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Multisymplectic Reduction for Proper Actions

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Abstract. We consider symmetries of the Dedonder equation arising from variational problems with partial derivatives. Assuming a proper action of the symmetry group, we identify a set of reduced equations on an open dense subset of the domain of definition of the fields under consideration. By continuity, the Dedonder equation is satisfied whenever the reduced equations are satisfied.

1 Introduction

Multisymplectic formulation of a variational problem is implicit in the works of Th. Dedonder [9], and Th. H. J. Lepage [17]. A comprehensive review of early works can be found in [12]. It was rediscovered in the late sixties in the context of relativistic field theories, mainly because it provides a covariant transition from the Lagrangian formulation to the infinite-dimensional Hamiltonian formalism. A comprehensive list of references can be found in [1], [13], [14], see also [16]. The current revival is related to the discovery of multisymplectic integrators, see [19], [20], [2], and references quoted there.

There are four levels of approach to second-order partial differential equations arising from variational problems:

- *variational approach* in which points are evolutions of the fields under consideration,
- *infinite-dimensional Hamiltonian approach* on the space of Cauchy data,
- *multisymplectic formulation* on the first jet bundle of the fields, and
- *Euler-Lagrange equations*.

The Euler-Lagrange equations of a variational problem are second-order differential equations. On the other hand, in the multisymplectic approach we are dealing with a system of equations in exterior differential forms on the first jet bundle, in which we can forget about the target map associating to a 1-jet the value of the fields at the same source point. For this reason, a study of symmetries of the theory which do not preserve the target map is easier in the multisymplectic approach than on the level of the Euler-Lagrange equations.

Reduction of symmetries of various types of dynamical systems has been studied quite extensively, see [5] and hundreds of references quoted there. In most cases

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one assumes that the symmetry group acts properly on the space on which one studies equations of motion. Hence, each of the approaches listed above can handle a different class of symmetry groups. For example, time dependent gauge transformations can be studied in the framework of the variational approach [18]. The space of Cauchy data approach enables one to discuss reduction of the group of time independent gauge transformations [21], [22]. Since the group of gauge transformations does not act properly on the first jet bundle, neither the multisymplectic formulation nor the Euler-Lagrange equations approach are suitable for the discussion of reduction of gauge symmetries, at least in terms of the commonly used techniques which require properness of the action. Finite-dimensional symplectic reduction of the zero level of the momentum map for an improper group action has been successfully treated in [23]. However, it requires techniques which we have not been able to extend to infinite-dimensional Hamiltonian systems.

If the fields under consideration are local sections $\sigma\colon M \to Q$ of a fibre bundle $\mu\colon Q \to M$, then the Lagrangian of the theory gives rise to an exact form Ω on the first jet bundle P of Q such that σ satisfies the Lagrange-Euler equations if and only if its jet extension $j^1\sigma$ satisfies the equation

$$(1) \qquad (j^1\sigma)^*(X \,\lrcorner\, \Omega) = 0$$

for all vector fields X tangent to the fibres of the source map $\alpha\colon P \to M$ [9].

Let $\beta\colon P \to Q$ denote the target map. A multisymplectic theory is said to be *regular* if, for every local section $\rho\colon M \to P$ of α such that

$$(2) \qquad \rho^*(X \,\lrcorner\, \Omega) = 0$$

for all vector fields X tangent to the fibres of α, the section $\sigma = \beta \circ \rho$ of μ satisfies the Lagrange-Euler equations. We refer to equation (2) as the Dedonder equation. Regular multisymplectic theories include the relativistic theory of scalar field, Yang-Mills theory, and general relativity[1] [1].

The aim of this paper is to discuss reduction of symmetries of multisymplectic theories under the assumption that the action of a finite-dimensional symmetry group G on P is proper, and G-orbits are contained in the fibres of the source map $\alpha\colon P \to M$. The reduction is performed in terms of a G-invariant Riemannian metric k on P, which is used to define directions normal to G-orbits.

In the multisymplectic approach conservation laws are expressed by the vanishing of the divergence of appropriate currents. Constants of motion are given by the integrals of conserved currents over Cauchy surfaces, and are defined only in the infinite-dimensional formulation in the space of Cauchy data. Moreover, the Dedonder equation (2) admits neither existence nor uniqueness theorem. In order to obtain existence and uniqueness theorems one also has to proceed to the Cauchy data space formulation. Hence, the presented multisymplectic reduction of symmetries does not lead to the same degree of simplification as in the case of Hamiltonian systems.

[1]The Hilbert Lagrangian of general relativity depends linearly on the second jets of the metric tensor. This requires a modification of the definition of the form Ω.

For each $p \in P$, there is a submanifold N of P consisting of points of the same orbit type (see the discussion following equation (12)). The G-invariant metric k on P enables us to decompose the tangent bundle to P along N into TN and its k-orthogonal complement $T^{\perp}N$. Moreover, TN can be split into the vertical component, ver TN, tangent to G-orbits in N and its k-orthogonal complement, hor TN, (see equation (15)). This gives rise to a splitting of the Dedonder equation into three equations. We identify the structure of each of these equations in terms of the geometry of the orbit space. The first reduced equation is of the form of an inhomogeneous Dedonder equation with the right hand side given by the interaction of the degrees of freedom in hor TN and ver TN. The second reduced equation is an invariant form of the conservation laws. The third reduced equation involves the structure of links of the stratification of the orbit space. In the case when the action of the symmetry group is free, the third reduced equation is vacuously satisfied. We illustrate the general theory with a simple example of a complex relativistic scalar field with a $U(1)$-invariant potential.

2 Relationship to Variational Problems

We consider here the multisymplectic dynamics of fields represented by local sections $\sigma\colon M \to Q$ of a fibre bundle $\mu\colon Q \to M$. We denote by $P = J^1(Q)$ the bundle of 1-jets of sections of μ, and by $\alpha\colon P \to M$ and $\beta\colon P \to Q$ the source and the target map, respectively. For each $x \in M$, elements of $\alpha^{-1}(x) \subset P$ are equivalence classes $j^1_x\sigma$ of local sections σ of μ, under the equivalence relation $\sigma \sim \sigma'$ if and only if the derived maps $T\sigma\colon TM \to TQ$ and $T\sigma'\colon TM \to TQ$ coincide at x, that is if $T_x\sigma = T_x\sigma'$. For each section σ of π with domain $U \subseteq M$, we denote by $j^1\sigma\colon U \to P$ the 1-jet extension of σ given by $j^1\sigma(x) = j^1_x\sigma$ for every $x \in U$.

We denote by $L\colon P \to \mathbb{R}$ the Lagrangian of the theory and by ϑ a volume form on M. There are several forms Ω on P such that a local section σ of π satisfies the Euler-Lagrange equations for L, if and only if its first jet extension $j^1\sigma$ satisfies equation (1) [17]. Here we take

$$\Omega = d\Theta, \tag{3}$$

where Θ is a form introduced by Dedonder [9], which satisfies the condition

$$u_1 \mathbin{\lrcorner} (u_2 \mathbin{\lrcorner} \Theta) = 0 \quad \forall u_1, u_2 \in \ker T\alpha. \tag{4}$$

We refer to Θ as the Dedonder form[2] for L, and to Ω as the corresponding multisymplectic form.

In terms of local coordinates (x^μ, q^A, p^A_μ) on P, a section $\sigma\colon M \to Q$ of π is given by $q^A = q^A(x)$ and its first jet extension $j^1\sigma$ by $q^A = q^A(x)$ and $p^A_\mu = q^A_{,\mu}(x)$, where $q^A_{,\mu}(x)$ denotes the derivative of $q^A(x)$ with respect to x^μ. We assume that the coordinates are such that locally $\vartheta = dx^1 \wedge \cdots \wedge dx^n \equiv d_nx$, and use the notation

[2] This form appears in the literature also as a Poincaré-Cartan form or a Hamilton-Cartan form. However, in [3], E. Cartan attributed it to Th. Dedonder.

$dx_\mu = \frac{\partial}{\partial x^\mu} \mathbin{\lrcorner} \vartheta$. If $L = L(x^\mu, q^A, p^A_\mu)$ is the coordinate expression for the Lagrangian, then

$$\Theta = \frac{\partial L}{\partial p^A_\mu} dq^A \wedge dx_\mu + \Big(L - p^A_\mu \frac{\partial L}{\partial p^A_\mu}\Big) d_n x. \tag{5}$$

Let $\rho\colon M \to P$ be a local section of the source map $\alpha\colon P \to M$. In local coordinates (x^μ, q^A, p^A_μ) it is given by $q^A = q^A(x)$ and $p^A_\mu = p^A_\mu(x)$. The Dedonder equations (2) for ρ are

$$\frac{\partial^2 L}{\partial p^B_\nu \partial p^A_\mu} q^A_{,\mu}(x) = \frac{\partial H}{\partial p^B_\nu},$$

$$\Big(\frac{\partial^2 L}{\partial q^B \partial p^A_\mu} - \frac{\partial^2 L}{\partial q^A \partial p^B_\mu}\Big) q^A_{,\mu}(x) = \frac{\partial H}{\partial q^B},$$

where $H = p^A_\mu \frac{\partial L}{\partial p^A_\mu} - L$.

3 Symmetries of an Abstract Dedonder Equation

We can abstract from most of the structure used in the derivation given in the preceding section. We consider a locally trivial fibration $\alpha\colon P \to M$, with $\dim M = n$, and an $(n+1)$-form Ω on P such that

$$d\Omega = 0, \tag{6}$$

and

$$u_1 \mathbin{\lrcorner} \big(u_2 \mathbin{\lrcorner} (u_3 \mathbin{\lrcorner} \Omega)\big) = 0 \quad \forall u_1, u_2, u_3 \in \ker T\alpha. \tag{7}$$

We refer to Ω as the multisymplectic form of the theory, and consider local sections ρ of $\alpha\colon P \to M$ satisfying the Dedonder equation

$$\rho^*(X \mathbin{\lrcorner} \Omega) = 0 \tag{8}$$

for all vector fields X on P tangent to the fibres of $\alpha\colon P \to M$.

Let

$$\Phi\colon G \times P \to P\colon (g, p) \mapsto \Phi_g(p) \equiv g \cdot p \tag{9}$$

be a proper action of a Lie group G on P which preserves Ω and induces the identity transformation on M. In other words,

$$\Phi_g^* \Omega = \Omega, \quad \text{and} \quad \alpha \circ \Phi_g = \alpha \tag{10}$$

for all $g \in G$. We denote by $\bar{P} = P/G$ the space of G orbits on P, and by $\pi\colon P \to \bar{P}$ the orbit map. By hypothesis, for each $p \in P$, the fibre of π through p is contained in the fibre of $\alpha\colon P \to M$ through p, and there exists a smooth map $\alpha_{\bar{P}}\colon \bar{P} \to M$ such that $\alpha = \alpha_{\bar{P}} \circ \pi$.

4 Orbit Type

The orbit space $\bar{P} = P/G$ of a proper action is a stratified space, [10], and it has a decomposition as a union of manifolds which are called strata. We shall use this decomposition to determine the reduced equations and the invariant form of conservation laws in each stratum.

For each $p \in P$, the isotropy group of p is

$$G_p = \{g \in G \mid gp = p\}. \tag{11}$$

Since the action of G on P is proper, all isotropy groups are compact. For each compact subgroup K of G, we consider the set

$$P_{(K)} = \{p \in P \mid G_p \text{ is conjugate to } K\}, \tag{12}$$

consisting of points of orbit type K. Each $P_{(K)}$ is a local submanifold of P. Moreover, the projection $\pi(P_{(K)}) \subseteq \bar{P}$ is locally a manifold [15], [6].

Let $\bar{N}$ be a connected component of $\pi(P_{(K)})$ and $N = \pi^{-1}(\bar{N})$. Since $\pi(P_{(K)})$ is locally a manifold, it follows that $\bar{N}$ is a manifold, and N is a submanifold of P. Let $\pi_N \colon N \to \bar{N}$ be the projection map defined by the restriction of π to N. Similarly, we denote by $\alpha_N \colon N \to M$ the restriction of α to N and by $\alpha_{\bar{N}} \colon \bar{N} \to M$ the unique map such that $\alpha_N = \alpha_{\bar{N}} \circ \pi_N$. The action of G on P preserves N, and it induces an action of G on N such that $\bar{N}$ is the space of G-orbits on N, and π_N is the orbit map.

Using the pull-back to N of the G-invariant metric k on P, we obtain a decomposition $TN = \text{ver } TN \oplus \text{hor } TN$. The action of G on N preserves ver TN and hor TN separately. The space of G-orbits in hor TN can be identified with $T\bar{N}$, and the orbit map with $T\pi_N$ restricted to hor TN. Let

$$V_{\bar{N}} = (\text{ver } TN)/G \tag{13}$$

denote the space of G-orbits in ver TN, and $\gamma_N \colon \text{ ver } TN \to V_{\bar{N}}$ the orbit map. The space $V_{\bar{N}}$ is a vector bundle over $\bar{N}$ with projection map $\vartheta_{\bar{N}} \colon V_{\bar{N}} \to \bar{N}$ such that $\vartheta_{\bar{N}} \circ \gamma_N = \pi_N \circ (\tau_{TN} \mid \text{ver } TN)$. Let $T^{\perp}N$ be the k-orthogonal complement of TN in the restriction $T_N P$ of TP to N. In other words, for each $p \in N$,

$$T_p^{\perp} N = \{v \in T_p P \mid \mathrm{k}(v, w) = 0 \; \forall w \in T_p N\}. \tag{14}$$

We have a direct sum decomposition

$$T_N P = T^{\perp} N \oplus \text{ver } TN \oplus \text{hor } TN. \tag{15}$$

Suppose that ρ is a local section of $\alpha \colon P \to M$ with domain U such that $\rho(U) \subseteq N$. We denote by $\rho_N \colon U \to N$ the local section of α_N defined by ρ. It is uniquely defined by the condition $\rho = \rho_N \circ \iota_N$, where $\iota_N \colon N \to P$ is the inclusion map. We denote by $\rho_{\bar{N}} \colon U \to \bar{N}$ the projection of ρ_N to a section of $\alpha_{\bar{N}} \colon \bar{N} \to M$. In other words,

$$\rho_{\bar{N}} = \pi_N \circ \rho_N. \tag{16}$$

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We can split $T\rho_N$ into its horizontal component, hor $T\rho_N$, with values in hor TN and the vertical component, ver $T\rho_N$, with values in ver TN, and write

$$T\rho_N = \text{hor } T\rho_N \oplus \text{ver } T\rho_N. \tag{17}$$

Clearly, $T\pi_N \circ \text{hor } T\rho_N = T\rho_{\bar{N}}$, so that all the information provided by $T\pi_N \circ$ hor $T\rho_N$ is already encoded in $\rho_{\bar{N}}$. Additional information is carried by

$$\rho_N^\sharp = \gamma_N \circ \text{ver } T\rho_N \colon TU \to V_{\bar{N}}. \tag{18}$$

We shall study the Dedonder equation (2) for ρ in terms of the decompositions (15) and (17).

The assumption that U is open, non-empty and $\rho(U) \subseteq N$ implies that $T_p\alpha$ maps $T_{\rho(U)}N$ onto $T_U M$ and $T\rho(TU) \subseteq T_{\rho(U)}N$, where $T_{\rho(U)}N$ denotes the restriction of TN to $\rho(U)$. Since $T_{\rho(U)}P = \ker_{\rho(U)} T\alpha \oplus T\rho(T_U M) = T_{\rho(U)}N \oplus T^{\perp}_{\rho(U)}N$ it follows that

$$\ker_{\rho(U)} T\alpha = \ker_{\rho(U)} T\alpha_N \oplus (T^{\perp}_{\rho(U)}N \cap \ker_{\rho(U)} T\alpha). \tag{19}$$

Consider the vector field X in the Dedonder equation (2). It appears in the equations only through its values at points of $\rho(U)$, and it has values in ker $T\alpha$. We can decompose the restriction X_N of X to points of N into its components corresponding to the decomposition (15) obtaining

$$X_N = \text{hor } X_N + \text{ver } X_N + X_N^{\perp}.$$

This gives rise to the decomposition of the Dedonder equation into three sets of equations

$$\rho_N^*(\text{hor } X_N \mathbin{\lrcorner} \Omega) = 0, \tag{20}$$

$$\rho_N^*(\text{ver } X_N \mathbin{\lrcorner} \Omega) = 0, \tag{21}$$

$$\rho_N^*(X_N^{\perp} \mathbin{\lrcorner} \Omega) = 0, \tag{22}$$

for all vector fields X_N with values in ker $T\alpha$.

5 The First Reduced Equation

For every $x \in U$, evaluating the left- hand side of equation (20) on $u_1, \dots, u_n \in T_x M$, and taking into account equation (7), we obtain

$$\begin{aligned}
&\rho_N^*(\text{hor } X_N(p) \mathbin{\lrcorner} \Omega)(u_1, \dots, u_n) \\
&\quad = \sum_{k=1}^{n} (-1)^k \big[\text{ver } T\rho_N(u_k) \mathbin{\lrcorner} (\text{hor } X_N(p) \mathbin{\lrcorner} \Omega)\big] \big(\text{hor } T\rho_N(u_1), \dots, \text{hor } T\rho_N(u_n)\big) \\
&\qquad + (\text{hor } X_N(p) \mathbin{\lrcorner} \Omega)\big(\text{hor } T\rho_N(u_1), \dots, \text{hor } T\rho_N(u_n)\big)
\end{aligned}$$

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where $p = \rho_N(x)$. Hence, equation (20) at $p = \rho(x)$ is equivalent to

(23)
$$\sum_{k=1}^{n}(-1)^k\big[\mathrm{ver}\, T\rho_N(u_k) \lrcorner \big(\mathrm{hor}\, X_N(p) \lrcorner \Omega\big)\big]\big(\mathrm{hor}\, T\rho_N(u_1), \dots, \mathrm{hor}\, T\rho_N(u_n)\big) + \big(\mathrm{hor}\, X_N(p) \lrcorner \Omega\big)\big(\mathrm{hor}\, T\rho_N(u_1), \dots, \mathrm{hor}\, T\rho_N(u_n)\big) = 0$$

for all vectors $\mathrm{hor}\, X_N(p) \in \ker T\alpha_N \cap \mathrm{hor}\, T_pN$. Let $\mathrm{hor}\, \Omega_N$ denote the horizontal part of the pull-back of Ω to N. In other words,

$$\mathrm{hor}\, \Omega_N(w_1, \dots, w_{n+1}) = \Omega(\mathrm{hor}\, w_1, \dots, \mathrm{hor}\, w_{n+1})$$

for every $w_1, \dots, w_{n+1} \in T_pN$, and $p \in N$. It is a G-invariant $(n+1)$-form on N, annihilated by vectors in $\mathrm{ver}\, TN = \ker T\pi_N$. Hence, it pushes forward to a unique $(n+1)$-form $\Omega_{\bar{N}}$ on $\bar{N}$ such that

$$\mathrm{hor}\, \Omega_N = \pi_N^* \Omega_{\bar{N}}. \tag{24}$$

Moreover,

$$\begin{aligned}\big(\mathrm{hor}\, X_N(p) \lrcorner \Omega\big)&\big(\mathrm{hor}\, T\rho(u_1), \dots, \mathrm{hor}\, T\rho(u_n)\big) \\ &= \big(T\pi_N\big(\mathrm{hor}\, X_N(p)\big) \lrcorner \Omega_{\bar{N}}\big)\big(T\rho_{\bar{N}}(u_1), \dots, T\rho_{\bar{N}}(u_n)\big).\end{aligned} \tag{25}$$

In order to interpret the second term on the left hand side of equation (23) we introduce an n-form Ξ_N on N with values in $(\mathrm{ver}\, TN)^*$ defined as follows. For $p \in N$, $w_1, \dots, w_n \in T_pN$, and $v \in \mathrm{ver}\, T_pP$,

$$\Xi_N(w_1, \dots, w_n)(v) = \Omega(v, \mathrm{hor}\, w_1, \dots, \mathrm{hor}\, w_n). \tag{26}$$

Since Ω is G-invariant, it follows that

$$\Xi\big(T\Phi_g(w_1), \dots, T\Phi_g(w_n)\big)\big(T\Phi_g(v)\big) = \Xi(w_1, \dots, w_n)(v)$$

for every $g \in G$. Hence, Ξ_N is G-equivariant. Moreover, as an n-form on N, Ξ_N is annihilated by vectors in $\mathrm{ver}\, TN$. Therefore, Ξ_N induces an n-form $\Xi_{\bar{N}}$ on the orbit space $\bar{N} = \pi(N)$ with values in $V_{\bar{N}}^*$ such that

$$\Xi_N(w_1, \dots, w_n)(v) = \Xi_{\bar{N}}\big(T\pi_N(w_1), \dots, T\pi_N(w_n)\big)\big(\gamma_N(v)\big). \tag{27}$$

We can now rewrite the first term in equation (23) in the form

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k&\big[\mathrm{ver}\, T\rho_N(u_k) \lrcorner (\mathrm{hor}\, X_N(p) \lrcorner \Omega)\big]\big(\mathrm{hor}\, T\rho_N(u_1), \dots, \mathrm{hor}\, T\rho_N(u_n)\big) \\ &= -\sum_{k=1}^{n}(-1)^k\big[\rho_{\bar{N}}^*((T\pi(\mathrm{hor}\, X_N(p))) \lrcorner \Xi_{\bar{N}})(u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n)(\rho_N^\sharp(u_k))\big].\end{aligned}$$

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In order to simplify the form of the result, for every vector field $X_{\bar{N}}$ on $\bar{N}$, we introduce the notation

$$(28)\quad \left(\rho_N^{\sharp} \mathbin{\dot{\wedge}} \rho_{\bar{N}}^*(X_{\bar{N}} \lrcorner\, \Xi_{\bar{N}})\right)(u_1, \ldots, u_n) = -\sum_{k=1}^{n}(-1)^k\left[\rho_{\bar{N}}^*(X_{\bar{N}} \lrcorner\, \Xi_{\bar{N}})(u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n)\left(\rho_N^{\sharp}(u_k)\right)\right].$$

Taking into account equations (23) and (25), we can rewrite equation (20) in the form

$$(29)\qquad \rho_{\bar{N}}^*(X_{\bar{N}} \lrcorner\, \Omega_{\bar{N}}) + \rho_N^{\sharp} \mathbin{\dot{\wedge}} \rho_{\bar{N}}^*(X_{\bar{N}} \lrcorner\, \Xi_{\bar{N}}) = 0$$

for all vector fields $X_{\bar{N}}$ on $\bar{N}$ tangent to the fibres of $\alpha_{\bar{N}} \colon \bar{N} \to M$.

6 The Second Reduced Equation

For each ξ in the Lie algebra $\mathfrak{g}$ of G, we denote by X_ξ the fundamental vector field on P generating the action of the 1-parameter subgroup $\exp t\xi$ of G. Since G is a symmetry group, the Lie derivative $\mathcal{L}_{X_\xi}\Omega$ of Ω with respect to X_ξ vanishes. Moreover, $d\Omega = 0$ implies that $X_\xi \lrcorner\, \Omega$ is closed. Suppose that $X_\xi \lrcorner\, \Omega$ is exact. In other words, assume that there exists an n-form Ψ_ξ on P such that

$$X_\xi \lrcorner\, \Omega = d\Psi_\xi.$$

If ρ is a local section of $\alpha \colon P \to M$ satisfying the Dedonder equation (8), then

$$(30)\qquad d\rho^*\Psi_\xi = 0.$$

Equation (30) is a multisymplectic version of the conservation law corresponding to the 1-parameter group of symmetries $\exp t\xi$.

Since the vertical distribution, ver TN, is spanned by the fundamental vector fields X_ξ, for $\xi \in \mathfrak{g}$, the equation (21) is equivalent to the totality of conservation laws. However, equation (30) requires an additional assumption that $X_\xi \lrcorner\, \Omega$ is exact. Moreover, the fundamental vector fields X_ξ need not be G-invariant. Hence, the conservation laws (30) do not lead directly to an equation in the orbit space. In order to exhibit the invariant content of equation (21), we introduce an $(n-1)$-form Σ_N on N with values in $\wedge^2(\text{ver } TN)^*$ defined as follows. For every $p \in N$, $v_1, v_2 \in \text{ver } T_pN$ and $w_1, \ldots, w_{n-1} \in T_pN$,

$$(31)\qquad \Sigma_N(w_1, \ldots, w_{n-1})(v_1, v_2) = \Omega(v_1, v_2, w_1, \ldots, w_{n-1}).$$

G-invariance of Ω implies that

$$\Sigma_N\left(T\Phi_g(w_1), \ldots, T\Phi_g(w_{n-1})\right)\left(T\Phi_g(v_1), T\Phi_g(v_2)\right) = \Sigma(w_1, \ldots, w_{n-1})(v_1, v_2)$$

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for every $g \in G$. Moreover, the condition (7) implies that, as an $(n-1)$ form on N, Σ_N is annihilated by vectors in ver TN. Hence, it gives rise to a unique $(n-1)$-form $\Sigma_{\bar{N}}$ on $\bar{N}$ with values in $\wedge^2 V_{\bar{N}}^*$ such that,

(32)
$$\Sigma_N(w_1,\dots,w_{n-1})(v_1,v_2) = \Sigma_{\bar{N}}\big(T\pi_N(w_1),\dots,T\pi_N(w_{n-1})\big)\big(\gamma_N(v_1),\gamma_N(v_2)\big).$$

For $x \in U$, let $p = \rho_N(x)$. The left hand side of equation (21) evaluated on $u_1,\dots,u_n \in T_xM$ reads

$$\begin{aligned}
&\rho_N^*\big(\operatorname{ver} X_N(p) \lrcorner \Omega\big)(u_1,\dots,u_n)\\
&\quad = \rho_{\bar{N}}^*\Xi_{\bar{N}}(u_1,\dots,u_n)\big(\gamma_N\big(\operatorname{ver} X_N(p)\big)\big)\\
&\qquad - \sum_{k=1}^{n}(-1)^k\rho_{\bar{N}}^*\Sigma_{\bar{N}}(u_1,\dots,u_{k-1},u_{k+1},\dots,u_n)\big(\gamma_N(\operatorname{ver} X_N(p)),\rho_N^\sharp(u_k)\big).
\end{aligned}$$

Introducing the notation

$$\begin{aligned}
(33)\qquad &(\rho_N^\sharp \mathbin{\dot\wedge} \rho_{\bar{N}}^*\Sigma_{\bar{N}})(u_1,\dots,u_n)(v)\\
&\quad = -\sum_{k=1}^{n}(-1)^k\rho_{\bar{N}}^*\Sigma_{\bar{N}}(u_1,\dots,u_{k-1},u_{k+1},\dots,u_n)\big(v,\rho_N^\sharp(u_k)\big),
\end{aligned}$$

we can rewrite equation (21) in the form

$$\rho_{\bar{N}}^*\Xi_{\bar{N}} + \rho_N^\sharp \mathbin{\dot\wedge} \rho_{\bar{N}}^*\Sigma_{\bar{N}} = 0. \tag{34}$$

Equation (34) is equivalent to the conservation law (30).

7 The Third Reduced Equation

In this subsection we are going to discuss equation (22). The bundle $T^\perp N$ over N is G-invariant. We denote the space of G-orbits in $T^\perp N$ by $T^c\bar{N}$. It is a cone bundle over $\bar{N}$ which encodes the information about links of the stratification structure of $\bar{P}$ [8]. Let $\kappa_N : T^\perp N \to T^c\bar{N}$ denote the orbit map, and $(T^c\bar{N})^*$ the space of homogeneous functions on $T^c\bar{N}$ of degree 1.

We introduce an n-form Γ_N on N with values in $(T^\perp N)^*$ defined as follows. For every $p \in N$, $u \in T_p^\perp N$, and $w_1,\dots,w_n \in T_pN$,

$$\Gamma_N(w_1,\dots,w_n)(u) = \Omega(u,\operatorname{hor} w_1,\dots,\operatorname{hor} w_n). \tag{35}$$

The G-invariance of Ω implies that, for every $g \in G$,

$$\Gamma_N\big(T\Phi_g(w_1),\dots,T\Phi_g(w_n)\big)\big(T\Phi_g(u)\big) = \Gamma_N(w_1,\dots,w_n)(u).$$

Moreover, Γ_N is annihilated by vectors in ver TN. In other words, $\Gamma_N(w_1,\dots,w_n) = 0$ if one of the vectors $w_1,\dots,w_n$ is in ver TN. The form Γ_N induces a unique n-form on $\bar{N}$ with values in $(T^c\bar{N})^*$ such that

$$\Gamma_N(w_1,\dots,w_n)(u) = \Gamma_{\bar{N}}\big(T\pi_N(w_1),\dots,T\pi_N(w_n)\big)\big(\kappa_N(u)\big). \tag{36}$$

Next, we introduce an $(n-1)$-form Δ_N on N with values in the space of linear maps from ver TN to $(T^{\perp}N)^*$ defined as follows. For every $p \in N$, $u \in T^{\perp}_p N$, $v \in \operatorname{ver} T_p N$ and $w_1, \dots, w_{n-1} \in T_p N$,

$$\Delta_N(w_1, \dots, w_{n-1})(v)(u) = \Omega(u, v, w_1, \dots, w_{n-1}). \tag{37}$$

The G-invariance of Ω implies that, for every $g \in G$,

$$\Delta_N\big(T\Phi_g(w_1), \dots, T\Phi_g(w_{n-1})\big)\big(T\Phi_g(v)\big)\big(T\Phi_g(u)\big) = \Delta_N(w_1, \dots, w_{n-1})(v)(u).$$

Moreover, Δ_N is annihilated by vectors in ver TN. Hence, Δ_N induces an $(n-1)$-form $\Delta_{\bar N}$ on $\bar N$ with values in the space of functions from $V_{\bar N}$ to $T^c\bar N$ such that

$$\Delta_N(w_1, \dots, w_{n-1})(v)(u) = \Delta_{\bar N}\big(T\pi_N(w_1), \dots, T\pi_N(w_{n-1})\big)\big(\gamma_N(v)\big)\big(\kappa_N(u)\big). \tag{38}$$

For $x \in U$, $p = \rho(x)$, we evaluate the left hand side of equation (22) on $u_1, \dots, u_n \in T_xM$, taking into account equations (7), (35), (36), (37) and (38), and obtain

$$\begin{aligned}
&\rho_N^*\big(X_N^{\perp}(p) \mathbin{\lrcorner} \Omega\big)(u_1, \dots, u_n)\\
&\quad = \big(X_N^{\perp}(p) \mathbin{\lrcorner} \Omega\big)\big(\operatorname{hor} T\rho_N(u_1), \dots, \operatorname{hor} T\rho_N(u_n)\big)\\
&\qquad + \sum_{k=1}^{n}(-1)^k \rho_N^*\big[\operatorname{ver} T\rho_N(u_k) \mathbin{\lrcorner} \big(X_N^{\perp}(p) \mathbin{\lrcorner} \Omega\big)\big](u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n).
\end{aligned}$$

Hence, using equations (35), (36), (37) and (38), we get

$$\begin{aligned}
&\rho_N^*\big(X_N^{\perp}(p) \mathbin{\lrcorner} \Omega\big)(u_1, \dots, u_n)\\
&\quad = \Gamma_{\bar N}\big(T\rho_{\bar N}(u_1), \dots, T\rho_{\bar N}(u_n)\big)\big(\kappa_N\big(X_N^{\perp}(p)\big)\big)\\
&\qquad + \sum_{k=1}^{n}(-1)^k \rho_{\bar N}^*\Delta_{\bar N}(u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n)\big(\rho_N^{\sharp}(u_k)\big)\big(\kappa_N\big(X_N^{\perp}(p)\big)\big).
\end{aligned}$$

Hence, equation (22) is equivalent to

$$\rho_{\bar N}^*\Gamma_{\bar N} + \rho_N^{\sharp} \mathbin{\dot\wedge} \rho_{\bar N}^*\Delta_{\bar N} = 0, \tag{39}$$

where $\rho_N^{\sharp} \mathbin{\dot\wedge} \rho_{\bar N}^*\Delta_{\bar N}$ is defined as in equation (33).

8 Reconstruction

The reduced equations (29), (34) and (39) are conditions on maps $\rho_{\bar N} \colon U \to \bar N$ and $\rho_N^{\sharp} \colon TU \to V_{\bar N}$ such that $\vartheta_{\bar N} \circ \rho_N^{\sharp} = \rho_{\bar N} \circ \tau_M$, where U is an open subset of M and $\tau_M \colon TM \to M$ is the tangent bundle projection.

Suppose we have $\rho_{\bar N} \colon U \to \bar N$ and $\rho_N^{\sharp} \colon TU \to V_{\bar N}$ satisfying equations (29), (34) and (39). The aim of reconstruction is to find a section $\rho_N \colon U \to N$ satisfying the Dedonder equation (2) such that

$$\rho_{\bar N} = \pi_N \circ \rho_N \quad \text{and} \quad \rho_N^{\sharp} = \gamma_N \circ \operatorname{ver} T\rho_N, \tag{40}$$

see equation (18). It is clear from the derivation of the reduced equations that if ρ_N satisfies equations (40), then it satisfies the Dedonder equation (2). Thus, it suffices to find ρ_N satisfying equations (40).

Equation $\rho_{\bar{N}} = \pi_N \circ \rho_N$ implies that ρ_N is a lift of $\rho_{\bar{N}}$ to a section of $\alpha_N \colon N \to M$. Lifts of sections are determined up to gauge transformations on N. In other words, if $\rho'_N \colon U \to N$ is any lift of $\rho_{\bar{N}}$, then $\rho_N = \phi_N \circ \rho'_N$ for some diffeomorphism $\phi_N \colon N \to N$ which commutes with the action of G on N and satisfies $\pi_N \circ \phi_N = \pi_N$. Thus, we may choose any lift ρ'_N of ρ_N, and need to determine the diffeomorphism ϕ_N so that

$$\rho^{\sharp}_N = \gamma_N \circ \operatorname{ver} T(\phi_N \circ \rho'_N). \tag{41}$$

Our results can be summarized in the following:

Theorem *A local section ρ_N of $\alpha_N \colon N \to M$ satisfies the Dedonder equation* (2) *if and only if $\rho_N = \phi_N \circ \rho'_N$ where $\rho_{\bar{N}} = \pi_N \circ \rho'_N$ and $\rho^{\sharp}_N = \gamma_N \circ \operatorname{ver} T(\phi_N \circ \rho'_N)$ satisfy the reduced equations* (29), (34) *and* (39).

9 A generic case

The orbit space $\bar{P}$ is stratified by connected components of local manifolds $\pi(P_{(K)})$, where K runs over compact subgroups of G and $P_{(K)}$ is the local submanifold of P given by equation (12). Hence, P is stratified by connected components of local manifolds $P_{(K)}$. In the above discussion we have assumed that ρ is a local section of $\alpha \colon P \to M$ with range contained in a single stratum. In general, this is not the case, and we may consider the partition of the domain of ρ given by sets $S_{(K)} = \rho^{-1}(P_{(K)})$, as K varies over compact subroups of G. By the properties of stratification, [15], see also [6], the union of connected components of sets $S_{(K)}$ which are open in M is dense in the domain of ρ.

Since ρ is smooth, it follows that it satisfies the Dedonder equation (2) if and only if it satisfies the Dedonder equation on an open dense subset of its domain. In particular, ρ satisfies the Dedonder equation if its restriction to each open connected component U of $S_{(K)}$ satisfies the Dedonder equation. Since U is connected and ρ is continuous, $\rho(U)$ is contained in a connected component N of $P_{(K)}$. We denote by $\rho_N \colon U \to N$ the section obtained from ρ by restrictiong its domain to U and co-domain to N. It follows from the discussion in the preceding sections that ρ_N satisfies the Dedonder equation if and only $\rho_{\bar{N}} = \pi_N \circ \rho$ and $\rho^{\sharp}_N = \gamma_N \circ \operatorname{ver} T\rho_N$ satisfy equations (29), (34) and (39).

We can summarize our result in the following

Singular Reduction Theorem *The domain of a smooth local section ρ of $\alpha \colon P \to M$ is contained in the closure of the union of open sets $U \subseteq M$ such that $\rho(U)$ is contained in a connected component N of $P_{(K)}$. The section ρ satisfies the Dedonder equation if and only if $\rho_{\bar{N}} = \pi_N \circ \rho_N$ and $\rho^{\sharp}_N = \gamma_N \circ \operatorname{ver} T\rho_N$ satisfy equations* (29), (34) *and* (39), *where $\rho_N \colon U \to N$ denote the section obtained from ρ by restrictiong its domain to U and co-domain to N.*

10 Regular Reduction

Reduction for a free and proper action of G on P is usually referred to as a regular reduction. In the case of a free and proper action, P has the structure of a (left) principal fibre bundle over $\bar{P}$ with structure group G and the projection map π. Therefore, there is only one isotropy group, namely the trivial subgroup $\{e\}$ of G consisting of the identity element in G. Hence, $P = P_{\{e\}} = P_{(\{e\})}$. Moreover, if P is connected, then $N = P$ and $\bar{N} = \bar{P}$.

The space of G-orbits in ver TP is naturally isomorphic to the adjoint bundle $P[\mathfrak{g}]$. We use the notation $\gamma\colon$ ver $TP \to P[\mathfrak{g}]$ and $\vartheta\colon P[\mathfrak{g}] \to \bar{P}$ for the orbit map and the vector bundle projection, respectively. As before, to each local section ρ of $\alpha\colon P \to M$ with domain U we associate $\rho_{\bar{P}} = \pi \circ \rho\colon U \to \bar{P}$ and $\rho^\sharp = \gamma \circ \operatorname{ver} T\rho\colon TU \to P[\mathfrak{g}]$.

Since $\bar{N} = \bar{P}$, we can write $\Omega_{\bar{N}} = \Omega_{\bar{P}}$, $\Xi_{\bar{N}} = \Xi_{\bar{P}}$ and $\Sigma_{\bar{N}} = \Sigma_{\bar{P}}$. Moreover, $T^\perp P = 0$ implies that the left hand side of equation (39) vanishes identically. Therefore, we obtain:

Regular Reduction Theorem *For a free and proper action of G on P, a smooth local section ρ of $\alpha\colon P \to M$ satisfies the Dedonder equation if and only if $\bar{\rho} = \pi \circ \rho$ and $\rho^\sharp = \gamma \circ \operatorname{ver} T\rho$ satisfy the reduced equations*

$$\rho_{\bar{P}}^* \Xi_{\bar{P}} + \rho^\sharp \dot{\wedge} \rho_{\bar{P}}^* \Sigma_{\bar{P}} = 0,$$

and

$$\rho_{\bar{P}}^*(X_{\bar{P}} \lrcorner \Omega_{\bar{P}}) + \rho^\sharp \dot{\wedge} \rho_{\bar{P}}^*(X_{\bar{P}} \lrcorner \Xi_{\bar{P}}) = 0,$$

for all vector fields $X_{\bar{P}}$ on $\bar{P}$ tangent to the fibres of $\alpha_{\bar{P}}\colon \bar{P} \to M$.

11 An Example

We consider here a relativistic complex scalar field φ with a potential V which depends on $|\varphi|^2 = \varphi\bar{\varphi}$. In the terminology of Section 2, $M = \mathbb{R}^4$ and $Q = M \times \mathbb{C}$ with $\mu\colon Q \to M$ given by the projection to the first factor. Let $(g^{\lambda\nu}) = \operatorname{diag}(1, -1, -1, -1)$, where $\lambda, \nu = 0, \dots, 3$, be the Minkowski metric tensor on M. The Lagrangian form on $P = \mathbb{R}^4 \times \mathbb{C} \times (\mathbb{C} \otimes \mathbb{R}^4)$ is given by

$$L = \Big(\frac{1}{2} p^\nu \bar{p}_\nu - V(z\bar{z})\Big)\, d_4x, \tag{42}$$

where $z \in \mathbb{C}$, $p_\mu \in \mathbb{C} \otimes \mathbb{R}^4$, and the Greek indices are raised in terms of $g^{\mu\nu}$, *i.e.*, $p^\mu = g^{\mu\nu} p_\nu$, and we use convention of summation over the repeated indices. The corresponding Dedonder form is

$$\Theta = \frac{1}{2}(p^\mu d\bar{z} + \bar{p}^\mu dz) \wedge dx_\mu - \Big(\frac{1}{2} p^\nu \bar{p}_\nu + V(z\bar{z})\Big)\, d_4x. \tag{43}$$

Hence,

$$\Omega = \frac{1}{2}(dp^\mu \wedge d\bar{z} + d\bar{p}^\mu \wedge dz) \wedge dx_\mu - \Big(\frac{1}{2} p^\nu d\bar{p}_\nu + \frac{1}{2} \bar{p}^\nu dp_\nu + dV(z\bar{z})\Big) \wedge d_4x. \tag{44}$$

The Dedonder equation (2) for a section ρ given by $z = z(x)$ and $p_\mu = p_\mu(x)$ yields

$$z_{,\mu}(x) = p_\mu(x), \quad \bar{z}_{,\mu}(x) = \bar{p}_\mu(x), \tag{45}$$

$$\frac{1}{2}g^{\mu\nu} p_{\mu,\nu} = -\frac{\partial V}{\partial \bar{z}}, \quad \frac{1}{2}g^{\mu\nu} \bar{p}_{\mu,\nu} = -\frac{\partial V}{\partial z}. \tag{46}$$

Equation (45) ensures that ρ is the first jet extension of its projection $\sigma = \beta \circ \rho$. Hence, the Dedonder equations (45) and (46) are equivalent to the Euler-Lagrange equations for the Lagrangian (42), given by

$$\frac{1}{2}g^{\mu\nu} z_{,\mu\nu} = -\frac{\partial V}{\partial \bar{z}}, \quad \frac{1}{2}g^{\mu\nu} \bar{z}_{,\mu\nu} = -\frac{\partial V}{\partial z}.$$

Introducing real variables r, t, and complex variables s_μ such that $z = re^{it}$ and $p_\mu = rs_\mu e^{it}$, we get

$$dz = ire^{it}dt + e^{it}dr, \quad dp_\mu = irs_\mu e^{it}dt + e^{it}s_\mu dr + e^{it}rds_\mu.$$

Hence,

$$\Theta = \frac{ir^2}{2}(\bar{s}^\mu - s^\mu)dt \wedge dx_\mu + \frac{r}{2}(\bar{s}^\mu + s^\mu)dr \wedge dx_\mu - \Big(\frac{1}{2}r^2 s^\nu \bar{s}_\nu + V(r^2)\Big)\, d_4x \tag{47}$$

and

$$\begin{aligned}\Omega = {} & \frac{ir^2}{2}d(\bar{s}^\mu - s^\mu) \wedge dt \wedge dx_\mu + ir(\bar{s}^\mu - s^\mu)dr \wedge dt \wedge dx_\mu \\ & + \frac{r}{2}d(\bar{s}^\mu + s^\mu)dr \wedge dx_\mu - d\Big(\frac{1}{2}r^2 s^\nu \bar{s}_\nu + V(r^2)\Big)\, d_4x.\end{aligned} \tag{48}$$

The Lagrangian (42) is relativistically invariant. However, the action of the Poincaré group does not satisfy the assumptions made here. On the other hand, the theory is also invariant under the action

$$\Phi\colon U(1) \times P \to P\colon (e^{i\theta}, (x^\mu, z, p_\mu)) \mapsto (x^\mu, e^{i\theta}z, e^{i\theta}p_\mu). \tag{49}$$

In the following we describe the reduction of $U(1)$ symmetry corresponding to the action (49) as a simple example of the general theory developed here.

The group $U(1)$ is compact. Hence the action Φ is proper, but it is not free. There are two isotropy groups: the whole group $U(1)$, and the trivial subgroup $\{1\}$ consisting of the identity in G. We have two connected components to consider:

$$N_{U(1)} = P_{U(1)} = P_{(U(1))} = \{(x^\mu, z, p_\mu) \in P \mid z = 0,\ p_\mu = 0\},$$

consisting of points in P fixed by the action of $U(1)$, and its complement

$$N_{\{1\}} = P_{\{1\}} = P_{(\{1\})} = \{(x^\mu, z, p_\mu) \in P \mid z\bar{z} + p_0\bar{p}_0 + p_i\bar{p}_i \neq 0\},$$

on which $U(1)$ acts freely.

We consider first $N_{U(1)}$. It is the image of the zero section of $\alpha\colon P \to M$. Hence, $\alpha_{N_{U(1)}}\colon N_{U(1)} \to M$ is injective, and $\ker T\alpha_{N_{U(1)}} = 0$. Hence, the left hand sides of equations (20) and (21) are identically zero. On the other hand, a section $\rho_{N_{U(1)}}$ of $\alpha_{N_{U(1)}}$ has values in $N_{U(1)} = P_{U(1)}$, which implies that $T\rho_{N_{U(1)}}(u)$ is $U(1)$-invariant. Hence, for every $x \in M$, $u_1, \dots, u_n \in T_xM$, $e^{i\theta} \in U(1)$, and $v \in T_p^{\perp}N$, where $p = \rho_{N_{U(1)}}(x)$, $U(1)$-invariance of Ω implies that

$$\begin{aligned}
\rho^*_{N_{U(1)}}\big(T\Phi_{e^{i\theta}}(v) \lrcorner\, \Omega\big)\,(u_1,\dots,u_n)& \\
&= \big(T\Phi_{e^{i\theta}}(v) \lrcorner\, \Omega\big)\,\big(T\rho_{N_{U(1)}}(u_1),\dots,T\rho_{N_{U(1)}}(u_n)\big)\\
&= \big(T\Phi_{e^{i\theta}}(v) \lrcorner\, \Omega\big)\,\big(T\Phi_{e^{i\theta}}(T\rho_{N_{U(1)}}(u_1)),\dots,T\Phi_{e^{i\theta}}(T\rho_{N_{U(1)}}(u_n))\big)\\
&= (v \lrcorner\, \Phi^*_{e^{i\theta}}\Omega)(u_1,\dots,u_n) = (v \lrcorner\, \Omega)(u_1,\dots,u_n)\\
&= \rho^*_{N_{U(1)}}(v \lrcorner\, \Omega)(u_1,\dots,u_n).
\end{aligned}$$

Hence,

$$\rho^*_{N_{U(1)}}\big(T\Phi_{e^{i\theta}}(v) \lrcorner\, \Omega\big) = \rho^*_{N_{U(1)}}(v \lrcorner\, \Omega)$$

for every $v \in T^{\perp}N_{U(1)}$ and every $e^{i\theta} \in U(1)$. However, the bundle $T^{\perp}N_{U(1)}$ consists of vectors in $T_{N_{U(1)}}N$ with vanishing average over $U(1)$ [8]. Hence, integrating over $U(1)$ we get

$$\begin{aligned}
\rho^*_{N_{U(1)}}(v \lrcorner\, \Omega) &= \frac{1}{2\pi}\int_1^{2\pi} \rho^*_{N_{U(1)}}\big(T\Phi_{e^{i\theta}}(v) \lrcorner\, \Omega\big)\, d\theta\\
&= \frac{1}{2\pi}\rho^*_{N_{U(1)}}\left(\left(\int_1^{2\pi}\big(T\Phi_{e^{i\theta}}(v)\big)\, d\theta\right) \lrcorner\, \Omega\right) = 0.
\end{aligned}$$

Thus, the left hand side of equation (22) vanishes identically.

It follows from the discussion above that every local section of $\alpha_{N_{U(1)}}$ satisfies the Dedonder equation. Since sections of $\alpha_{N_{U(1)}}$ are zero sections, we conclude that every zero section satisfies the Dedonder equation. This result is evident from equations (45) and (46).

We consider now a section $\rho_{N_{\{e\}}}$ of $\alpha_{N_{\{e\}}}\colon N_{\{e\}} \to M$. Since the action of $U(1)$ on $N_{\{e\}}$ is free, we are dealing with regular reduction. Observe first that $N_{\{e\}}$ is open in P. Hence, $T^{\perp}N_{\{e\}} = 0$, which implies that the left hand side of equation (22) vanishes identically.

In order to continue our discussion, we need to introduce a $U(1)$-invariant Riemannian metric k on P. We choose

$$\text{k} = dx^0 \otimes dx^0 + dx^i \otimes dx^i + dz \otimes d\bar{z} + dp_0 \otimes d\bar{p}_0 + dp_i \otimes d\bar{p}_i, \tag{50}$$

where we have adopted the convention of summation over the repeated index $i = 1, 2, 3$. The restriction of k to $N_{\{e\}}$, described in terms of the coordinates (x^{μ}, t, r, s_{μ}), is

$$\text{k} = dx^0 \otimes dx^0 + dx^i \otimes dx^i + r^2(1 + s_0\bar{s}_0 + s_i\bar{s}_i)dt \otimes dt + (1 + s_0\bar{s}_0 + s_i\bar{s}_i)dr \otimes dr + r^2(ds_0 \otimes d\bar{s}_0 + ds_i \otimes d\bar{s}_i) + rdr \otimes (s_0 d\bar{s}_0 + \bar{s}_0 ds_0 + s_i d\bar{s}_i + \bar{s}_i ds_i). \tag{51}$$

The orbit space $\bar{N}_{\{e\}}$ is parametrized by the variables $(x^\mu, r, s_\mu, \bar{s}_\mu)$. The vertical distribution, ver $TN_{\{e\}}$, is spanned by the vector field $X = i\frac{\partial}{\partial t}$ on $N_{\{e\}}$. Hence, ver $TN_{\{e\}}$ is 1-dimensional, and equation (31) yields $\Sigma_{N_{\{e\}}} = 0$, which implies

$$\Sigma_{\bar{N}_{\{e\}}} = 0. \tag{52}$$

Similarly, equation (26), (27) and (48) yield

$$\Xi_{\bar{N}_{\{e\}}} = -\frac{ir^2}{2} d(\bar{s}^\mu - s^\mu) \wedge dx_\mu - ir(\bar{s}^\mu - s^\mu) dr \wedge dx_\mu. \tag{53}$$

We denote by Υ the unique 1-form on P such that $\langle \Upsilon | w \rangle = \text{k}(i\frac{\partial}{\partial t}, w)$ for all $w \in TN_{\{e\}}$. Taking into account equation (51) we get

$$\Upsilon = -ir^2(1 + s_0\bar{s}_0 + s_i\bar{s}_i)dt.$$

Since $\langle \Upsilon | i\frac{\partial}{\partial t} \rangle = r^2(1 + s_0\bar{s}_0 + s_i\bar{s}_i)$, equation (48) yields

$$\begin{aligned} \operatorname{hor} \Omega_{N_{\{e\}}} &= \Omega - \frac{1}{r^2(1 + s_0\bar{s}_0 + s_i\bar{s}_i)} \Upsilon \wedge \Big(i\frac{\partial}{\partial t} \mathbin{\lrcorner} \Omega \Big) \\ &= \frac{r}{2} d(\bar{s}^\mu + s^\mu) \wedge dr \wedge dx_\mu - d\Big(\frac{1}{2} r^2 s^\nu \bar{s}_\nu + V(r^2) \Big) d_4x. \end{aligned} \tag{54}$$

Taking into account equation (24) we get

$$\Omega_{\bar{N}_{\{e\}}} = \frac{r}{2} d(\bar{s}^\mu + s^\mu) \wedge dr \wedge dx_\mu - d\Big(\frac{1}{2} r^2 s^\nu \bar{s}_\nu + V(r^2) \Big) d_4x. \tag{55}$$

The vector field $i\frac{\partial}{\partial t}$ on $N_{\{e\}}$ generates the action of $U(1)$, and it induces a trivialization of the bundle $V_{\bar{N}_{\{e\}}}$. Each $v \in$ ver $T_p N_{\{e\}}$ is of the form $v = s(i\frac{\partial}{\partial t})$, for some $s \in \mathbb{R}$, and $\gamma_{N_{\{e\}}}(v) = \big(\pi_{N_{\{e\}}}(p), s\big)$. The section $\rho_{N_{\{e\}}}$ under consideration is given by $x^\mu \mapsto \big(x^\mu, r(x), t(x), s_\mu(x), \bar{s}_\mu(x)\big)$. It gives rise to a section $\rho_{\bar{N}_{\{e\}}} : x^\mu \mapsto \big(x^\mu, r(x), s_\mu(x), \bar{s}_\mu(x)\big)$ of $\alpha_{\bar{N}_{\{e\}}}$. In the reduced equations (29) and (34) $\rho_{\bar{N}}$ and $\rho^\sharp_N$ are independent variables. Hence, we can write $\rho^\sharp_{N_{\{e\}}} = y_\mu dx^\mu$, where y_μ are unknowns to be determined. Taking into account equations (53) and (55), we get from equation (29) the following equations:

$$-\frac{r}{2}(s^\mu_{,\mu} + \bar{s}^\mu_{,\mu}) - \big(2rs^\mu \bar{s}_\mu + V'(r^2) \big) - iry_\mu(\bar{s}^\mu - s^\mu) = 0, \tag{56}$$

$$\frac{r}{2} r_{,\mu} + r^2 \bar{s}_\mu + \frac{ir^2}{2} y_\mu = 0. \tag{57}$$

Similarly, taking into account equations (52) and (53), we obtain from equation (34)

$$g^{\mu\nu}\left(r^2(\bar{s}_{\nu,\mu} - s_{\nu,\mu}) + 2r(\bar{s}_\nu - s_\nu)r_{,\mu}\right) = 0. \tag{58}$$

Equation (39) does not introduce any additional conditions because its left hand side vanishes identically.

If we have $r(x)$, $s_\mu(x)$ and $y_\mu(x)$ satisfying equations (56), (57) and (58), we can reconstruct the section ρ as follows. Assume that ρ is given by $z = r(x)e^{it(x)}$, and $p_\mu(x) = r(x)s_\mu(x)e^{it(x)}$, where the phase factor $t(x)$ is to be determined. Then,

$$T\rho\Big(\frac{\partial}{\partial x^\mu}\Big) = \frac{\partial}{\partial x^\mu} + r_{,\mu}\frac{\partial}{\partial r} + t_{,\mu}\frac{\partial}{\partial t} + s_{\nu,\mu}\frac{\partial}{\partial s_\nu} + \bar{s}_{\nu,\mu}\frac{\partial}{\partial \bar{s}_\nu},$$

and

$$k\left(i\frac{\partial}{\partial t}, T\rho\Big(\frac{\partial}{\partial x^\mu}\Big)\right) = ir^2(1 + s_0\bar{s}_0 + s_i\bar{s}_i)t_{,\mu},$$

which implies that

$$\operatorname{ver} T\rho\Big(\frac{\partial}{\partial x^\mu}\Big) = t_{,\mu}\frac{\partial}{\partial t}.$$

Hence, $\rho^\sharp = t_{,\mu}dx^\mu$, which yields the reconstruction equation $t_{,\mu} = y_\mu$. Integrating this equation we get $t(x)$ up to a constant. In other words, we get the section ρ up to the action of $U(1)$.

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