

## ON MOMENTS OF DOWNWARD PASSAGE TIMES FOR SPECTRALLY NEGATIVE LÉVY PROCESSES

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### Abstract

The existence of moments of first downward passage times of a spectrally negative Lévy process is governed by the general dynamics of the Lévy process, i.e. whether it is drifting to  $+\infty$ ,  $-\infty$ , or oscillating. Whenever the Lévy process drifts to  $+\infty$ , we prove that the  $\kappa$ th moment of the first passage time (conditioned to be finite) exists if and only if the  $(\kappa + 1)$ th moment of the Lévy jump measure exists. This generalizes a result shown earlier by Delbaen for Cramér–Lundberg risk processes. Whenever the Lévy process drifts to  $-\infty$ , we prove that all moments of the first passage time exist, while for an oscillating Lévy process we derive conditions for non-existence of the moments, and in particular we show that no integer moments exist.

*Keywords:* Conjugate subordinator; Cramér–Lundberg risk process; exit time; fluctuation theory; first hitting time; fractional calculus; moments; ruin theory; spectrally negative Lévy process; subordinator; time to ruin.

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### 1. Introduction

Let  $X = (X_t)_{t \geq 0}$  be a spectrally negative Lévy process, i.e. a Lévy process that does not exhibit positive jumps, starting at zero. In this article we study moments of the *first (downward) passage time of  $-x$* ,  $x \geq 0$ , of the process  $X$ , i.e. moments of

$$\tau_x^- := \inf\{t > 0: X_t < -x\}, \quad (1.1)$$

conditioned on finiteness of this stopping time.

The first passage time  $\tau_x^-$ , sometimes also referred to as the *exit time*, of (spectrally negative) Lévy processes is a well-known object that has been studied by many authors; see [5, Section 9.5] for a general overview. However, most results are limited on giving a representation of the Laplace transform of the first passage time.

In the case of a Brownian motion with drift  $p \in \mathbb{R}$ , due to the continuity of the paths, the first passage time  $\tau_x^-$  coincides with the *first hitting time* of  $-x$ , that is, with  $\tau_x^{-,*} = \inf\{t > 0: X_t = -x\}$ . In this special case,  $\tau_x^-$  is known to have the Laplace transform

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(cf. [12, Eq. I.(9.1)])

$$\mathbb{E}[e^{-q\tau_x^-}] = \exp[-(\sqrt{p^2 + 2q} + p)x], \quad x \geq 0, q > 0, \tag{1.2}$$

and its distribution is given explicitly (cf. [12, Eq. I.(9.2)]) as

$$\mathbb{P}(\tau_x^- \in dz) = \frac{x}{\sqrt{2\pi z^3}} \exp\left[-\frac{(x + pz)^2}{2z}\right] dz, \quad x, z \geq 0,$$

where in both formulas we assumed the process to be standardized, i.e. such that  $\sigma^2 = 1$ .

For general spectrally negative Lévy processes, the first hitting time and the first passage time can be related via the undershoot  $X_{\tau_x^-} + x \leq 0$ , as shown in [4]. In particular, for spectrally negative  $\alpha$ -stable processes ( $1 < \alpha < 2$ ) this relation reads (cf. [17])

$$\tau_x^{-,*} = \tau_x^- + |X_{\tau_x^-} + x|^\alpha \cdot \hat{\tau}_x^{+,*}, \tag{1.3}$$

where  $\hat{\tau}_x^{+,*}$  is an independent copy of the first upwards hitting time  $\tau_x^{+,*} = \inf\{t > 0 : X_t = x\}$ . Note that (1.3) is a direct consequence of the self-similarity (of index  $1/\alpha$ ) of  $\alpha$ -stable processes, which implies further that  $(X_{\tau_x^-} + x) \stackrel{d}{=} x \cdot (X_{\tau_1^-} - 1)$ .

The hitting time  $\tau_x^{-,*}$  of a spectrally one-sided stable process has been studied, e.g. in [7, 11, 17]. In particular, in [17] fractional moments and a series representation of the density  $\tau_x^{-,*}$  are provided.

The first downward passage time  $\tau_x^-$  has also been extensively studied in the field of actuarial mathematics, where the spectrally negative Lévy process  $X$  is interpreted as a *risk process* and shifted to start at  $x \geq 0$ . Then, due to the space homogeneity of the Lévy process,  $\tau_x^-$  coincides with the *time of ruin*, i.e. the first time the process passes the value zero. The most prominent example for such a risk process is the classical *Cramér–Lundberg model*, where  $X$  is chosen to be a spectrally negative compound Poisson process, i.e.

$$X_t = x + pt - \sum_{i=1}^{N_t} S_i, \quad t \geq 0. \tag{1.4}$$

Here,  $x \geq 0$  is interpreted as the *initial capital*,  $p > 0$  denotes a constant *premium rate*, the Poisson process  $(N_t)_{t \geq 0}$  represents the *claim counting process*, and the independent and identically distributed (i.i.d.) positive random variables  $\{S_i, i \in \mathbb{N}\}$  are the *claim size variables* which are independent of  $(N_t)_{t \geq 0}$ .

For this model, under the profitability assumption  $\mathbb{E}[X_1] > 0$ , it is shown in [3], for all  $\kappa > 0$ , that the  $\kappa$ th moment of the ruin time exists if and only if the  $(\kappa + 1)$ th moment of the claim size distribution exists. In this paper we generalize this result to arbitrary spectrally negative Lévy processes. Note that, while the proof given in [3] relies on results on the speed of convergence of random walks, we use a completely different approach here via fractional differentiation of Laplace transforms. In particular, our approach allows us to relate the existence of  $\mathbb{E}[(\tau_x^-)^\kappa | \tau_x^- < \infty]$  with the existence of the  $\kappa$ th moment of the subordinator  $(\tau_x^+)_{x \geq 0}$  of upwards passage times  $\tau_x^+ = \inf\{t > 0 : X_t > x\}$  at a specific random time. As a by-product, we show that  $(\tau_x^+)_{x \geq 0}$  is a special subordinator and identify its conjugate subordinator.

Before presenting and proving our main theorem on the existence of moments of the first passage time in Section 3, we collect various preliminary results on (spectrally negative) Lévy processes and fractional derivatives in Section 2.

## 2. Preliminaries

Throughout this article let  $X = (X_t)_{t \geq 0}$  be a Lévy process, i.e. a càdlàg stochastic process with independent and stationary increments, defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ . It is well known that the Lévy process  $X$  is fully characterized by its *characteristic exponent*  $\Psi$ , which is defined via  $e^{-t\Psi(\theta)} = \mathbb{E}[e^{i\theta X(t)}]$  and takes the form

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta y} + i\theta y \mathbf{1}_{\{|y| < 1\}}) \Pi^*(dy), \quad \theta \in \mathbb{R},$$

for constants  $a \in \mathbb{R}$ ,  $\sigma^2 \geq 0$ , and a measure  $\Pi^*$  on  $\mathbb{R} \setminus \{0\}$  satisfying  $\int_{\mathbb{R}} (1 \wedge y^2) \Pi^*(dy) < \infty$ . The measure  $\Pi^*$  is called the *Lévy measure* or *jump distribution* of  $X$ , while  $(\sigma^2, a, \Pi^*)$  is the *characteristic triplet* of  $X$ .

If  $X$  has no upwards jumps, i.e. if  $\Pi^*((0, \infty)) = 0$ , then  $X$  is called *spectrally negative*. In this case, it is handy to use the *Laplace exponent*  $\psi(\theta) := (1/t) \log \mathbb{E}[e^{\theta X_t}]$ ,  $\theta \geq 0$ , of  $-X$  instead of the characteristic exponent, which can then be written in the form

$$\psi(\theta) = c\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(0, \infty)} (e^{-\theta y} - 1 + \theta y \mathbf{1}_{\{y < 1\}}) \Pi(dy), \quad (2.1)$$

where  $c = -a \in \mathbb{R}$ ,  $\sigma^2 \geq 0$ , and  $\Pi(dy) = \Pi^*(-dy)$  is the mirrored version of the jump measure, which is therefore defined on  $(0, \infty)$ .

The Laplace exponent  $\psi$  admits some useful properties. Clearly,  $\psi(0) = 0$ , and  $\lim_{\theta \rightarrow \infty} \psi(\theta) = \infty$ . On  $(0, \infty)$  the function  $\psi$  is infinitely often differentiable and strictly convex. Lastly, as  $\psi$  is nothing more than the cumulant-generating function of  $X_1$ , it carries information on the moments of  $X$ . In particular, it is well known (cf. [15, Corollary 25.8]) that, for any  $\kappa > 0$ ,

$$\mathbb{E}[|X_1|^\kappa] < \infty \text{ if and only if } \int_{|y| \geq 1} |y|^\kappa \Pi(dy) < \infty,$$

and for  $\kappa = k \in \mathbb{N}_0$  this in turn implies

$$|\partial^k \psi(0+) | := |\psi^{(k)}(0+) | < \infty. \quad (2.2)$$

Note that throughout this article  $\partial_q^k f(q, z)$  denotes the  $k$ th derivative of a function  $f$  with respect to  $q$ , while  $\partial_q := \partial_q^1$ . In the case of only one parameter, we will usually omit the subscript.

We will also use the Laplace exponent's right inverse, which we always denote by

$$\Phi(q) := \sup\{\theta \geq 0 : \psi(\theta) = q\}, \quad q \geq 0.$$

From the mentioned properties of  $\psi$ , it follows immediately that  $\Phi(0) = 0$  if and only if  $\psi'(0+) \geq 0$ , while  $\Phi(0) > 0$  if and only if  $\psi'(0+) < 0$ . The function  $q \mapsto \Phi(q)$  is strictly monotone increasing on  $[0, \infty)$ , infinitely often differentiable on  $(0, \infty)$ , and it is the well-defined inverse of  $\psi(\theta)$  on the interval  $[\Phi(0), \infty)$ , i.e.  $\Phi(\psi(\theta)) = \theta$  and  $\psi(\Phi(q)) = q$  for all  $\theta \in [\Phi(0), \infty)$ ,  $q \geq 0$ . Thus, applying the chain rule on  $q \mapsto q = \psi(\Phi(q))$  immediately yields

$$\Phi'(q) = \partial_q \Phi(q) = \frac{1}{\psi'(\Phi(q))}, \quad q \geq 0, \quad (2.3)$$

where the case  $q = 0$  is interpreted in the limiting sense  $q \downarrow 0$ .

Finally, note that, by definition,

$$\lim_{q \downarrow 0} \frac{q}{\Phi(q)} = \begin{cases} \psi'(0+) & \text{if } \psi'(0+) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For proofs of the stated properties and a more thorough discussion of Lévy processes in general we refer the reader to [9, 15].

As announced in the introduction, we are interested in the first downward passage time  $\tau_x^-$  of  $-x$ ,  $x \geq 0$ , as defined in (1.1), or, more precisely, in the first passage time given that the process passes through  $-x$ , i.e.

$$(\tau_x^- \mid \tau_x^- < \infty). \tag{2.4}$$

Note that in the case that  $\psi'(0+) = \mathbb{E}[X_1] \in [-\infty, 0]$  we have  $\tau_x^- = (\tau_x^- \mid \tau_x^- < \infty)$  as  $X$  enters the negative half-line almost surely. In the case that  $\psi'(0+) > 0$  the term *passage time* will typically be used for the conditioned quantity (2.4).

To avoid trivialities we exclude the case that  $X$  is a pure drift, which implies a deterministic first passage time. Hence, we always have  $\mathbb{P}(\tau_x^- < \infty) > 0$ . Moreover, we exclude the hitting level  $x = 0$  whenever  $X_t$  is of unbounded variation, as in this case  $\tau_0^- = 0$  almost surely.

To study  $\tau_x^-$  (or  $(\tau_x^- \mid \tau_x^- < \infty)$ ) we will use the concept of scale functions. Recall that for any  $q \geq 0$  the  $q$ -scale function  $W^{(q)}: \mathbb{R} \rightarrow [0, \infty)$  of the spectrally negative Lévy process  $X$  is the unique function such that, for  $x \geq 0$ , its Laplace transform satisfies

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}$$

for all  $\beta > \Phi(q)$ . For  $x < 0$  we set  $W^{(q)}(x) = 0$ . Furthermore, the *integrated  $q$ -scale function*  $Z^{(q)}: \mathbb{R} \rightarrow [0, \infty)$  is given by  $Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) dy$ , and it fulfils, cf. [9, Theorem 8.1],

$$\mathbb{E}[e^{-q\tau_x^-} \mathbf{1}_{\{\tau_x^- < \infty\}}] = Z^{(q)}(x) - \frac{q}{\Phi(q)} \cdot W^{(q)}(x), \quad x \in \mathbb{R}, \quad q \geq 0. \tag{2.5}$$

Taking the limit  $q \downarrow 0$ , this immediately implies  $\mathbb{P}(\tau_x^- < \infty) = 1 - (0 \vee \psi'(0+)) \cdot W^{(0)}(x)$ ,  $x \in \mathbb{R}$ , where we use the standard notation  $\vee$  to denote the maximum.

Observe that the functions  $q \mapsto W^{(q)}(x)$  and  $q \mapsto Z^{(q)}(x)$  may be extended analytically to  $\mathbb{C}$ , which means especially that they are infinitely often differentiable with bounded derivatives on every compact set  $K \subset \mathbb{C}$ . This implies in particular that limits of type  $q \downarrow 0$  exist. Again, we refer to [9] for missing proofs and further details. For detailed accounts of scale functions and their numerous applications, we also refer to [2, 8].

Lastly, let us recall that fractional moments of non-negative random variables can be computed via fractional differentiation of the corresponding Laplace transform. More precisely, define, for any  $\kappa \in (0, 1)$ , the *Marchaud fractional derivative* of a function  $f(z)$ ,  $z \geq 0$ ,

$$\mathbf{D}_z^\kappa f(z) = \frac{\kappa}{\Gamma(1 - \kappa)} \int_z^\infty \frac{f(z) - f(u)}{(u - z)^{\kappa+1}} du, \tag{2.6}$$

cf. [14, (5.58)], while, for  $\kappa \geq 1$  with  $n := \lfloor \kappa \rfloor$  denoting the largest integer less than or equal to  $\kappa$ ,  $\mathbf{D}_z^\kappa f(z) = \partial_z^n \mathbf{D}_z^{\kappa-n} f(z)$ . Then, cf. [19, Theorem 1], for any non-negative random variable  $T$  with Laplace transform  $g(z) = \mathbb{E}[e^{-zT}]$ ,  $z \geq 0$ , the  $\kappa$ th absolute moment of  $T$  exists if and only if  $\mathbf{D}_z^\kappa g(0)$  exists, in which case

$$\mathbb{E}[T^\kappa] = \mathbf{D}_z^\kappa g(0). \tag{2.7}$$

This allows us to derive the following lemma.

**Lemma 2.1.** *For any  $\kappa > 0$  and  $x \geq 0$ , the  $\kappa$ th moment of the first downward passage time  $\tau_x^- \mid \tau_x^- < \infty$  of a spectrally negative Lévy process is given by*

$$\mathbb{E}[(\tau_x^-)^\kappa \mid \tau_x^- < \infty] = \frac{1}{\mathbb{P}(\tau_x^- < \infty)} \cdot \left[ \mathbf{D}_q^\kappa \left( Z^{(q)}(x) - \frac{q}{\Phi(q)} \cdot W^{(q)}(x) \right) \right]_{q=0},$$

and it exists if and only if the right-hand side exists and is finite.

*Proof.* As  $\mathbb{E}[e^{-q\tau_x^-} \mathbf{1}_{\{\tau_x^- < \infty\}}] = \mathbb{E}[e^{-q\tau_x^-} \mid \tau_x^- < \infty] \cdot \mathbb{P}(\tau_x^- < \infty)$ , the claim follows immediately from (2.7) and (2.5).  $\square$

### 3. Existence of moments

In [3], Delbaen showed in a classical Cramér–Lundberg model (1.4) that is *profitable*, i.e. with  $\psi'(0+) > 0$ , that for any  $\kappa > 0$  the  $\kappa$ th moment of the ruin time exists if and only if the  $(\kappa + 1)$ th moment of the claim sizes exists. Delbaen’s proof relies on results on the speed of convergence of random walks. In this paper we use an alternative approach via fractional derivatives of Laplace transforms to prove an extension of the result in [3] to any spectrally negative Lévy process. Moreover, we additionally consider the *non-profitable* settings of  $\psi'(0+) \leq 0$ .

Our main result in this section thus reads as follows. Note that although part (i) of Theorem 3.1 can be derived from [1, Proposition 1.1 and Theorem 1.1], we give a short proof below for the reader’s convenience.

**Theorem 3.1.** *Let  $(X_t)_{t \geq 0}$  be a spectrally negative Lévy process with Laplace exponent  $\psi$  as in (2.1), and let  $\tau_x^-$  denote its first passage time of  $-x$  for  $x \geq 0$ .*

- (i) *If  $\psi'(0+) < 0$ , then, for any  $x \geq 0$ , there exists  $q^* > 0$  such that  $\mathbb{E}[e^{q^* \tau_x^-}] < \infty$ , which implies that, for any  $x \geq 0$  and  $\kappa \geq 0$ ,  $\mathbb{E}[(\tau_x^-)^\kappa] < \infty$ .*
- (ii) *If  $\psi'(0+) > 0$ , then, for any  $x \geq 0$  and  $\kappa > 0$ ,  $\mathbb{E}[(\tau_x^-)^\kappa \mid \tau_x^- < \infty] < \infty$  if and only if  $\int_{[1, \infty)} y^{\kappa+1} \Pi(dy) < \infty$ .*
- (iii) *Assume  $\psi'(0+) = 0$ .*
  - (a) *If there exists  $\kappa^* \in (0, 1]$  such that  $\int_{[1, \infty)} y^{\kappa^*+1} \Pi(dy) = \infty$ , then, for any  $x \geq 0$  and  $\kappa \geq \kappa^*$ ,*

$$\mathbb{E}[(\tau_x^-)^\kappa] = \infty. \tag{3.1}$$

- (b) *If  $\psi''(0+) < \infty$ , then (3.1) holds for any  $x \geq 0$  and  $\kappa > \frac{1}{2}$ .*

*In particular, (3.1) holds for any  $x \geq 0$  and  $\kappa \geq 1$ .*

**Remark 3.1.** Note that, a priori, the above theorem needs no restrictions concerning possible choices of the location parameter  $c \in \mathbb{R}$  of  $(X_t)_{t \geq 0}$ . However, since, by [15, Example 25.12],  $c - \int_{[1, \infty)} y \Pi(dy) = \mathbb{E}[X_1] = \psi'(0+)$ , in cases (ii) and (iii) the assumption  $\psi'(0+) \geq 0$  implies that  $c \geq \int_{[1, \infty)} y \Pi(dy) \geq 0$ . In particular,  $c < 0$  is a valid choice only in case (i).

**Remark 3.2.** At first glance, Theorem 3.1(iii) suggests that, for an oscillating process  $(X_t)_{t \geq 0}$ , no fractional moments of the first passage time of zero exist. This, however, is not true in general and we provide two counterexamples:

- (a) Consider a (standardized) Brownian motion without drift for which, by (1.2),  $\mathbb{E}[e^{-q \cdot \tau_x^-}] = \exp[-\sqrt{2q} \cdot x]$ ,  $x \geq 0$ . Then,  $\mathbb{P}(\tau_x^- < \infty) = 1$  and, from (2.6) and (2.7), we obtain, for any  $\kappa \in (0, 1)$ ,

$$\mathbb{E}[(\tau_x^-)^\kappa] = [\mathbf{D}_q^\kappa e^{-\sqrt{2q} \cdot x}]_{q=0} = \frac{\kappa}{\Gamma(1 - \kappa)} \int_0^\infty \frac{1 - e^{-\sqrt{2u} \cdot x}}{u^{\kappa+1}} du,$$

which is finite if and only if  $\kappa \in (0, \frac{1}{2})$ . In particular, in this case (3.1) holds for any  $\kappa \geq \frac{1}{2}$ , which shows that Theorem 3.1(iii)(b) is near to being sharp.

- (b) Consider a spectrally negative,  $\alpha$ -stable Lévy process  $(X_t)_{t \geq 0}$  with index  $\alpha \in (1, 2)$  such that the Laplace exponent of  $-X$  is given by  $\psi(\theta) = \theta^\alpha$  and, in particular,  $\psi'(0) = 0$ . For such a process it has been shown in [17, Proposition 4 and the subsequent remark] that the first passage time  $\tau_x^-$  admits finite fractional moments, namely  $\mathbb{E}[(\tau_x^-)^\kappa] < \infty$  if and only if  $\kappa \in (-1, 1 - (1/\alpha))$ .

The identification of the threshold  $\kappa^* \leq 1$  such that  $\mathbb{E}[(\tau_x^-)^\kappa] < \infty$ ,  $\kappa < \kappa^*$ , and  $\mathbb{E}[(\tau_x^-)^\kappa] = \infty$  for  $\kappa \geq \kappa^*$  for a general oscillating and spectrally negative Lévy process seems difficult; the chosen approach for our proof of Theorem 3.1 only yields the sufficient condition for (3.1) as stated in Theorem 3.1(iii)(a). Moreover, the above example of a Brownian motion clearly shows that the threshold  $\kappa^*$  cannot solely depend on the Lévy measure. We therefore leave this question open for future research.

*Proof of Theorem 3.1(i).* Recall that if  $\psi'(0+) < 0$ , then  $\Phi(0) > 0$  and, by the convexity of  $\psi$ , we obtain  $\psi'(\Phi(0)) > 0$ . Furthermore, the Laplace exponent  $\psi$  extended to  $\mathbb{C}$  is analytic on  $\{x \in \mathbb{C} : \text{Re}(z) > 0\}$ , and hence it is analytic in a neighborhood of  $\Phi(0)$ . Consequently, the inverse function theorem of complex analysis, cf. [13, Theorem 10.30], implies that the inverse of  $\psi(\theta)$ , i.e.  $\Phi(q)$ , is also analytic in a neighborhood of zero. By (2.5), this yields that  $\mathbb{E}[e^{-q \tau_x^-}]$  is analytic in a neighborhood of zero as well, and thus  $\mathbb{E}[e^{q^* \tau_x^-}] < \infty$  for some  $q^* > 0$  as claimed. □

To prove the second and third parts of Theorem 3.1 we start with a simple lemma that reduces the problem of existence of moments of the first passage time to finiteness of (fractional) derivatives of a certain function in zero.

**Lemma 3.1.** *Set  $\eta(q) := q/\Phi(q)$ ,  $q > 0$ . Then, for any  $x \geq 0$  and  $\kappa > 0$ ,  $\mathbb{E}[(\tau_x^-)^\kappa \mid \tau_x^- < \infty] < \infty$  if and only if  $\lim_{q \downarrow 0} |\mathbf{D}_q^\kappa \eta(q)| < \infty$ .*

*Proof.* It follows immediately from Lemma 2.1 that  $\mathbb{E}[(\tau_x^-)^\kappa \mid \tau_x^- < \infty] < \infty$  if and only if  $\lim_{q \downarrow 0} \mathbf{D}_q^\kappa (Z^{(q)}(x) - \eta(q) \cdot W^{(q)}(x)) < \infty$ . However,  $q \mapsto W^{(q)}(x)$  and  $q \mapsto Z^{(q)}(x)$  are infinitely often differentiable with bounded derivatives on  $[0, \infty)$ . Hence, linearity of the (fractional) derivative reduces the problem to characterization of the finiteness of the derivative  $\lim_{q \downarrow 0} \mathbf{D}_q^\kappa (\eta(q) \cdot W^{(q)}(x))$ .

Observe that the definition of the Marchaud derivative is equivalent to the Liouville derivative for sufficiently good functions; see [14, Remark 5.3] for details. We may

therefore apply the product rule for fractional Liouville derivatives (cf. [18, p. 206]) to  $\eta(q) \cdot W^{(q)}(x)$ . Recalling again that  $q \mapsto W^{(q)}(x)$  is infinitely often differentiable with bounded derivatives on every compact  $K \subset \mathbb{C}$ , and that  $W^{(q)}(x) > 0$  for any  $x > 0$ , we conclude that  $\lim_{q \downarrow 0} \mathbf{D}_q^k(\eta(q) \cdot W^{(q)}(x)) < \infty$  if and only if  $\lim_{q \downarrow 0} \mathbf{D}_q^k \eta(q) < \infty$ , as claimed.

If  $x = 0$  note that  $W^{(q)}(0) > 0$  if and only if  $(X_t)_{t \geq 0}$  is of bounded variation (cf. [2, (25)]), and in this case the above argumentation yields the result.  $\square$

The remainder of the proof of Theorem 3.1 relies on the interpretation of  $\eta(q)$  as the Laplace exponent of a certain killed subordinator, as shown in the next proposition. Recall that a *subordinator*  $(Y_t)_{t \geq 0}$  is a Lévy process with non-decreasing paths whose Laplace exponent  $\varphi(\theta) = -(1/t) \log \mathbb{E}[e^{-\theta Y_t}]$  is of the form  $\varphi(\theta) = \tilde{c} \cdot \theta + \int_0^\infty (1 - e^{-\theta y}) \tilde{\Pi}(dy)$  for  $\theta \geq 0$ , a drift  $\tilde{c} \geq 0$ , and a measure  $\tilde{\Pi}$  such that  $\int_{(0, \infty)} (1 \wedge y) \tilde{\Pi}(dy) < \infty$ . A *killed subordinator*  $(Y_t)_{t \geq 0}$  is defined via

$$Y_t = \begin{cases} \tilde{Y}_t & \text{if } t < \mathbf{e}_\beta, \\ \zeta & \text{if } t \geq \mathbf{e}_\beta, \end{cases}$$

where  $(\tilde{Y}_t)_{t \geq 0}$  is a subordinator,  $\mathbf{e}_\beta$  is an independent  $\text{Exp}(\beta)$ -distributed time,  $\beta > 0$ , and  $\zeta$  denotes some *cemetery state*. As usual, we interpret  $\beta = 0$  as  $\mathbf{e}_\beta = \infty$  corresponding to no killing. The Laplace exponent  $\varphi_Y$  of a killed subordinator is given by  $\varphi_Y(\theta) = -\log \mathbb{E}[e^{-\theta Y_1}] = -\log \mathbb{E}[e^{-\theta \tilde{Y}_1} \cdot \mathbf{1}_{\{1 < \mathbf{e}_\beta\}}] = \beta + \varphi_{\tilde{Y}}(\theta)$  for the Laplace exponent  $\varphi_{\tilde{Y}}(\theta)$  of  $(\tilde{Y}_t)_{t \geq 0}$ .

Further, for any  $x \geq 0$ , let  $\tau_x^+ := \inf\{t > 0 : X_t > x\}$  be the first upwards passage time of  $x$ , i.e. the first time that  $X_t$  is above  $x$ . It is well known (cf. [9, Theorem 3.12]) that, for all  $q \geq 0$ ,

$$\mathbb{E}[e^{-q \tau_x^+} \cdot \mathbf{1}_{\{\tau_x^+ < \infty\}}] = e^{-\Phi(q)x}, \quad x \geq 0. \quad (3.2)$$

If, furthermore,  $\mathbb{E}[X_1] = \psi'(0+) \geq 0$ , then  $(\tau_x^+)_{x \geq 0}$  is a subordinator with Laplace exponent  $\Phi(q)$ , cf. [9, Corollary 3.14].

**Proposition 3.1.** *Assume  $\psi'(0+) \geq 0$ , and define a killed subordinator  $(Y_t)_{t \geq 0}$ , independent of  $(\tau_x^+)_{x \geq 0}$ , through its Laplace exponent  $\varphi(\theta) = -t^{-1} \log \mathbb{E}[e^{-\theta Y_t}]$  by setting*

$$\varphi(\theta) := \psi'(0+) + \frac{\sigma^2}{2} \theta + \int_0^\infty (1 - e^{-\theta y}) \Pi((y, \infty)) dy, \quad \theta > 0.$$

*Then,  $(\tau_Y^+)_{t \geq 0}$  is a killed subordinator with Laplace exponent*

$$-\frac{1}{t} \log \mathbb{E}[e^{-q \tau_Y^+}] = \eta(q), \quad q \geq 0. \quad (3.3)$$

*Proof.* An application of [10, Theorem 1] on the subordinator  $(Y_t)_{t \geq 0}$  implies that there exists a spectrally negative Lévy process, the so-called *parent process*, with Laplace exponent  $\theta \cdot \varphi(\theta)$  and whose characteristic triplet coincides with that of  $(X_t)_{t \geq 0}$ . Thus,  $\varphi(\theta) = \psi(\theta)/\theta$ . Equation (3.3) is now a direct consequence of [15, Theorem 30.1] and the fact that

$$\varphi(\Phi(q)) = \frac{\psi(\Phi(q))}{\Phi(q)} = \frac{q}{\Phi(q)} = \eta(q).$$

$\square$

**Remark 3.3.** Note that the above proposition implies that, as long as  $\psi'(0+) \geq 0$ , the subordinator  $(\tau_x^+)_{x \geq 0}$  is a *special subordinator* since its *conjugate* Laplace exponent  $q/\Phi(q) = \eta(q)$  is shown to be the Laplace exponent of a (killed) subordinator. See, e.g., [9, Section 5.6] or [16, Chapter 11] for general information on special subordinators and their Laplace exponents, which are also known as *special Bernstein functions*.

Combining Lemma 3.1, Proposition 3.1, and (2.7), it is an immediate consequence that, assuming  $\psi'(0+) \geq 0$ , for all  $\kappa > 0$  and  $x \geq 0$ ,

$$\mathbb{E}[(\tau_x^-)^\kappa \mid \tau_x^- < \infty] < \infty \quad \text{if and only if} \quad \mathbb{E}[(\tau_{Y_1}^+)^\kappa] < \infty. \tag{3.4}$$

In order to find suitable conditions for the right-hand side of (3.4), we next prove a general statement concerning the existence of moments of a subordinated subordinator.

**Proposition 3.2.** *Let  $(Z_t)_{t \geq 0}$  be a non-zero subordinator, and let  $(Y_t)_{t \geq 0}$  be a (possibly killed) non-zero subordinator, independent of  $(Z_t)_{t \geq 0}$ . If  $\mathbb{E}[Z_1] < \infty$ , then, for all  $\kappa > 0$ ,  $\mathbb{E}[Z_{Y_1}^\kappa] < \infty$  if and only if  $\mathbb{E}[Z_1^\kappa] < \infty$  and  $\mathbb{E}[Y_1^\kappa] < \infty$ . If  $\mathbb{E}[Z_1] = \infty$  and  $\kappa \in (0, 1)$ , then  $\mathbb{E}[Z_{Y_1}^\kappa] < \infty$  implies  $\mathbb{E}[Z_1^\kappa] < \infty$  and  $\mathbb{E}[Y_1^\kappa] < \infty$ .*

To prove this proposition, we need the following lemma.

**Lemma 3.2.** *Let  $(Z_t)_{t \geq 0}$  be a non-zero subordinator with  $\mathbb{E}[Z_1] < \infty$ . If  $\mathbb{E}[Z_1^\kappa] < \infty$  for some  $\kappa > 0$ , then  $\mathbb{E}[Z_t^\kappa] < \infty$  for all  $t \geq 0$  and the mapping  $t \mapsto \mathbb{E}[Z_t^\kappa]$ ,  $t \geq 1$ , is of polynomial order  $\kappa$ .*

*Proof.* First note that, by Hölder’s inequality, for all  $n \in \mathbb{N}$ ,  $a_1, \dots, a_n \geq 0$ , and  $r \geq 1$ ,

$$(a_1 + \dots + a_n)^r \leq n^{r-1} \cdot (a_1^r + \dots + a_n^r), \tag{3.5}$$

while for  $r \leq 1$  the inequality in (3.5) holds with  $\geq$  instead of  $\leq$ .

Let  $\varphi$  be the Laplace exponent of the subordinator  $(Z_t)_{t \geq 0}$ . By [15, Corollary 25.8], finiteness of  $\mathbb{E}[Z_1^\kappa]$  for some  $\kappa > 0$  implies finiteness of  $\mathbb{E}[Z_t^\kappa]$  for all  $t \geq 0$ .

By our assumptions,  $\mathbb{E}[Z_1] = \varphi'(0+) \in (0, \infty)$  and it follows that, cf. [15, Example 25.12],  $\mathbb{E}[Z_t] = t \cdot \varphi'(0+) = t \cdot \mathbb{E}[Z_1]$ . Now let  $\kappa \geq 1$ . Then, by Jensen’s inequality we conclude that  $\mathbb{E}[Z_t^\kappa] \geq \mathbb{E}[Z_t]^\kappa = t^\kappa \cdot \mathbb{E}[Z_1]^\kappa$ , which yields a lower bound of degree  $\kappa$ . In order to show an upper bound, set  $n := \lceil t \rceil$  such that  $t/n =: c_n \in [\frac{1}{2}, 1]$ , and let  $\xi_i$  be i.i.d. copies of  $Z_1$ . Then, due to the infinite divisibility and monotonicity of  $Z$ , we have

$$\begin{aligned} \mathbb{E}[Z_t^\kappa] &\leq \mathbb{E}[Z_n^\kappa] = \mathbb{E} \left[ \left( \sum_{i=1}^n \xi_i \right)^\kappa \right] \\ &\leq \mathbb{E} \left[ n^{\kappa-1} \cdot \sum_{i=1}^n \xi_i^\kappa \right] = n^\kappa \cdot \mathbb{E}[\xi_1^\kappa] = t^\kappa \cdot c_n^{-\kappa} \cdot \mathbb{E}[Z_1^\kappa] \leq t^\kappa \cdot 2^\kappa \cdot \mathbb{E}[Z_1^\kappa], \end{aligned} \tag{3.6}$$

where we used (3.5) for the second inequality.

To prove the statement for  $\kappa \in (0, 1)$ , note that  $(\cdot)^\kappa$  is concave. Hence, Jensen’s inequality yields an upper bound in this case. The lower bound follows analogously to (3.6), setting  $n := \lfloor t \rfloor$ , and applying the variant of (3.5) for  $r \leq 1$ . □



*Proof of Proposition 3.2.* We write  $\nu_Z, b_Z$  for the Lévy measure and drift of  $Z$ , respectively, and likewise  $\nu_Y, b_Y$  for the Lévy measure and drift of  $Y$ . Then, as  $(Z_{Y_t})_{t \geq 0}$  is a (killed) subordinator with Lévy measure  $\nu$ , say,  $\mathbb{E}[Z_{Y_1}^\kappa] < \infty$  is equivalent (cf. [15, Corollary 25.8]) to  $\int_{[1, \infty)} z^\kappa \nu(dz) < \infty$ , where the Lévy measure  $\nu$  of the subordinated process is given (cf. [15, Theorem 30.1]) by  $\nu(B) = b_Y \nu_Z(B) + \int_{(0, \infty)} \mu^s(B) \nu_Y(ds)$  for any Borel set  $B$  in  $(0, \infty)$ , where  $\mu = \mathcal{L}(Z_1)$  denotes the distribution of  $Z_1$ . Thus,

$$\begin{aligned} \int_{[1, \infty)} z^\kappa \nu(dz) &= b_Y \int_{[1, \infty)} z^\kappa \nu_Z(dz) + \int_{[1, \infty)} z^\kappa d\left(\int_{(0, \infty)} \mu^s(z) \nu_Y(ds)\right) \\ &= b_Y \int_{[1, \infty)} z^\kappa \nu_Z(dz) + \int_{(0, \infty)} \int_{[1, \infty)} z^\kappa \mu^s(dz) \nu_Y(ds), \end{aligned} \tag{3.7}$$

where all the terms are non-negative and hence the sum that appears is finite if and only if both summands are finite. From [15, Corollary 25.8] we know that  $\int_{[1, \infty)} z^\kappa \nu_Z(dz) < \infty$  if and only if  $\mathbb{E}[Z_1^\kappa] < \infty$  if and only if  $\mathbb{E}[Z_s^\kappa] < \infty$  for all  $s \geq 0$ . Thus, assume  $\int_{[1, \infty)} z^\kappa \nu_Z(dz) < \infty$  from now on, which implies  $\int_{[1, \infty)} z^\kappa \mu^s(dz) = \mathbb{E}[\mathbf{1}_{\{Z_s \geq 1\}} Z_s^\kappa] < \infty$ . Furthermore,

$$\begin{aligned} \int_{(0, \infty)} \int_{[1, \infty)} z^\kappa \mu^s(dz) \nu_Y(ds) &= \int_{(0, \infty)} \mathbb{E}[\mathbf{1}_{\{Z_s \geq 1\}} Z_s^\kappa] \nu_Y(ds) \\ &= \int_{(0, 1)} \mathbb{E}[\mathbf{1}_{\{Z_s \geq 1\}} Z_s^\kappa] \nu_Y(ds) \\ &\quad + \int_{[1, \infty)} \mathbb{E}[Z_s^\kappa] \nu_Y(ds) - \int_{[1, \infty)} \underbrace{\mathbb{E}[\mathbf{1}_{\{Z_s < 1\}} Z_s^\kappa]}_{\in [0, 1]} \nu_Y(ds), \end{aligned} \tag{3.8}$$

where the left-hand side of the equation is finite if and only if the right-hand side is finite. Here, the last integral, as well as the sum of all three, is non-negative.

Consider the first integral. We have that  $\mathbb{E}[\mathbf{1}_{\{Z_s \geq 1\}} Z_s^\kappa] = \mathbb{P}(Z_s \geq 1) \cdot \mathbb{E}[Z_s^\kappa \mid Z_s \geq 1]$ , where  $\mathbb{E}[Z_s^\kappa \mid Z_s \geq 1] =: C_1(s) < \infty$ ,  $s \in [0, \infty)$ , since we assumed  $\mathbb{E}[Z_s^\kappa] < \infty$ . From [15, Lemma 30.3], it follows that  $\mathbb{P}(Z_s \geq 1) \leq C_2 s$  for some  $C_2 \in (0, \infty)$ . Thus, setting  $C_1 := \sup_{s \in (0, 1)} C_1(s) < \infty$ ,

$$\int_{(0, 1)} \mathbb{E}[\mathbf{1}_{\{Z_s \geq 1\}} Z_s^\kappa] \nu_Y(ds) \leq C_1 C_2 \int_{(0, 1)} s \nu_Y(ds)$$

is finite because  $Y$  is a subordinator, which implies  $\int_{(0, \infty)} (1 \wedge y) \nu_Y(dy) < \infty$ .

For the second integral, note that, by Lemma 3.2, the mapping  $s \mapsto \mathbb{E}[Z_s^\kappa]$  is of polynomial order  $\kappa$  for all  $s \geq 1$ . Thus, it follows that the second summand in (3.8), and hence also the second summand in (3.7), is finite if and only if  $\int_{[1, \infty)} s^\kappa \nu_Y(ds) < \infty$  and  $\int_{[1, \infty)} z^\kappa \nu_Z(dz) < \infty$ . This finishes the proof of the claimed equivalence.

In the case  $\mathbb{E}[Z_1] = \infty$  and  $\kappa \in (0, 1)$ , we cannot apply Lemma 3.2 to find a necessary and sufficient condition for the finiteness of the second summand in (3.8). However, an inspection of the proof of Lemma 3.2 shows that even in this case the mapping  $s \mapsto \mathbb{E}[Z_s^\kappa]$  can be bounded

from below by a function of polynomial order  $\kappa$  for all  $s \geq 1$ . Thus, finiteness of the second summand in (3.8) still implies  $\int_{[1,\infty)} z^\kappa \nu_Z(dz) < \infty$ , and the finiteness of all the summands in (3.7) implies  $\int_{[1,\infty)} s^\kappa \nu_Y(ds) < \infty$  and  $\int_{[1,\infty)} z^\kappa \nu_Z(dz) < \infty$ , as claimed.  $\square$

Let us now concentrate on the case  $\psi'(0+) > 0$  treated in Theorem 3.1(ii), where in the light of (3.4) it remains to be proved that  $\mathbb{E}[(\tau_Y^+)^k] < \infty$  is equivalent to  $\int_{[1,\infty)} y^{\kappa+1} \Pi(dy) < \infty$ . To show this, we need the following useful connection between the existence of integer moments of  $\tau_1^+$  and  $X_1$ .

**Lemma 3.3.** *Assume that  $\psi'(0+) > 0$ . Then, for all  $k \in \mathbb{N}_0$ ,*

$$\lim_{q \downarrow 0} |\Phi^{(k)}(q)| < \infty \quad \text{if and only if} \quad \lim_{q \downarrow 0} |\psi^{(k)}(q)| < \infty. \tag{3.9}$$

*Proof.* We prove the statement by induction. Clearly, for  $k = 0$  there is nothing to show. For  $k = 1$  it follows from the assumption  $\psi'(0+) > 0$  and the fact that  $(X_t)_{t \geq 0}$  is spectrally negative, that  $\psi'(0+) \in (0, \infty)$ . By (2.3) we thus conclude that  $\Phi'(0+) = 1/\psi'(0+) \in (0, \infty)$ , and the equivalence is trivially fulfilled. Further, for  $k = 2$  we compute, via (2.3),

$$\Phi''(q) = \partial_q \left( \frac{1}{\psi'(\Phi(q))} \right) = -\frac{\psi''(\Phi(q))}{\psi'(\Phi(q))^3}, \quad q > 0,$$

such that

$$\Phi''(0+) = -\frac{\psi''(0+)}{\psi'(0+)^3},$$

which proves the claim for  $k = 2$ .

Assume now that (3.9) holds for all  $\ell = 1, \dots, n - 1$ . If there exists  $\ell' \in \{1, \dots, n - 1\}$  such that both sides of (3.9) are infinite, then, for all  $\ell \in \{\ell', \dots, n - 1\}$ , both terms are infinite as well. Therefore we assume that both sides are finite for all  $\ell = 1, \dots, n - 1$ .

By the definition of  $\Phi$  we have  $\psi(\Phi(q)) = q$  for all  $q \geq 0$ , and hence  $\partial_q^n \psi(\Phi(q)) = 0$  for all  $n \geq 2$ . Using Faà di Bruno’s formula (cf. [6, (2.2)]) for  $n \geq 2$ , we therefore conclude that  $0 = \sum_{k=1}^n \psi^{(k)}(\Phi(q)) \cdot B_{n,k}(\Phi'(q), \dots, \Phi^{(n-k+1)}(q))$ , where the functions  $B_{n,k}$  denote the partial Bell polynomials. Thus, we get

$$\begin{aligned} \Phi^{(n)}(q) &= B_{n,1}(\Phi^{(n)}(q)) = \frac{-1}{\psi'(\Phi(q))} \cdot \sum_{k=2}^n \psi^{(k)}(\Phi(q)) \cdot B_{n,k}(\Phi'(q), \dots, \Phi^{(n-k+1)}(q)) \\ &= \frac{-1}{\psi'(\Phi(q))} \cdot \sum_{j=1}^{n-1} \psi^{(n+1-j)}(\Phi(q)) \cdot B_{n,n+1-j}(\Phi'(q), \dots, \Phi^{(j)}(q)), \end{aligned}$$

where the left-hand side is finite if and only if the right-hand side is finite. However, the right-hand side is finite in the limit  $q \downarrow 0$  if and only if

$$\begin{aligned} \lim_{q \downarrow 0} \left| \frac{1}{\psi'(\Phi(q))} \cdot \psi^{(n)}(\Phi(q)) \cdot B_{n,n}(\Phi'(q)) \right| &= \lim_{q \downarrow 0} \left| \frac{1}{\psi'(\Phi(q))} \cdot \psi^{(n)}(\Phi(q)) \cdot \Phi'(q)^n \right| \\ &= \left| \frac{\psi^{(n)}(0+)}{\psi'(0+)^{n+1}} \right| < \infty, \end{aligned}$$

since all the other summands are finite in the limit  $q \downarrow 0$  by assumption.  $\square$

We are now in a position to present the proof of Theorem 3.1(ii).

*Proof of Theorem 3.1(ii).* Assume  $\psi'(0+) > 0$ . By Lemma 3.1, Proposition 3.1, and (2.7), we see immediately that, for all  $\kappa > 0$ ,  $\mathbb{E}[(\tau_x^-)^\kappa \mid \tau_x^- < \infty] < \infty$  if and only if  $\mathbb{E}[(\tau_{Y_1}^+)^\kappa] < \infty$ . Further applying Proposition 3.2, it follows that

$$\mathbb{E}[(\tau_{Y_1}^+)^\kappa] < \infty \quad \text{if and only if} \quad \mathbb{E}[(\tau_1^+)^\kappa] < \infty \text{ and } \mathbb{E}[Y_1^\kappa] < \infty, \tag{3.10}$$

since  $\mathbb{E}[\tau_1^+] = \Phi'(0+) = 1/\psi'(0+) < \infty$  as noted in the proof of Lemma 3.3. Furthermore,  $\mathbb{E}[Y_1^\kappa] < \infty$  is equivalent to finiteness of

$$\int_{[1,\infty)} y^\kappa \Pi((y, \infty)) dy = \frac{1}{\kappa + 1} \int_{[1,\infty)} y^{\kappa+1} \Pi(dx) - \frac{1}{\kappa + 1} \Pi((1, \infty)) \tag{3.11}$$

by partial integration. Thus,  $\mathbb{E}[Y_1^\kappa] < \infty$  if and only if  $\mathbb{E}[|X_1|^{\kappa+1}] < \infty$ , and  $\mathbb{E}[|X_1|^{\kappa+1}] < \infty$  is shown to be a necessary condition for  $\mathbb{E}[(\tau_x^-)^\kappa \mid \tau_x^- < \infty] < \infty$ . However,  $\mathbb{E}[|X_1|^{\kappa+1}] < \infty$  is a sufficient condition as well, since it implies  $\mathbb{E}[|X_1|^k] < \infty$  for  $k = \lfloor \kappa + 1 \rfloor \geq 1$ . This in turn implies  $\mathbb{E}[(\tau_1^+)^\kappa] < \infty$  by (2.2) and Lemma 3.3, which then yields  $\mathbb{E}[(\tau_1^+)^\kappa] < \infty$ , since  $\kappa < k$ . Thus, both conditions on the right-hand side of (3.10) hold if and only if  $\mathbb{E}[|X_1|^{\kappa+1}] < \infty$ , which finishes the proof.  $\square$

Finally, we consider the oscillating case of  $\psi'(0+) = 0$ . Again, in the light of (3.4) we need to investigate the existence of  $\mathbb{E}[(\tau_{Y_1}^+)^\kappa]$ , where this time we restrict ourselves to finding conditions for  $\mathbb{E}[(\tau_{Y_1}^+)^\kappa] = \infty$ .

*Proof of Theorem 3.1(iii).* Assume that  $\psi'(0+) = 0$ . Consider case (a),  $\kappa \in (0, 1]$ . From (3.4) we have  $\mathbb{E}[(\tau_x^-)^\kappa] = \infty$  if and only if  $\mathbb{E}[(\tau_{Y_1}^+)^\kappa] = \infty$ , and, by Proposition 3.2, for  $\kappa \in (0, 1)$  the latter follows in particular if  $\mathbb{E}[Y_1^\kappa] = \infty$ . This, however, is equivalent to  $\int_{[1,\infty)} y^\kappa \Pi((y, \infty)) dy = \infty$  and, via (3.11), it is furthermore equivalent to  $\int_{[1,\infty)} y^{\kappa+1} \Pi(dy) = \infty$ .

Consider now the case  $\kappa = 1$ , i.e.  $\mathbb{E}[Y_1] = \infty$ . From (2.3) it follows that  $\Phi'(0+) = \mathbb{E}[\tau_1^+] = \infty$ , and an inspection of the proof of Proposition 3.2 reveals that in this setting also  $\mathbb{E}[\tau_{Y_1}^+] = \infty$ . This again implies the statement.

For case (b), we consider a fixed  $\kappa \in (\frac{1}{2}, 1)$  and prove (3.1) for the chosen  $\kappa$ . This will immediately also imply the statement for any  $\kappa \geq 1$ .

As before, from (3.4) we have  $\mathbb{E}[(\tau_x^-)^\kappa] = \infty$  if and only if  $\mathbb{E}[(\tau_{Y_1}^+)^\kappa] = \infty$ , where, by Proposition 3.2, the latter follows if  $\mathbb{E}[(\tau_1^+)^\kappa] = \infty$ . Here, by (3.2), (2.7), and (2.6),

$$\begin{aligned} \mathbb{E}[(\tau_1^+)^\kappa] &= [\mathbf{D}_q^\kappa e^{-\Phi(q)}]_{q=0} = \left[ \frac{\kappa}{\Gamma(1-\kappa)} \int_q^\infty \frac{e^{-\Phi(q)} - e^{-\Phi(u)}}{u^{\kappa+1}} du \right]_{q=0} \\ &= \frac{\kappa}{\Gamma(1-\kappa)} \int_0^\infty \frac{1 - e^{-\Phi(u)}}{u^{\kappa+1}} du, \end{aligned} \tag{3.12}$$

where the left-hand side is finite if and only if the right-hand side is finite.

As  $\Phi$  is monotonically increasing with  $\Phi(0) = 0$  and  $\Phi(u) \xrightarrow{u \rightarrow \infty} \infty$ , we clearly have, for all  $\varepsilon > 0$ ,

$$\int_\varepsilon^\infty \frac{1 - e^{-\Phi(u)}}{u^{\kappa+1}} du \leq \int_\varepsilon^\infty \frac{1}{u^{\kappa+1}} du < \infty.$$

Thus, by (3.12), we have

$$\mathbb{E}[(\tau_1^+)^{\kappa}] < \infty \quad \text{if and only if} \quad \int_0^\varepsilon \frac{1 - e^{-\Phi(u)}}{u^{\kappa+1}} < \infty \text{ for some } \varepsilon > 0. \tag{3.13}$$

By Taylor’s expansion, the term  $1 - e^{-\Phi(u)}$  is of the same order as  $u\Phi'(u)e^{-\Phi(u)}$  as  $u \downarrow 0$ . Moreover, by (2.3),

$$\lim_{u \downarrow 0} \frac{u\Phi'(u)}{u^\kappa} = \lim_{u \downarrow 0} \frac{\Phi'(u)}{u^{\kappa-1}} = \lim_{u \downarrow 0} \frac{u^{1-\kappa}}{\psi'(\Phi(u))}.$$

Recall that  $\kappa \in (\frac{1}{2}, 1)$  and  $\psi''(0+) < \infty$ . By a twofold application of l’Hospital’s rule, we get

$$\begin{aligned} \lim_{u \downarrow 0} \frac{u^{1-\kappa}}{\psi'(\Phi(u))} &= \lim_{u \downarrow 0} \frac{(1-\kappa) \cdot u^{-\kappa}}{\psi''(\Phi(u)) \cdot \Phi'(u)} = \frac{(1-\kappa)}{\psi''(0+)} \cdot \lim_{u \downarrow 0} \frac{\psi'(\Phi(u))}{u^\kappa} \\ &= \frac{(1-\kappa)}{\psi''(0+)} \cdot \lim_{u \downarrow 0} \frac{\psi''(\Phi(u)) \cdot \Phi'(u)}{\kappa \cdot u^{\kappa-1}} = \frac{(1-\kappa)}{\kappa} \cdot \lim_{u \downarrow 0} \frac{u^{1-\kappa}}{\psi'(\Phi(u))}. \end{aligned}$$

As  $\kappa \neq \frac{1}{2}$  this can only be true if

$$\lim_{u \downarrow 0} \frac{u^{1-\kappa}}{\psi'(\Phi(u))} = \lim_{u \downarrow 0} \frac{\psi'(\Phi(u))}{u^\kappa} = \text{either } 0 \text{ or } \infty,$$

which in turn implies

$$\lim_{u \downarrow 0} \frac{u\Phi'(u)}{u^\kappa} = \lim_{u \downarrow 0} \frac{u^{1-\kappa}}{\psi'(\Phi(u))} \cdot \frac{\psi'(\Phi(u))}{u^\kappa} = \lim_{u \downarrow 0} u^{1-2\kappa} = \infty.$$

Thus also

$$\lim_{u \downarrow 0} \frac{1 - e^{-\Phi(u)}}{u^\kappa} = \lim_{u \downarrow 0} \frac{u\Phi'(u)e^{-\Phi(u)}}{u^\kappa} = \infty,$$

and in particular for any  $C > 0$  there exists  $u_0 > 0$  such that  $\frac{1 - e^{-\Phi(u)}}{u^\kappa} > C$  for all  $u < u_0$ . Hence,

$$\int_0^\varepsilon \frac{1 - e^{-\Phi(u)}}{u^{\kappa+1}} du \geq \int_0^{u_0 \wedge \varepsilon} \frac{1 - e^{-\Phi(u)}}{u^{\kappa+1}} du \geq \int_0^{u_0 \wedge \varepsilon} \frac{C \cdot u^\kappa}{u^{\kappa+1}} du = C \cdot \int_0^{u_0 \wedge \varepsilon} \frac{1}{u} du = \infty.$$

By (3.13) this implies  $\mathbb{E}[(\tau_1^+)^{\kappa}] = \infty$ , and thus the statement.

Lastly, note that (3.1) for all  $x > 0$ ,  $\kappa \geq 1$  is a direct consequence of (a) and (b), as either  $\psi''(0+) < \infty$ , in which case we can apply (b), or  $\psi''(0+) = \infty$ , which is equivalent to  $\int_{[1, \infty)} y^2 \Pi(dy) = \infty$  and hence  $\kappa^* = 1$  is a possible choice in (a). □

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### References

- [1] AURZADA, F., IKSANOV, A. AND MEINERS, M. (2015). Exponential moments of first passage times and related quantities for Lévy processes. *Math. Nachr.* **288**, 1921–1938.
- [2] AVRAM, F., GRAHOVAC, D. AND VARDAR-ACAR, C. (2020). The W, Z scale functions kit for first passage problems of spectrally negative Lévy processes, and applications to control problems. *ESAIM Prob. Statist.* **24**, 454–525.
- [3] DELBAEN, F. (1990). A remark on the moments of ruin time in classical risk theory. *Insurance Math. Econom.* **9**, 121–126.
- [4] DONEY, R. A. (1991). Hitting probabilities for spectrally positive Lévy processes. *J. London Math. Soc.* **44**, 566–576.
- [5] DONEY, R. A. (2007). *Fluctuation Theory for Lévy processes* (Lect. Notes Math. **1897**). Springer, New York.
- [6] JOHNSON, W. P. (2002). The curious history of Faà di Bruno’s formula. *Amer. Math. Monthly* **109**, 217–234.
- [7] KUZNETSOV, A., KYPRIANOU, A. E., PARDO, J. C. AND WATSON, A. R. (2014). The hitting time of zero for a stable process. *Electron. J. Prob.* **19**, 1–26.
- [8] KUZNETSOV, A., KYPRIANOU, A. E. AND RIVERO, V. (2013). The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters II*, eds S. Cohen, A. Kuznetsov, A. E. Kyprianou and V. Rivero (Lect. Notes Math. **2061**). Springer, New York, pp. 97–186.
- [9] KYPRIANOU, A. E. (2014). *Fluctuations of Lévy processes with Applications*, 2nd edn. Springer, New York.
- [10] KYPRIANOU, A. AND RIVERO, V. (2008). Special, conjugate and complete scale functions for spectrally negative Lévy processes. *Electron. J. Prob.* **13**, 1672–1701.
- [11] PESKIR, G. (2008). The law of the hitting times to points by a stable Lévy process with no negative jumps. *Electron. Commun. Prob.* **13**, 653–659.
- [12] ROGERS, L. C. G. AND WILLIAMS, D. (2000). *Diffusions, Markov Processes and Martingales*, Vol 1, 2nd edn. Cambridge University Press.
- [13] RUDIN, W. (1987). *Real and Complex Analysis*, 3rd edn. McGraw-Hill, New York.
- [14] SAMKO, S. G., KILBAS, A. A. AND MARICHEV, O. I. (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Newark, NJ.
- [15] SATO, K. (2013). *Lévy Processes and Infinitely Divisible Distributions*, 2nd edn. Cambridge University Press.
- [16] SCHILLING, R. L., SONG, R. AND VONDRACEK, Z. (2012). *Bernstein Functions: Theory and Applications*, 2nd edn. De Gruyter, Berlin.
- [17] SIMON, T. (2011). Hitting densities for spectrally positive stable processes. *Stochastics* **83**, 203–214.
- [18] UCHAIKIN, V. V. (2013). *Fractional Derivatives for Physicists and Engineers*, Vol. I. Springer, New York.
- [19] WOLFE, S. J. (1975). On moments of probability distribution functions. In *Fractional Calculus and its Applications*, ed B. Ross (Lect. Notes Math. **457**). Springer, New York, pp. 306–316.