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INCOMPRESSIBLE SURFACES IN THE BOUNDARY OF A HANDLEBODY—AN ALGORITHM

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Our first result is a decomposition theorem for free groups relative to a set of elements. This enables us to formulate several algebraic conditions, some necessary and some sufficient, for various surfaces in the boundary of a 3-dimensional handlebody to be incompressible. Moreover, we show that there exists an algorithm to determine whether or not these algebraic conditions are met.

Many of our algebraic ideas are similar to those of Shenitzer [**3**]. Conversations with Professor Roger Lyndon were helpful in the initial development of these results, and he reviewed an earlier version of this paper, suggesting Theorem 1 (iii) and its proof. Our notation and techniques are standard (cf. [**1**], [**2**]). A set X of elements in a finitely generated free group F is a *basis* if it is a minimal generating set, and $X^{\pm 1}$ denotes the set of all elements in X, together with their inverses. A *cyclic word* in F is the set of all cyclic permutations of a cyclically reduced word; thus cyclic words are in one-to-one correspondence with conjugacy classes in F, which we refer to as *cyclic elements*. If w is a cyclic element in F and F' is a subgroup, we abuse our notation slightly to write $w \in F'$ whenever $w \cap F' \neq \emptyset$. If $W \subset F$ is a set of words (or cyclic words) in the basis X, the *incidence graph* $J(W)$ is defined to have vertex set W and an undirected edge between two vertices whenever there is an $x \in X$ such that either x or x^{-1} lies in each of the corresponding words. The set W is *connected with respect to* X if $J(W)$ is connected and it is *connected* if it is connected with respect to each basis of F. *Components* are defined accordingly. A second graph, the *star graph* $\sum(W)$, is defined to have vertex set $X^{\pm 1}$ and an undirected edge connecting vertices x and y for each occurrence of a sequence of letters $x^{-1}y$ or $y^{-1}x$ in W.

The automorphism group of F is generated by two types of *Whitehead automorphisms*; (a) automorphisms σ which permute elements of $X^{\pm 1}$, and (b) automorphisms τ, which, for some fixed $d \in X^{\pm 1}$ and some set $A \subset X^{\pm 1}$, with $d \in A$ and $d^{-1} \notin A$, take each $a \in A$ to either ad or $d^{-1}ad$ and fix each element of $X^{\pm 1} - A$. Whitehead automorphisms of this second type are denoted by $\tau = (A, d)$, and, since $\sigma^{-1}\tau\sigma = \sigma^{-1}(A, d)\sigma = (A\sigma, d\sigma)$, they generate a normal subgroup of automorphisms. The *outer automorphism group* of F is the group of all automorphisms modulo the

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inner automorphisms. The symbol $|w|_X$ denotes the length of the word (or cyclic word) w with respect to X, and $|W|_X = \sum |w|_X$, the sum taken over all $w \in W$. If ν is any automorphism of F, we shall consistently compare $|w|_X$ with $|w|_{X\nu}$ rather than (equivalently) with $|w\nu|_X$. If $|W|_Y$ is minimal in $\langle Y \rangle$, we say $|W|_Y$ is minimal.

THEOREM 1. *Let W be a finite set of elements (resp., cyclic elements) in the finitely generated free group F. There exists a decomposition $F = F_0 * F_1 * \ldots * F_n$, with $F_i \neq 1$ if $i \neq 0$, such that*

(i) $W \subset \bigcup_{i \neq 0} F_i$;

(ii) *$W_i = W \cap F_i$ is non-empty and connected for $1 \leqq i \leqq n$;*

(iii) *The stabilizer of W in the automorphism group (outer automorphism group) of F is the direct product of the stabilizers of the components W_i in the (outer) automorphism groups of the F_i, $1 \leqq i \leqq n$;*

(iv) *If $F = G_1 * G_2$ and $W \subset G_1 \cup G_2$, then each of $W \cap G_1$ and $W \cap G_2$ is the union of certain of the components W_i, and the corresponding F_i are (conjugate to) free factors of G_1 and G_2, respectively;*

(v) *If Y is any basis for F such that $|W|_Y$ is minimal, then each F_i has a basis equal to (conjguate to) some $Y_i \subset Y$; and*

(vi) *If Y is any basis for F such that $|W|_Y$ is minimal, then the number of basis elements used in so expressing W is minimal.*

Proof. Let X be any basis for F such that $|W|_X$ is minimal. By 1.7.1 of [**2**], there exist partitions $X_0, X_1, \ldots, X_n$ of X and $W_1, W_2, \ldots, W_n$ of W such that each W_i is connected with respect to X, X_0 is the set of all $x \in X$ such that neither x nor x^{-1} occurs in any $w \in W$, and X_i, $1 \leqq i \leqq n$, is the set of all $x \in X$ such that either x or x^{-1} occurs in some $w \in W_i$. Let $F_i = \langle X_i \rangle$ for $0 \leqq i \leqq n$, so $W_i = W \cap F_i \neq \emptyset$, $W \subset \bigcup_{i \neq 0} F_i$ and (i) is established. Since any automorphism of F_i may be extended to all of F by making it the identity on F_j, $j \neq i$, we see that $|W|_X$ minimal implies $|W_i|_{X_i}$ minimal for $1 \leqq i \leqq n$. Note that, by definition, W_i is also connected with respect to X_i.

LEMMA. *Let $|W_i|_{X_i}$ be minimal, let W_i be connected with respect to X_i, let $\tau = (A, d)$ with $d \notin X_i^{\pm 1}$, and assume $|W_i|_{X_i} \geqq |W_i|_{X\tau}$. If W consists of cyclic words, then τ either fixes F_i or conjugates F_i by d, while if W consists of ordinary words, then τ fixes F_i.*

Proof. First assume W consists of cyclic words. Since $J(W_i)$ is connected, 1.7.7. of [**2**] assures us that the star graph $\sum(W_i)$ is connected on the vertex set $X_i^{\pm 1}$. By 1.4.16 of [**2**], either $X_i^{\pm 1} \subset A$ or $X_i^{\pm 1} \subset X^{\pm 1} - A$. In the former case, τ conjugates F_i by d, while in the latter, τ is the identity on F_i. Now assume W consists of ordinary words. If F_i is cyclic, the proof is as above; otherwise, let $\hat{F} = F * \langle z \rangle$, $\hat{F}_i = F_i * \langle z \rangle$, and let u be the cyclic word $z^2 w_1 z^2 w_2 \ldots z^2 w_m$, where $W_i = \{w_1, w_2, \ldots, w_m\}$.

We extend τ to $\hat{\tau}$ on $\hat{F}$ by defining $z\hat{\tau} = z$, so $|u|_{X \cup \{z\}} = |u|_{X\tau \cup \{z\}}$. By the proof of 1.7.7 of [**2**], the star graph $\sum(u)$ is connected, and we are reduced to the case above. However, since $z\hat{\tau} = z$, τ must be the identity on F_i.

Next, assume $|W_i|_{X_i}$ is minimal. If $|W_i|_X$ were not minimal, there would exist, by Whitehead's Theorem (1.4.20 and 1.4.24 of [**2**]), a Whitehead automorphism $\tau = (A, d)$ such that $|W_i|_{X\tau} < |W_i|_X = |W_i|_{X_i}$. By the lemma, we would have $d \in X_i^{\pm 1}$, contradicting the minimality of $|W_i|_{X_i}$. Thus $|W_i|_{X_i}$ minimal implies $|W_i|_X$ minimal. Similarly, $|W_i|_X$ minimal for $1 \leqq i \leqq n$ implies $|W|_X$ minimal. Thus, $|W|_X$ is minimal if and only if $|W_i|_X$ is minimal for $1 \leqq i \leqq n$; i.e., if and only if $|W_i|_{X_i}$ is minimal for $1 \leqq i \leqq n$. Whitehead's Theorem assures us that if $|W|_X = |W|_{X\tau}$ is minimal, then τ is the product of level Whitehead automorphisms; i.e., $\tau = \tau_1\tau_2 \ldots \tau_k$, and each τ_j is level on W. Hence, by the above, τ_j is level on each W_i. Moreover, since those τ_j of the second type generate a normal subgroup, we may assume all but possibly τ_k are of the second type. But any stabilizer of W is level on W, and thus on each W_i, so (iii) follows immediately from the lemma.

Now let Y be any basis such that $|W|_Y$ is minimal. Whitehead automorphisms of the first type merely permute $Y^{\pm 1}$, so we may assume $Y = X\tau_i \ldots \tau_k$, where each τ_i is a level Whitehead automorphism of the second type, and (v) follows from the lemma. To prove (iv), let $Y = Y_1 \cup Y_2$ be a basis for F such that $G_1 = \langle Y_1 \rangle$, $G_2 = \langle Y_2 \rangle$, and each of $|W \cap G_1|_{Y_1}$ and $|W \cap G_2|_{Y_2}$ are minimal. As above, $|W|_Y$ is minimal, and (iv) follows from the lemma. To prove (vi), assume $Y = Y_1 \cup Y_2$ such that $W \subset \langle Y_1 \rangle$ and the number of elements in Y_1 is minimal. Then (vi) follows from (iv). Finally, to prove (ii), merely apply the proof of (iv) with W_i replacing W.

If the set W of elements (or cyclic elements) is not contained in any proper free factor of F, and if W is connected, we say *W binds F*. If $F = G_1 * G_2$ and $W \subset G_1 \cup G_2$, then *W respects the splitting $F = G_1 * G_2$*. Implicit in the Whitehead Theorem (1.4.20 and 1.4.24 of [**2**]) is an algorithm for determining whether or not a finite set W binds F or respects a splitting of F. By Theorem 1, we need merely apply successive Whitehead automorphisms (they are finite in number) until $|W|$ is minimal and observe the situation at this point.

Now we turn to topological considerations. The symbols $\partial(\ldots)$, $N(\ldots)$, $(\ldots)^0$, and $\mathrm{Cl}(\ldots)$ denote the boundary, regular neighborhood, interior, and closure, respectively, of the object $(\ldots)$, and H_n denotes an orientable 3-dimensional handlebody of genus n. If $B = \{B_1, \ldots, B_n\} \subset H_n$ is any collection of properly embedded, pairwise disjoint disks such that $\mathrm{Cl}(H_n - N(B))$ is a 3-cell, then B corresponds, in a natural way, to a basis for $\pi_1(H_n, p)$, and all bases are so derived (cf. [**4**]). Such sets B

enable us to translate geometric questions into algebraic ones. If $\gamma \subset H_n$ is any oriented closed curve, then γ determines a cyclic element of $\pi_1(H_n, p)$, and we let γ denote both the curve and the corresponding cyclic element. Since we deal only with cyclic elements, the basepoint p is immaterial and we shall suppress it, writing $\pi_1(H_n)$.

THEOREM 2. *Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\} \subset \partial H_n$ be a set of pairwise disjoint, noncontractible (in H_n), oriented simple closed curves which respects the splitting $\pi_1(H_n) = F_1 * F_2$. Let $\Gamma_i = \Gamma \cap F_i$, $i = 1, 2$. There exists a set $D \subset H_n$ of properly embedded incompressible disks such that $D \cap \Gamma = \emptyset$ and D separates each element of Γ_1 from each element of Γ_2.*

Proof. Assume F_1 has rank k and F_2 has rank l. Let V_k denote a wedge of the k circles $c_1 \ldots c_k$ with the point c_0 in common, and let V_l denote a wedge of the l circles $c_{k+1} \ldots c_n$ with the point c_0' in common. Connect V_k to V_l by an arc from c_0 to c_0' with the point v in its interior, yielding a complex V. Note that $\pi_1(V) = \pi_1(V_k) * \pi_1(V_l)$, $\pi_2(V) = \pi_2(H_n) = 0$, and there is an isomorphism

$$f_*: \pi_1(H_n) \rightarrow \pi_1(V)$$

such that $f_*(F_1) = \pi_1(V_k)$ and $f_*(F_2) = \pi_1(V_l)$. By standard arguments (cf. [**1**, p. 66]), f_* is induced by a map $f: H_n \rightarrow V$. Now consider $f^{-1}(v)$. By [**5**, Lemma 1.1], we may move f via a homotopy until $f^{-1}(v)$ consists of a finite number of incompressible disks. Assume Γ has been moved by an isotopy in ∂H_n until $f^{-1}(v) \cap \Gamma$ is minimal. If $f^{-1}(v) \cap \Gamma \neq \emptyset$, then, since Γ respects the given splitting of $\pi_1(H_n)$, there must exist an arc α in some γ_i such that $\alpha \cap f^{-1}(v) = \partial\alpha$, the loop $f(\alpha)$ is contractible in V, and α does not cobound, with the closure of an arc in $\partial f^{-1}(v) - \alpha$, a 2-cell in ∂H_n. Let $C = N(\alpha)$, a 3-cell, so $C \cap f^{-1}(v)$ consists of two spanning disks in C, the boundaries of which cobound an annulus A in ∂C. Let $B = \mathrm{Cl}(A \cap H_n^0)$, a disk, and let $B' \subset C$ be a properly embedded (in C) disk such that

$$\partial B' \subset A \text{ and } B' \cap B = f^{-1}(v) \cap B.$$

Modify f via a homotopy (cf. [**1**, p. 67]) to f' defined as follows. Let $f'|H_n - C^0 = f|H_n - C^0$, and $f'(B') = v$. Now B' meets the disk $C \cap \partial H_n$ in two arcs β_1' and β_2' which cobound, with the arcs β_1 and β_2, respectively, of $B \cap \partial H_n$, disks in $C \cap \partial H_n$. Since the loop $f(\alpha)$ is trivial, so is the loop $f(\beta_i)$, and we can extend f' across the disk in ∂H_n bounded by $\beta_i \cup \beta_i'$ for $i = 1, 2$. If δ_i is a component of $\mathrm{Cl}(\partial(C \cap \partial H_n) - B)$, then $[f(\delta_i)]$ is trivial and both $f(\delta_1)$ and $f(\delta_2)$ lie on the same side of w. Thus, we may extend f' to all of ∂H_n, and since $\pi_2(V) = 0$, to the remaining open 3-cells in H_n, so that

$$f'^{-1}(v) = (f^{-1}(v) - (f^{-1}(v) \cap C)) \cup B'.$$

If α connects distinct components of $f^{-1}(v)$, we stop, because $f'^{-1}(v)$ will consist entirely of disks. If $\partial\alpha$ was in one component of $f^{-1}(v)$, then $f'^{-1}(v)$ will be compressible; however, f' may then be altered by an additional homotopy, fixed on ∂H_n, which again makes the preimage of v incompressible disks. In either case we will have replaced f by a map f' homotopic to it, and $f'^{-1}(v) \cap \Gamma$ will be reduced. Hence, we may assume $f^{-1}(v) \cap \Gamma = \emptyset$, and if $D = f^{-1}(v)$, our construction assures us that D separates Γ as required.

COROLLARY 1. *Let* $\Gamma = \{\gamma_1, \ldots, \gamma_m\} \subset \partial H_n$ *be a set of pairwise disjoint, oriented simple closed curves. Then* $S = \mathrm{Cl}(\partial(H_n) - N(\Gamma))$ *is incompressible if and only if* Γ *binds* $\pi_1(H_n)$ *and no* γ_i *is contractible in* ∂H_n.

Proof. If S is compressible, there exists, by Dehn's Lemma, a properly embedded disk $D \subset H_n$ with $\partial D \subset S$ and ∂D noncontractible in S. Since no element of ∂S is trivial in ∂H_n, D cannot be boundary parallel in H_n. If D separates H_n, then $\pi_1(H_n)$ may be split nontrivially along D, and this splitting is respected by ∂S, and hence by Γ. If D doesn't separate, then ∂S is contained in

$$\pi_1(\mathrm{Cl}(H_n) - N(D)) \approx F_{n-1},$$

a proper factor of $\pi_1(H_n)$, so Γ cannot bind $\pi_1(H_n)$.

If Γ does not bind and no γ_i is contractible in ∂H_n, we can, by Theorem 2, find a collection of incompressible disks D which misses Γ and ∂S, making S compressible.

COROLLARY 2. *Let* $S \subset \partial H_n$ *be an arbitrary connected surface with* $\partial S \neq \emptyset$ *and with no component of* ∂S *contractible in* H_n.

(a) *If* ∂S *binds* $\pi_1(H_n)$, *then* S *is incompressible.*

(b) *If* ∂S *respects a splitting of* $\pi_1(H_n)$ *such that at least one element of* ∂S *is in each factor, then* S *is compressible.*

(c) *If* S *is incompressible, then* ∂S *is a connected subset of* $\pi_1(H_n)$.

Proof. (a) This proof is identical to the first half of the proof of Corollary 1.

(b) By Theorem 2 we find a set D of disks which reflects the split. Since D separates components of ∂S, and since S is connected, S must be compressible.

(c) If ∂S were not connected in $\pi_1(H_n)$ we could find a splitting of $\pi_1(H_n)$ respected by the set, with components of ∂S on each side of the splitting. Thus, as in (b), S would be compressible.

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