} \right\|_{\Lambda^p\otimes_\gamma X} \left\| \left(y_n^{(0)}\right)_{n=1}^{n_0} \right\|_{\Lambda^q\otimes_\delta Y} < \|u\|_{(\gamma_p,\delta_q)} + \epsilon/2,$$

with

$$\left\| \left(a_n^{(0)}\right)_{n=1}^{n_0} \right\|_{\Lambda^p\otimes_\gamma X} \le \left(\|u\|_{(\gamma_p,\delta_q)} + \epsilon/2\right)^{1/2},$$
$$\left\| \left(y_n^{(0)}\right)_{n=1}^{n_0} \right\|_{\Lambda^q\otimes_\delta Y} \le \left(\|u\|_{(\gamma_p,\delta_q)} + \epsilon/2\right)^{1/2}.$$

From the choices of the representations of $u_{i+1}-u_i$, it follows that for any $k\in\mathbb{N}$,

$$\Big\| u-\Big(u_0+\sum_{i=1}^{k}\sum_{n=1}^{n_i} a_n^{(i)}\otimes y_n^{(i)}\Big)\Big\|_{(\gamma_p,\delta_q)} = \|u-u_k\|_{(\gamma_p,\delta_q)} < (1/2)^{2k+1}\epsilon,$$

i.e., the series $\big(u_0+\sum_{i=1}^{\infty}\sum_{n=1}^{n_i} a_n^{(i)}\otimes y_n^{(i)}\big)$ converges to u in (γ_p,δ_q)-norm. At this point, it is less cumbersome to relabel some of the sequences. Consider the composed sequences given by

$$(a_n) := (a_1^{(0)},\dots,a_{n_0}^{(0)},a_1^{(1)},\dots,a_{n_1}^{(1)},\dots),$$
$$(y_n) := (y_1^{(0)},\dots,y_{n_0}^{(0)},y_1^{(1)},\dots,y_{n_1}^{(1)},\dots).$$

Let $n'_k = n_0+n_1+\dots+n_k$ for each $k\in\mathbb{N}\cup\{0\}$. Then, for each $k\in\mathbb{N}\cup\{0\}$,

$$\|(a_i)_{i=1}^{n'_k}\|_{\Lambda^p\widetilde{\otimes}_\gamma X} \le \sum_{i=0}^{k}\|(a_1^{(i)},\dots,a_{n_i}^{(i)})\|_{\Lambda^p\widetilde{\otimes}_\gamma X} \le (\|u\|_{(\gamma_p,\delta_q)}+\epsilon/2)^{1/2}+\epsilon^{1/2}$$

and, similarly,

$$\|(y_i)_{i=1}^{n'_k}\|_{c_0\widetilde{\otimes}_\delta Y} \le (\|u\|_{(\gamma_p,\delta_q)}+\epsilon/2)^{1/2}+\epsilon^{1/2}.$$

Also, for each $k\in\mathbb{N}\cup\{0\}$,

$$\sum_{i=0}^{k}\sum_{n=1}^{n_i} a_n^{(i)}\otimes y_n^{(i)} = \sum_{i=1}^{n'_k} a_i\otimes y_i.$$

Thus, the series $\sum_{i=1}^{\infty} a_i\otimes y_i$ converges to u in (γ_p,δ_q)-norm.

To show that $(a_n)\in\Lambda^p\,\widetilde{\otimes}_\gamma X$ and $(y_n)\in\Lambda^q\,\widetilde{\otimes}_\delta Y$, let

$$\xi_n := \begin{cases} (1/2^i)^{1/2} & \text{for } n'_i < n \le n'_{i+1},\ i=0,1,2,\dots,\\ 1 & \text{for } 1\le n\le n'_0. \end{cases}$$

Then $(\xi_n)\in c_0$. For large $m,n\in\mathbb{N}$ with $m\le n$, select $j,\ell\in\mathbb{N}$ such that $n'_j < m \le n \le n'_{j+\ell}$.

At this point, specific information about the norms γ and δ is required. We now consider the particular cases for $\|\cdot\|_{(\gamma_p,\delta_q)}$.

To prove (i), consider the case where X and Y are Banach lattices, $\gamma=|\epsilon|$ and $\delta=M$. Since $\|\cdot\|_M=\|\cdot\|_{(\epsilon_0,|\epsilon|_1)}$ has (REP) [21, Theorem 4.1], we may assume that $p\neq 1$.

Let $\bar a_i := a_i/\xi_i$, for all $i\in\mathbb{N}$ and $B(X')_+=\{a'\in X'_+ \mid \|a'\|\le 1\}$. Then

$$\sup\big\{\|(\langle|\bar a_i|,a'\rangle)_{i=m}^{n}\|_{\Lambda^p} \;\big|\; a'\in B(X')_+\big\} \le \sum_{k=j}^{j+\ell}\sup\big\{\|(\langle|\bar a_i|,a'\rangle)_{i=n'_k+1}^{n'_{k+1}}\|_{\Lambda^p} \;\big|\; a'\in B(X')_+\big\}.$$

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By definition of (ξ_n) it follows that

$$
\begin{aligned}
&\sum_{k=j}^{j+\ell} \sup\left\{ \|(\langle|\bar{a}_i|, a'\rangle)_{i=n'_k+1}^{n'_{k+1}}\|_{\Lambda^p} \;\middle|\; a' \in B(X')_+ \right\} \\
&\le \sum_{k=j}^{j+\ell} (1/\xi_k)\ \sup\left\{ \|(\langle|a_i|, a'\rangle)_{i=n'_k+1}^{n'_{k+1}}\|_{\Lambda^p} \;\middle|\; a' \in B(X')_+ \right\} \\
&= \sum_{k=j}^{j+\ell} 2^{k/2} \left\| (a_{n'_k+1}, a_{n'_k+2}, \dots, a_{n'_{k+1}}) \right\|_{\Lambda^p \otimes_{|\epsilon|} X} \\
&= \sum_{k=j}^{j+\ell} 2^{k/2} \left\| \left(a_i^{(k+1)}\right)_{i=1}^{n'_{k+1}} \right\|_{\Lambda^p \otimes_{|\epsilon|} X} \\
&\le \sum_{k=j}^{j+\ell} 2^{k/2} ((1/4)^k \epsilon)^{1/2} = \epsilon^{1/2} \sum_{k=j}^{j+\ell} (1/2)^{k/2}.
\end{aligned}
$$

This shows that $\left(\langle|\bar{a}_i|, a'\rangle\right)_{i\le n}$ is a Cauchy sequence in Λ^p for each $a' \in B(X')_+$ which converges to $\left(\langle|\bar{a}_i|, a'\rangle\right)_{i=1}^{\infty} \in \Lambda^p$. Thus $(\bar{a}_n) \in \Lambda_p^{|\text{weak}|}(X)$; consequently, $(a_n) \in \Lambda^p \,\widetilde{\otimes}_{|\epsilon|}\, X$.

Let $\bar{y}_i := y_i/\xi_i$ for all $i \in \mathbb{N}$. Fix n and select s such that $n'_s < n \le n'_{s+1}$. Then

$$
\left(\sum_{i=1}^{n} |\bar{y}_i|^q \right)^{1/q} \le \left(\sum_{i=1}^{n'_{s+1}} |\bar{y}_i|^q \right)^{1/q}.
$$

We also have in the Banach lattice $Y(\Lambda^p)$ that

$$
\begin{aligned}
\left| \left(2^{1/2} y_1^{(s+1)}, \dots, 2^{(s+1)/2} y_{n_{s+1}}^{(s+1)}\right) \right| &= \left(2^{1/2} \left| y_1^{(s+1)} \right|, \dots, 2^{(s+1)/2} \left| y_{n_{s+1}}^{(s+1)} \right| \right) \\
&\le 2^{(s+1)/2} \left| \left(y_1^{(s+1)}, \dots, y_{n_{s+1}}^{(s+1)} \right) \right|,
\end{aligned}
$$

and since the M-norm is a Riesz norm,

$$
\left\| (2^{1/2} y_1^{(s+1)}, \dots, 2^{(s+1)/2} y_{n_{s+1}}^{(s+1)}) \right\|_{\Lambda^q \otimes_M Y} \le 2^{(s+1)/2} \left\| (y_1^{(s+1)}, \dots, y_{n_{s+1}}^{(s+1)}) \right\|_{\Lambda^q \otimes_M Y}.
$$

Consequently,

$$
\begin{aligned}
\left\| (\overline{y}_i)_{i=1}^{n} \right\|_{\Lambda^q \otimes_M Y} &\le \left\| (\overline{y}_i)_{i=1}^{n'_{s+1}} \right\|_{\Lambda^q \otimes_M Y} \\
&\le \sum_{i=0}^{s+1} \left\| (\overline{y}_1^{(i)}, \dots, \overline{y}_{n_i}^{(i)}) \right\|_{\Lambda^q \otimes_M Y}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{s} \big\| (y_1^{(i)}, \dots, y_{n_i}^{(i)}) \big\|_{\Lambda^q \otimes_M Y} \\
&\qquad + \big\| (2^{1/2} y_1^{(s+1)}, \dots, 2^{(s+1)/2} y_{n_{s+1}}^{(s+1)}) \big\|_{\Lambda^q \otimes_M Y} \\
&\le \sum_{i=0}^{s} \big\| (y_1^{(i)}, \dots, y_{n_i}^{(i)}) \big\|_{\Lambda^q \otimes_M Y} + 2^{(s+1)/2} \big\| (y_1^{(s+1)}, \dots, y_{n_{s+1}}^{(s+1)}) \big\|_{\Lambda^q \otimes_M Y} \\
&\le (\|u\|_{(|\epsilon|_p, M_q)} + \epsilon/2)^{1/2} + \epsilon^{1/2} + 2^{(s+1)/2}((1/4)^s \epsilon)^{1/2} \\
&\le (\|u\|_{(|\epsilon|_p, M_q)} + \epsilon/2)^{1/2} + \epsilon^{1/2} + \epsilon^{1/2},
\end{aligned}
$$

from which we get $\sup_{n \in \mathbb{N}} \|(\overline{y}_i)_{i=1}^n\|_{\Lambda^q \otimes_M Y} < \infty$. This shows that $(\overline{y}_i) \in Y(\Lambda^q)$; consequently, $(y_i) \in \Lambda^q \,\widetilde{\otimes}_M\, Y$.

Furthermore,

$$
\begin{aligned}
\|(a_i)\|_{\Lambda^p \widetilde{\otimes}_{|\epsilon|} X} &\le (\|u\|_{(|\epsilon|_p, M_q)} + \epsilon/2)^{1/2} + \epsilon^{1/2}, \\
\|(y_i)\|_{\Lambda^q \widetilde{\otimes}_M Y} &\le (\|u\|_{(|\epsilon|_p, M_q)} + \epsilon/2)^{1/2} + 2\epsilon^{1/2}.
\end{aligned}
$$

Hence,

$$
\|(a_i)\|_{\Lambda^p \widetilde{\otimes}_{|\epsilon|} X} \|(y_i)\|_{\Lambda^q \widetilde{\otimes}_M Y} \le \|u\|_{(|\epsilon|_p, M_q)} + f(\epsilon), \tag{6.2}
$$

where f is a positive real valued function with $f(\epsilon) \to 0$ as $\epsilon \to 0$. By (6.1) and (6.2), the norm equality holds.

The proof that $\|\cdot\|_{(M_2, M_2)}$ has (REP) is similar.

To prove (ii), let X be a Banach space and Y a Banach lattice, $\gamma = {}^tM$ and $\delta = M$. We may again assume $p \neq 1$. Using the same notation as in the first part of the proof, we claim that $(\|a_i\|)_{i \le n}$ is a Cauchy sequence in Λ^p. For large $m, n \in \mathbb{N}$ with $m \le n$, select $j, \ell \in \mathbb{N}$ such that $n'_j < m \le n \le n'_{j+\ell}$; then

$$
\|(a_i)_{i=m}^n\|_{\Lambda^p \widetilde{\otimes}_{{}^tM} X} \le \sum_{k=j}^{j+\ell} \Big\| \big(a_i^{(k+1)}\big)_{i=1}^{n'_{k+1}} \Big\|_{\Lambda^p \otimes_{{}^tM} X} \le \sum_{k=j}^{j+\ell} ((1/4)^k \epsilon)^{1/2} \le 2\epsilon^{1/2}.
$$

Hence $(\|a_i\|)_{i=1}^{\infty} \in \Lambda^p$, from which we get $(a_i) \in \Lambda_p^{\text{strong}}(X) = \Lambda^p \,\widetilde{\otimes}_{{}^tM}\, X$ and

$$
\|(a_i)\|_{\Lambda^p \widetilde{\otimes}_{{}^tM} X} \le \|u\|_{({}^tM_p, M_q)} + \epsilon^{1/2} + 2\epsilon^{1/2}.
$$

The proof may now be completed in a similar way as in the preceding cases.

The proof of (iii) follows in an analogous way as that of (i) and (ii). ■

7 (γ_p, δ_q)-Nuclear Operators

Let X and Y be Banach spaces and let α be a reasonable cross norm on $X' \otimes Y$. The canonical embedding $\phi\colon X' \otimes_\alpha Y \to \mathcal{L}(X, Y)$ is continuous with norm ≤ 1.

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Let $\phi_\alpha\colon X' \widetilde{\otimes}_\alpha Y \to \mathcal{L}(X,Y)$ be the continuous extension of ϕ. The elements of $\mathcal{N}^\alpha(X,Y) := \phi_\alpha(X' \widetilde{\otimes}_\alpha Y)\big/\phi_\alpha^{-1}(0)$ are called the α-*nuclear operators* from X into Y, and $\|\cdot\|_{\mathcal{N}^\alpha(X,Y)}$, defined by

$$\|T\|_{\mathcal{N}^\alpha(X,Y)} = \inf\big\{ \|v\|_\alpha \bigm| v \in X' \widetilde{\otimes}_\alpha Y \text{ and } \phi_\alpha(v) = T\big\}$$

is called the α-*nuclear norm* of T.

We are interested in $\mathcal{N}^\alpha(X,Y)$, where α is one of the norms

$$\|\cdot\|_{(M_2,M_2)}, \quad \|\cdot\|_{(|\epsilon|_p,M_q)}, \quad \|\cdot\|_{({}^tM_p,M_q)}, \quad \|\cdot\|_{(M_p,{}^tM_q)}$$

(and where X and Y are appropriately chosen as Banach spaces or Banach lattices in order for these choices of norms to be meaningful).

To do so, we first consider $\mathcal{N}^\gamma(X,Y)$, where γ is one of the "component" norms that occur in the above choices of α, *i.e.*, γ is one of $|\epsilon|$, ${}^tM$ or M.

Let E and F be Banach lattices. It is well known and easy to verify that the range of the canonical embedding $\phi\colon E' \otimes F \to \mathcal{L}(E,F)$ is contained in $\mathcal{L}^r(E,F)$, and since $|\epsilon|$ is the norm on $E' \otimes F$ induced by the $\|\cdot\|_r$ norm, we have that $\phi_{|\epsilon|}\colon E' \widetilde{\otimes}_{|\epsilon|} F \to \mathcal{N}^{|\epsilon|}(E,F)$ is a surjective Riesz isometry. Furthermore, if X and Y are Banach spaces, then $\phi_{{}^tM}\colon E' \widetilde{\otimes}_{{}^tM} Y \to \mathcal{N}^{{}^tM}(E,Y)$ is a surjective isometry (and in the case of Y being a Banach lattice, $\phi_{{}^tM}$ is a surjective Riesz isometry); and $\phi_M\colon X' \widetilde{\otimes}_M F \to \mathcal{N}^M(X,F)$ is a surjective isometry (and in the case of X being a Banach lattice, ϕ_M is a surjective Riesz isometry).

We now proceed with the task at hand of considering $\mathcal{N}^\alpha(X,Y)$, where α is one of the norms

$$\|\cdot\|_{(M_2,M_2)}, \quad \|\cdot\|_{(|\epsilon|_p,M_q)}, \quad \|\cdot\|_{({}^tM_p,M_q)}, \quad \|\cdot\|_{(M_p,{}^tM_q)}.$$

Proposition 7.1 *Let $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \le p \le \infty$. Suppose $\|\cdot\|_{(\gamma_p,\delta_q)}$ is one of the following norms*

(a) $\|\cdot\|_{(M_2,M_2)}$ *or* $\|\cdot\|_{(|\epsilon|_p,M_q)}$ *on $X' \otimes Y$, where X, Y are Banach lattices;*
(b) $\|\cdot\|_{({}^tM_p,M_q)}$ *on $X' \otimes Y$, where X is a Banach space and Y is a Banach lattice;*
(c) $\|\cdot\|_{(M_p,{}^tM_q)}$ *or* $\|\cdot\|_{(|\epsilon|_p,{}^tM_q)}$ *on $X' \otimes Y$, where X is a Banach lattice and Y is a Banach space.*

Then the following statements are equivalent for $T \in \mathcal{L}(X,Y)$:

(i) $T \in \mathcal{N}^{(\gamma_p,\delta_q)}(X,Y)$.
(ii) *There exist $(x_i') \in \Lambda_\gamma^{p,c}(X')$ and $(y_i) \in \Lambda_\delta^{q,c}(Y)$ such that $T = \sum_{i=1}^\infty \langle x_i', \cdot\rangle y_i$, moreover,*

$$\|T\|_{\mathcal{N}^{(\gamma_p,\delta_q)}(X,Y)} = \inf \|(x_i')\|_{\Lambda^p \widetilde{\otimes}_\gamma X'} \|(y_i)\|_{\Lambda^q \widetilde{\otimes}_\delta Y},$$

where the inf *is taken over all such representations of T.*
(iii) *There exist $R \in \mathcal{N}^{{}^t\gamma}(X, \Lambda^p)$ and $S \in \mathcal{N}^\delta(\Lambda^p, Y)$ such that $T = S \circ R$; moreover,*

$$\|T\|_{\mathcal{N}^{(\gamma_p,\delta_q)}(X,Y)} = \inf \|S\|_\delta \|R\|_{{}^t\gamma},$$

where the inf *is taken over all such representations of T.*

Proof (a) Let E and F be Banach lattices and consider the norm $\|\cdot\|_{(M_2,M_2)}$.

(i) $\Leftrightarrow$ (ii). Note that $T \in \mathcal{N}^{(M_2,M_2)}(E,F)$ if and only if there exists $u \in E' \widetilde{\otimes}_{(M_2,M_2)} F$ such that $\phi_{(M_2,M_2)}(u) = T$. Since $\|\cdot\|_{(M_2,M_2)}$ satisfies property (REP), by Theorem 6.2, $u \in E' \widetilde{\otimes}_{(M_2,M_2)} F$ if and only if there exist $(x_i') \in \ell^{2,c}_{M}(E')$ and $(y_i) \in \ell^{2,c}_{M}(F)$ such that $u = \sum_{i=1}^{\infty} x_i' \otimes y_i$. The definition and continuity of $\phi_{(M_2,M_2)}$ guarantee that $T = \phi_{(M_2,M_2)}(u)$ if and only if $T = \sum_{i=1}^{\infty} \langle x_i', \cdot\,\rangle y_i$.

The stated norm equality follows from the definition of $\|T\|_{\mathcal{N}^{(M_2,M_2)}(E,F)}$ and the fact that $\|\cdot\|_{(M_2,M_2)}$ satisfies property (REP).

(ii) $\Leftrightarrow$ (iii). We note that $(x_i') \in \ell^{2,c}_{M}(E')$ if and only if $\sum_{i=1}^{\infty} x_i' \otimes e_i \in E' \widetilde{\otimes}_{{}^tM} \ell^2$. Because $\phi_{{}^tM} \colon E' \widetilde{\otimes}_{{}^tM} \ell^2 \to \mathcal{N}^{{}^tM}(E,\ell^2)$ is a surjective isometry, $\sum_{i=1}^{\infty} x_i' \otimes e_i$ is in one-to-one correspondence with $R_{(x_i')} \in \mathcal{N}^{{}^tM}(E,\ell^2)$, where $R_{(x_i')} = \sum_{i=1}^{\infty} \langle x_i', \cdot\,\rangle e_i = (\langle x_i', \cdot\,\rangle)_{i=1}^{\infty}$, and $\|(x_i')\|_M = \|R_{(x_i')}\|_{{}^tM}$.

Similarly, $(y_i) \in \ell^{2,c}_{M}(F)$ if and only if $\sum_{i=1}^{\infty} e_i \otimes y_i \in \ell^2 \widetilde{\otimes}_M F$. But $\phi_M \colon E' \widetilde{\otimes}_M \ell^2 \to \mathcal{N}^M(E,\ell^2)$ is a surjective isometry, so $\sum_{i=1}^{\infty} e_i \otimes y_i$ is in one-to-one correspondence with $S_{(y_i)} \in \mathcal{N}^M(\ell^2,F)$, where $S_{(y_i)}\big((\lambda_i)\big) = \sum_{i=1}^{\infty} \lambda_i y_i$, and $\|(y_i)\|_M = \|S_{(y_i)}\|_M$.

By construction of $R_{(x_i')}$ and $S_{(y_i)}$, it follows that $S_{(y_i)} \circ R_{(x_i')} = \sum_{i=1}^{\infty} \langle x_i', \cdot\,\rangle y_i$.

The equivalence of (ii) and (iii), together with the stated norm equalities, now follows.

The proofs of the equivalence of (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) for the other norms stated in the proposition follow almost verbatim as in the preceding case, with the obvious changes in notation. ∎

The following two known results are special cases of the norms under consideration. We use them to prove Theorem 7.4, our main result.

Theorem 7.2 (cf. [3, 21, 29]) *Let E be a Banach lattice and Y a Banach space. Then the following statements are equivalent for $T \in \mathcal{L}(E,Y)$:*

(i) $T \in \mathcal{N}^{(|\epsilon|_1,\epsilon_0)}(E,Y)$.

(ii) *T has a representation*

$$T = \sum_{i=1}^{\infty} \langle x_i', \cdot\,\rangle y_i,$$

where $(x_i') \in (\ell^1 \widetilde{\otimes}_{|\epsilon|} E')_+$ and $(y_i) \in c_0 \widetilde{\otimes}_{\epsilon} Y$. Moreover,

$$\|T\|_{\mathcal{N}^{(|\epsilon|_1,\epsilon_0)}(E,Y)} = \inf \|(x_i')\|_{\ell^1 \widetilde{\otimes}_{|\epsilon|} E'} \|(y_i)\|_{c_0 \widetilde{\otimes}_{\epsilon} Y},$$

where the inf *is taken over all such representations of T.*

(iii) *There exist $S \in \mathcal{L}^r_+(E,\ell^1)$ and $R \in \mathcal{L}(\ell^1,Y)$ such that S and R are compact and $T = R \circ S$. Further, $\|T\|_{\mathcal{N}^{(|\epsilon|_1,\epsilon_0)}(E,Y)} = \inf \|R\|\,\|S\|_r$, where the* inf *is taken over all such factorizations of T.*

(iii′) *There exist $S \in \mathcal{L}^r_+(E,\ell^1)$ and $R \in \mathcal{L}(\ell^1,Y)$ such that S is compact and $T = R \circ S$. Further, $\|T\|_{\mathcal{N}^{(|\epsilon|_1,\epsilon_0)}(E,Y)} = \inf \|R\|\,\|S\|_r$, where the* inf *is taken over all such factorizations of T.*

(iv) *There exist a measure space (Ω,Σ,μ), $S \in \mathcal{L}^r_+(E,L^1(\mu))$ and $R \in \mathcal{L}(L^1(\mu),Y)$ such that R is compact and $T = R \circ S$. Further, $\|T\|_{\mathcal{N}^{(|\epsilon|_1,\epsilon_0)}(E,Y)} = \inf \|R\|\,\|S\|_r$, where the* inf *is taken over all such factorizations of T.*

Theorem 7.3 (cf. [3, 21, 29]) *Let X be a Banach space and F a Banach lattice. Then the following statements are equivalent for $T \in \mathcal{L}(X, F)$:*

(i) $T \in \mathcal{N}^{(\epsilon_0, |\epsilon|_1)}(X, F)$.

(ii) *T has a representation*

$$T = \sum_{i=1}^{\infty} \langle x_i', \cdot \rangle y_i,$$

where $(x_i') \in c_0 \widetilde{\otimes}_\epsilon X'$ and $(y_i) \in (\ell^1 \widetilde{\otimes}_{|\epsilon|} F)_+$. Moreover,

$$\|T\|_{\mathcal{N}^{(\epsilon_0, |\epsilon_1|)}(X,F)} = \inf \|(x_i')\|_{c_0 \widetilde{\otimes}_\epsilon X'} \|(y_i)\|_{\ell^1 \widetilde{\otimes}_{|\epsilon|} F},$$

where the inf *is taken over all such factorizations of T.*

(iii) *There exist $S \in \mathcal{L}(X, c_0)$ and $R \in \mathcal{L}_+^r(c_0, F)$ such that S and R are compact and $T = R \circ S$. Further, $\|T\|_{\mathcal{N}^{(\epsilon_0, |\epsilon|_1)}(X,F)} = \inf \|R\|_r \|S\|$, where the* inf *is taken over all such factorizations of T.*

(iii′) *There exist $S \in \mathcal{L}(X, c_0)$ and $R \in \mathcal{L}_+^r(c_0, F)$ such that R is compact and $T = R \circ S$. Further, $\|T\|_{\mathcal{N}^{(\epsilon_0, |\epsilon|_1)}(X,F)} = \inf \|R\|_r \|S\|$, where the* inf *is taken over all such factorizations of T.*

(iv) *There exist a compact Hausdorff space Ω, $S \in \mathcal{L}(X, C(\Omega))$ and $R \in \mathcal{L}_+^r(C(\Omega), F)$ such that S is compact and $T = R \circ S$. Further, $\|T\|_{\mathcal{N}^{(\epsilon_0, |\epsilon|_1)}(X,F)} = \inf \|R\|_r \|S\|$, where the* inf *is taken over all such factorizations of T.*

Let E and F be Banach lattices and

$$\mathcal{K}^r(E, F) := \{T \colon E \to F \mid T = T_1 - T_2 \text{ for some } T_1, T_2 \in \mathcal{K}_+(E, F)\}.$$

We recall from [21] that if E is a Banach lattice and $\frac{1}{p} + \frac{1}{q} = 1$, then $\mathcal{K}^r(\Lambda^q, E)$ is also a Banach lattice for $1 \le p \le \infty$ and $\Lambda^p \widetilde{\otimes}_{|\epsilon|} E = \mathcal{K}^r(\Lambda^q, E)$ for $1 \le p < \infty$. We also recall from [20] that $|\epsilon|$ is symmetric, *i.e.* ${}^t|\epsilon| = |\epsilon|$.

As an example of using Proposition 7.1 and Theorem 6.2 in conjunction with Theorems 7.2 and 7.3, we characterize the nuclear operators associated with the renormed space of bounded operators, as in Theorem 3.3, and the nuclear operators associated with the p-convex, p-concave and positive p-summing operators.

Theorem 7.4 *Let E and F be Banach lattices, X and Y be Banach spaces and $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $T \in \mathcal{L}(E, F)$, then the following statements are equivalent:*

(a) $T \in \mathcal{N}^{(M_2, M_2)}(E, F)$.

(b) *There exist $T_1 \in \mathcal{L}_+(E, \ell^1)$, $T_2 \in \mathcal{L}(\ell^1, \ell^2)$, $T_3 \in \mathcal{L}(\ell^2, c_0)$ and $T_4 \in \mathcal{L}_+(c_0, F)$ such that $T = T_4 \circ T_3 \circ T_2 \circ T_1$, T_1 and T_4 are compact. Moreover, $\|T\|_{\mathcal{N}^{(M_2, M_2)}(X,F)} = \inf \|T_4\| \|T_3\| \|T_2\| \|T_1\|$, where the* inf *is taken over all such factorizations of T.*

(c) *There exist a measure space (Ω_1, Σ, μ), a compact Hausdorff space Ω_2, $T_1 \in \mathcal{L}_+(E, L^1(\mu))$, $T_2 \in \mathcal{L}(L^1(\mu), \ell^2)$, $T_3 \in \mathcal{L}(\ell^2, C(\Omega_2))$ and $T_4 \in \mathcal{L}_+(C(\Omega_2), F)$ such that $T = T_4 \circ T_3 \circ T_2 \circ T_1$ and T_2 and T_3 are compact. Moreover, $\|T\|_{\mathcal{N}^{(M_2, M_2)}(X,F)} = \inf \|T_4\| \|T_3\| \|T_2\| \|T_1\|$, where the* inf *is taken over all such factorizations of T.*

(ii) *If* $1 < p < \infty$ *and* $T \in \mathcal{L}(X, F)$*, then the following statements are equivalent:*

(a) $T \in \mathcal{N}^{({}^tM_p, M_q)}(X, F)$.

(b) *There exist* $T_1 \in \mathcal{L}(X, c_0)$, $T_2 \in \mathcal{L}_+(c_0, \ell^p)$, $T_3 \in \mathcal{L}(\ell^p, c_0)$ *and* $T_4 \in \mathcal{L}_+(c_0, F)$ *such that* $T = T_4 \circ T_3 \circ T_2 \circ T_1$*, and* T_2 *and* T_4 *are compact. Moreover,* $\|T\|_{\mathcal{N}^{({}^tM_p, M_q)}(X,F)} = \inf \|T_4\|\|T_3\|\|T_2\|\|T_1\|$*, where the* inf *is taken over all such factorizations of* T.

(c) *There exist a compact Hausdorff spaces* Ω_1, Ω_2, $T_1 \in \mathcal{L}_+(E, C(\Omega_1))$, $T_2 \in \mathcal{L}(C(\Omega_1), \ell^p)$, $T_3 \in \mathcal{L}(\ell^p, C(\Omega_2))$ *and* $T_4 \in \mathcal{L}_+(C(\Omega_2), F)$ *such that* $T = T_4 \circ T_3 \circ T_2 \circ T_1$*, and* T_2 *and* T_3 *are compact. Moreover,* $\|T\|_{\mathcal{N}^{({}^tM_p, M_q)}(X,F)} = \inf \|T_4\|\|T_3\|\|T_2\|\|T_1\|$*, where the* inf *is taken over all such factorizations of* T.

(ii′) *If* $1 < p < \infty$ *and* $T \in \mathcal{L}(E, Y)$*, then the following statements are equivalent:*

(a) $T \in \mathcal{N}^{(M_p, {}^tM_q)}(E, Y)$.

(b) *There exist* $T_1 \in \mathcal{L}_+(E, \ell^1)$, $T_2 \in \mathcal{L}(\ell^1, \ell^p)$, $T_3 \in \mathcal{L}_+(\ell^p, \ell^1)$ *and* $T_4 \in \mathcal{L}(\ell^1, Y)$ *such that* $T = T_4 \circ T_3 \circ T_2 \circ T_1$*, and* T_1 *and* T_3 *are compact. Moreover,* $\|T\|_{\mathcal{N}^{(M_p, {}^tM_q)}(E,Y)} = \inf \|T_4\|\|T_3\|\|T_2\|\|T_1\|$*, where the* inf *is taken over all such factorizations of* T.

(c) *There exist measure spaces* $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$, $T_1 \in \mathcal{L}_+(E, L^1(\mu_1))$, $T_2 \in \mathcal{L}(L^1(\mu_1), \ell^p)$, $T_3 \in \mathcal{L}_+(\ell^p, L^1(\mu_2))$ *and* $T_4 \in \mathcal{L}(L^1(\mu_2), F)$ *such that* $T = T_4 \circ T_3 \circ T_2 \circ T_1$*, and* T_2 *and* T_4 *are compact. Moreover,*

$$\|T\|_{\mathcal{N}^{({}^tM_p, M_q)}(E,Y)} = \inf \|T_4\|\|T_3\|\|T_2\|\|T_1\|,$$

where the inf *is taken over all such representations of* T.

(iii) *If* $1 < p < \infty$ *and* $T \in \mathcal{L}(E, Y)$*, then the following statements are equivalent:*

(a) $T \in \mathcal{N}^{(|\epsilon|_p, {}^tM_q)}(E, Y)$.

(b) *There exist* $T_1 \in \mathcal{L}(E, \ell^p)$, $T_3 \in \mathcal{L}_+(\ell^p, \ell^1)$ *and* $T_4 \in \mathcal{L}(\ell^1, Y)$ *such that* $T = T_4 \circ T_3 \circ T_1$*, and* T_1 *and* T_3 *are compact. Moreover,* $\|T\|_{\mathcal{N}^{(|\epsilon|_p, {}^tM_q)}(E,Y)} = \inf \|T_4\|\|T_3\|\|T_1\|$*, where the* inf *is taken over all such representations of* T.

(c) *There exist a measure space* $(\Omega_2, \Sigma_2, \mu_2)$, $T_1 \in \mathcal{L}(E, \ell^p)$, $T_2 \in \mathcal{L}_+(\ell^p, L^1(\mu_2))$ *and* $T_3 \in \mathcal{L}(L^1(\mu_2), F)$ *such that* $T = T_3 \circ T_2 \circ T_1$, T_1 *is compact and* T_2 *is compact. Moreover,* $\|T\|_{\mathcal{N}^{(|\epsilon|_p, {}^tM_q)}(E,Y)} = \inf \|T_3\|\|T_2\|\|T_1\|$ *, where the* inf *is taken over all such factorizations of* T.

Proof (i) (a) $\Leftrightarrow$ (b). Note that $T \in \mathcal{N}^{(M_2, M_2)}(E, F)$ if and only if there exist $R \in \mathcal{N}^{{}^tM}(E, \ell^2)$ and $S \in \mathcal{N}^M(\ell^2, F)$ such that $T = S \circ R$; moreover,

$$\|T\|_{\mathcal{N}^{(M_2, M_2)}(E,F)} = \inf \|S\|_M \|R\|_{{}^tM},$$

where the inf is taken over all such factorizations of T, by Proposition 7.1. But $R \in \mathcal{N}^{{}^tM}(E, \ell^2)$ if and only if there exist $T_1 \in \mathcal{L}_+^r(E, \ell^1)$ and $T_2 \in \mathcal{L}(\ell^1, \ell^2)$ such that T_1 is compact and $R = T_2 \circ T_1$; further, $\|R\|_{\mathcal{N}^{{}^tM}(E,\ell^2)} = \inf \|T_2\|\,\|T_1\|_r$, where the inf is taken over all such representations of R, by Proposition 7.3. Also, $S \in \mathcal{N}^M(\ell^2, F)$ if and only if there exist $T_3 \in \mathcal{L}(\ell^2, c_0)$ and $T_4 \in \mathcal{L}_+^r(c_0, F)$ such that T_4 is compact and

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$T = T_4 \circ T_3$; further, $\|S\|_{\mathcal{N}^M(\ell^2,F)} = \inf \|T_4\|_r \|T_3\|$, where the inf is taken over all such factorizations of T, by Proposition 7.3 (i) $\Leftrightarrow$ (iii$'$). Consequently, (a) is equivalent to (b).

It follows in a similar way, by using Proposition 7.2(i) $\Leftrightarrow$ (iv) and Proposition 7.3 (i) $\Leftrightarrow$ (iv), that (a) is equivalent to (c).

The proofs of the other statements follow by analogous arguments, noting that the remarks preceding Theorem 7.4, concerning $\mathcal{K}^r(E,F)$ and the symmetry of $|\epsilon|$, pertain to the proof of (iii). ■

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