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New directions in convex analysis: the differentiability of convex functions on topological linear spaces

BERNICE SHARP

My thesis is that topological linear spaces are a more natural setting for the study of convex functions than Banach spaces. Two areas of research which are currently almost distinct are synthesised: the differentiability of functions on topological linear spaces and the differentiability of convex functions on Banach spaces.

A *bornology* $\mathcal{M}$, on a topological linear space X, is a nonempty class of bounded subsets of X which contains all singletons. A real valued convex function on D is differentiable at $x \in D$ whenever the difference quotient converges uniformly over elements of $\mathcal{M}$ to a continuous linear function. Fréchet, Hadamard and Gateaux differentiability correspond to choices of bounded sets, compact sets and singletons for $\mathcal{M}$. This follows Averbukh and Smolyanov [**3**].

The classification of Banach spaces according to the differentiability properties of its continuous convex functions began with Asplund [**1**]. Eight classifications are defined for topological linear spaces; all of the classifications except Fréchet Minkowski Differentiability Space are used in the literature for Banach spaces.

Mazur's Theorem, that every separable Banach space is Weak Asplund, is generalised: a separable, Baire, topological linear space is a Weak Asplund Space. The space $C(\mathsf{R})$ of continuous real valued functions of a real variable with the topology of compact convergence is therefore a Weak Asplund Space; other examples are also given.

It is shown that for locally Lipschitz functions, *a fortiori* for convex functions, Gateaux and Fréchet differentiability coincide on spaces in which all bounded subsets are relatively compact; for a normed space this is of little interest, since in this case the space is finite dimensional, but a large class of locally convex spaces has this property, for example Montel Fréchet spaces, and the space of holomorphic functions on the complex unit disk. Such spaces are thus Asplund spaces. Changing the topology on a space changes the differentiability properties of its continuous convex functions to a surprising extent. Any topological linear space with a weak topology is shown to be

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an Asplund space. A marked contrast is seen: for example, with the norm topology it is well known that ℓ_∞ admits continuous convex functions which are nowhere even Gateaux differentiable; with the weak or weak* topology ℓ_∞ is an Asplund Space. At the opposite end of the topological spectrum, an example is given of an inductive limit of Asplund Spaces which is not a Gateaux Differentiability Space.

For S a topological space, $C(S)$ denotes the set of continuous real valued functions on S endowed with the topology of compact convergence. An example is given of a continuous convex function on $C(\mathbb{R})$ which is nowhere Fréchet differentiable, so $C(\mathbb{R})$ is not a Fréchet Differentiability Space. In [6] Čoban and Kenderov study the Gateaux differentiability of continuous convex functions on the Banach space $C(S)$ for S compact; these theorems are extended by considering various topological spaces S, and results for Fréchet differentiability are given. The class of spaces $C(S)$ which are Fréchet Differentiability Spaces is very small; it is shown that in this case, every compact subset of S consists only of isolated points and their cluster points, which generalises a result of Namioka and Phelps [10]. Some illuminating examples are given and some open questions are suggested: in particular $C(\mathbb{Q})$, the continuous functions on the rationals, is not classified. (This has since been answered, in a joint paper with Dr Roger Eyland.)

The nature of the subdifferential set of a continuous convex function is examined, and a characterisation of varying strengths of differentiability is given in terms of the convergence of subdifferentials in the appropriate topology in the dual space. These results form a basis for proofs of the characterisations of differentiability spaces.

It is shown that for locally convex spaces, six classes of differentiability space suffice: every continuous convex function is Fréchet, or Gateaux, differentiable on a dense set of its open convex domain if this property holds for every continuous gauge.

For Banach spaces it was proved by Namioka and Phelps [10] that Asplund Spaces are characterised by an elegant geometric property of the dual space: X is Asplund if and only if every weak* compact convex subset of X^* is the weak* closed convex hull of its weak* strongly exposed points. Recently a similar theorem was proved for Gateaux Differentiability Spaces ([11], 6.2). With an appropriate definition of "weak* strongly exposed point", a generalisation of these theorems is given: a locally convex space is a Fréchet (Gateaux) Differentiability Space if and only if the polar of every neighbourhood of the origin is the weak* closed convex hull of its weak* strongly exposed (weak* exposed) points.

There remain many open questions. Some arise as generalisations of well known results about Banach spaces; others are peculiar to this milieu.

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Catholic College of Education, Sydney
40 Edward St
North Sydney NSW 2060
Australia

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