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First Occurrence for the Dual Pairs $\big(U(p,q),U(r,s)\big)$

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Abstract. We prove a conjecture of Kudla and Rallis about the first occurrence in the theta correspondence, for dual pairs of the form $\big(U(p,q),U(r,s)\big)$ and most representations.

1 Introduction

Let (G,G') be a reductive dual pair in $\mathrm{Sp}=\mathrm{Sp}(2n,\mathbb{R})$, and let ω be the oscillator representation of $\widetilde{\mathrm{Sp}}$, the connected two-fold cover of Sp [H1]. If $\tilde{G}$ and $\tilde{G}'$ are the inverse images of G and G' in $\widetilde{\mathrm{Sp}}$ by the covering map, and π and π' are irreducible admissible representations of $\tilde{G}$ and $\tilde{G}'$ respectively, we say that π *corresponds* to π' if (on the level of Harish-Chandra modules), $\pi\otimes\pi'$ may be realized as a quotient of ω. Roger Howe [H3] showed that this correspondence defines a bijection between subsets of the admissible duals of $\tilde{G}$ and $\tilde{G}'$. An interesting problem is to find this *Howe* (or *theta*) correspondence explicitly, *i.e.*, decide which representations occur, and how they match up in the correspondence. Since the oscillator representation is a genuine representation of the metaplectic cover of Sp (*i.e.*, does not factor to the linear group), every representation which occurs in the Howe correspondence is also a genuine representation of the respective two-fold cover.

Consider dual pairs of the form $\big(U(p,q),U(r,s)\big)$ in $\mathrm{Sp}=\mathrm{Sp}\big(2(p+q)(r+s),\mathbb{R}\big)$. For each of these pairs, the cover of $U(p,q)$ depends only on the parity of $r+s$; similarly for the cover of $U(r,s)$. We are interested in the following question. Fix p and q, a parity ϵ of $r+s$ defining a cover $\tilde{U}(p,q)$, and a representation $\pi\in\tilde{U}(p,q)\widehat{\ }_{\text{genuine}}$, the genuine admissible dual of $\tilde{U}(p,q)$. For which choices of r,s does π occur in the correspondence for the dual pair $\big(U(p,q),U(r,s)\big)$?

If $\pi\in\tilde{U}(p,q)\widehat{\ }_{\text{genuine}}$ and π corresponds to $\pi'\in\tilde{U}(r,s)\widehat{\ }_{\text{genuine}}$, we write $\theta_{r,s}(\pi)=\pi'$ and sometimes say π *lifts to* $U(r,s)$. In this case, we refer to π' as the *theta lift* of π. If π does not occur in the correspondence, we write $\theta_{r,s}(\pi)=0$.

For any integer k, consider the collection of groups $\{U(r,s):r-s=k\}$ with associated hermitian (or skew-hermitian) spaces $\{W_{r,s}:r-s=k\}$. We call this set the k-th Witt tower. One may picture these groups arranged in a column, with the smallest (compact) group at the bottom. Notice that each space in a given Witt tower is obtained by adding a split space to a fixed anisotropic space; *e.g.*, if $k\geq 0$ then

$$W_{r,s}=W_{k,0}+W_{s,s}.$$

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Using the mixed model [H2] of the oscillator representation, one can show that if $\theta_{r,s}(\pi) \neq 0$, then π lifts to every group above $U(r,s)$ in the $(r-s)$-th Witt tower. This fact is called *persistence* (Kudla). Moreover, in the *stable range*, *i.e.*, if $\min\{r,s\} \geq p+q$, every genuine irreducible representation of $\tilde{U}(p,q)$ occurs (see *e.g.* [M], [L1]). Consequently, our question may be reformulated as follows: in each Witt tower, which is the group of least rank (or the space of least dimension) to which π lifts, *i.e.*, what is the *first occurrence* of π in each tower? Since the parity ϵ is just the parity of the dimension of the hermitian space (which is fixed in each Witt tower), we consider either only even dimensional 'target' spaces, *i.e.*, $r+s$ even, or only odd dimensional spaces, *i.e.*, $r+s$ odd.

If W is a hermitian or skew-hermitian space with signature (r,s), we define the *normalized determinant of* W by

$$\eta(W) = \eta(r,s) = (-1)^s(-1)^{\frac{(r+s)(r+s-1)}{2}}.$$

We call a hermitian or skew-hermitian space W *type* + if $\eta(W)=1$, and *type* $-$ if $\eta(W)=-1$. Notice that in a fixed dimension $r+s$, the normalized determinant varies with the parity of s; for example, in dimension 5, $(5,0)$, $(3,2)$, and $(1,4)$ are of type +, while $(4,1)$, $(2,3)$, and $(0,5)$ are of type $-$. In dimension 6, $(6,0)$, $(4,2)$, $(2,4)$, and $(0,6)$ are of type $-$, while $(5,1)$, $(3,3)$, and $(1,5)$ are of type +. Notice also that η is constant on the set of spaces in each Witt tower, so that we can define the normalized determinant $\eta(k)$ of the k-th Witt tower in the obvious way.

Definition 1.1 Let p, q, ϵ, and $\pi \in \tilde{U}(p,q)^{\widehat{}}_{\text{genuine}}$ be given as above.

(a) For k an integer such that $(-1)^k = \epsilon$, we define the *first occurrence index for* π *in the* k*-th Witt tower* by $m_k(\pi) = \min\{r+s : r-s=k \text{ and } \theta_{r,s}(\pi) \neq 0\}$;

(b) The *first occurrence index of* π is $m(\pi) = \min\{m_k(\pi) : (-1)^k = \epsilon\}$;

(c) The *type* $\pm$ *first occurrence index for* π is $m_{\pm}(\pi) = \min\{m_k(\pi) : (-1)^k = \epsilon \text{ and } \eta(k) = \pm 1\}$.

Kudla and Rallis [KR] have formulated a conjecture about the first occurrence for orthogonal-symplectic dual pairs over nonarchimedean fields. A natural analogue is the following

Conjecture 1.2 *Let* π *be a genuine irreducible admissible representation of* $\tilde{U}(p,q)$*. Then*

$$m_+(\pi) + m_-(\pi) = 2p + 2q + 2. \tag{1.3}$$

The truth of the conjecture for all representations $\pi \in \tilde{U}(p,q)^{\widehat{}}_{\text{genuine}}$ with $\epsilon = (-1)^{p+q+1}$ and satisfying a certain condition in terms of the inducing data follows from the main result of [P2]. We give a precise statement in Section 6.

In this paper, we prove the following result.

Theorem 1.4 *Conjecture 1.2 is true for the cases where* π *is a discrete series representation, or irreducibly induced from a discrete series representation.*

To illustrate this result let us consider the discrete series of $\tilde{U}(p,q) = \tilde{U}(2,1)$ with $r+s$ odd. In this case, all covers are the connected two-fold covers. Genuine discrete series representations π of $\tilde{U}(2,1)$ may be given by Harish-Chandra (HC) parameters of the form

$\lambda = (a, b; c)$ with $a, b, c \in \mathbb{Z} + \frac{1}{2}$, all three entries distinct, and $a > b$. HC parameters for $\tilde{U}(r, s)$ are given in a similar way. We distinguish six cases, and in each of them we identify $m_+(\pi)$, $m_-(\pi)$ and the corresponding signatures, as well as the theta lifts. Notice that we have $m_+(\pi) + m_-(\pi) = 2p + 2q + 2 = 8$.

Case 1: $a, b, c > 0$**.** In this case, $m_+(\pi) = 3$ and $m_-(\pi) = 5$. For the type $+$ first occurrence, π lifts to $U(2, 1)$, and $\theta_{2,1}(\pi) = \pi$. For the type $-$ first occurrence, we have the following situation: if $b \geq \frac{3}{2}$ then π lifts to $U(4, 1)$, and if $c \geq \frac{3}{2}$ then π lifts to $U(2, 3)$ (notice that at least one of these conditions must hold). If $b, c \geq \frac{3}{2}$ then these theta lifts are discrete series with HC parameters $(a, b, \frac{1}{2}, -\frac{1}{2}; c)$ and $(a, b; c, \frac{1}{2}, -\frac{1}{2})$ respectively. Otherwise, the theta lifts should be (see Conjecture 4.3) limits of discrete series with these same parameters.

Case 2: $a, b > 0; c < 0$**.** In this case, $m_-(\pi) = 3$ and $m_+(\pi) = 5$. For type $-$, π lifts to $U(3, 0)$, and $\theta_{3,0}(\pi)$ is the discrete series with HC parameter (a, b, c), *i.e.*, the representation with highest weight $(a - 1, b, c + 1)$. For type $+$, π lifts to $U(3, 2)$, and if $b, -c \geq \frac{3}{2}$ then π also lifts to $U(5, 0)$. In this last case, the theta lifts are discrete series with HC parameters $(a, b, c; \frac{1}{2}, -\frac{1}{2})$ and $(a, b, \frac{1}{2}, -\frac{1}{2}, c)$ respectively; if $b = \frac{1}{2}$ or $c = -\frac{1}{2}$ then $\theta_{3,2}(\pi)$ should be (see Conjecture 4.3) a limit of discrete series with the same parameter.

Case 3a: $a = \frac{1}{2}$**,** $b = -\frac{1}{2}$**,** $c > 0$**.** In this case, $m_-(\pi) = 1$ and $m_+(\pi) = 7$. For type $-$, π lifts to $U(0, 1)$, and $\theta_{0,1}(\pi)$ is the character of $\tilde{U}(0, 1)$ with weight c. For type $+$, π lifts to both $U(4, 3)$ and $U(2, 5)$. The theta lifts are non-tempered representations and should be (see Conjecture 4.3) constituents of representations induced from relative limits of discrete series on parabolic subgroups with Levi factors isomorphic to $\tilde{U}(3, 2) \times \mathbb{C}^\times$ and $\tilde{U}(1, 4) \times \mathbb{C}^\times$ respectively.

Case 3b: $a, c > 0, b < 0$***, and*** $a \geq \frac{3}{2}$ ***or*** $b \leq -\frac{3}{2}$**.** In this case, $m_-(\pi) = 3$ and $m_+(\pi) = 5$. For type $-$, π lifts to $U(1, 2)$, and $\theta_{1,2}(\pi)$ is the discrete series with HC parameter $(a; c, b)$. For type $+$ we have that if $a \geq \frac{3}{2}$ then π lifts to $U(3, 2)$, and if $b \leq -\frac{3}{2}$ and $c \geq \frac{3}{2}$ then π lifts to $U(1, 4)$. If $a, -b, c \geq \frac{3}{2}$ then the theta lifts are the discrete series with HC parameters $(a, \frac{1}{2}, -\frac{1}{2}; c, b)$ and $(a; c, \frac{1}{2}, -\frac{1}{2}, b)$ respectively; otherwise the theta lifts should be (see Conjecture 4.3) limits of discrete series with the same parameters.

The Cases 4 $(a > 0, b, c < 0)$, 5 $(a, b < 0, c > 0)$, and 6 $(a, b, c < 0)$ are similar to and may be obtained from Cases 3, 2, and 1 respectively, using the behavior of the theta correspondence with respect to contragredients (see Proposition 2.9). For example, in Case 5 we have $m_+(\pi) = 3$ and $m_-(\pi) = 5$, and π lifts to $U(0, 3)$ and $U(2, 3)$, as well as $U(0, 5)$ if $a \leq -\frac{3}{2}$ and $c \geq \frac{3}{2}$; the theta lifts are discrete series or limits of discrete series with parameters (c, a, b), $(\frac{1}{2}, -\frac{1}{2}; c, a, b)$, and $(c, \frac{1}{2}, -\frac{1}{2}, a, b)$ respectively.

For the proof of Theorem 1.4, we first use a doubling argument due to Kudla and Rallis to show for all $\pi \in \widehat{\tilde{U}(p, q)}_{\text{genuine}}$ that if π occurs with two groups from different Witt towers, then the sum of the associated dimensions is at least $2(p + q) + 2$, which implies one inequality for (1.3). A similar argument then yields that

$$m_k(\pi) + m_{k'}(\pi) = 2p + 2q + 2 \Rightarrow |k - k'| \leq 2.$$

Assuming the validity of the conjecture, the two earliest first occurrences will therefore be in neighboring Witt towers. Notice that such Witt towers have opposite normalized determinants.

Next we turn our attention to discrete series representations and use results due to J.-S. Li to identify the values r, s corresponding to $m_+(\pi)$ and $m_-(\pi)$ to prove the conjecture for this case. The theta lift in the earliest first occurrence (at rank $m(\pi)$) is again a discrete series, as is the case for the second earliest first occurrence (at rank $2p+2q+2-m(\pi)$) if the infinitesimal character is sufficiently regular [L2]. For all other cases, we formulate and give evidence in favor of a conjecture which identifies the theta lift in question. Using an explicit example, we show that if $m_k(\pi) > m(\pi)$, then the theta lift of a discrete series representation in first occurrence in the k-th Witt tower need not be a tempered representation (see [Ro]).

Finally, we use the induction principle ([K], [M], [AB], [P1]) to show that the truth of the conjecture for discrete series representations implies its truth for most representations.

2 Some Facts About the Correspondence

The Space of Joint Harmonics

The following discussion is in [H3]. Let (G, G') be a reductive dual pair in $\mathrm{Sp} = \mathrm{Sp}(2n, \mathbb{R})$. Let $U \cong U(n)$, K, and K' be maximal compact subgroups of Sp, G, and G' respectively. Assume that $K \cdot K' \subset U$. Let $\widetilde{\mathrm{Sp}}$ be the nontrivial two-fold cover of Sp, and denote $\tilde{G}$, $\tilde{U}$, $\tilde{K}$, *etc.*, the corresponding two-fold covers of the subgroups. Let ω be the oscillator representation of $\widetilde{\mathrm{Sp}}$. Let $\mathcal{P}$ be the space of polynomials on $\mathbb{C}^n$. In the Fock model of ω, the space of $\tilde{U}$-finite vectors of the Harish-Chandra module associated to ω may be identified with $\mathcal{P}$, in such a way that the action of $\tilde{U}$ (and $\tilde{K}, \tilde{K}'$) preserves the degree of polynomials. Consequently, for any nonnegative integer d and any $\tilde{K}$-type ($\tilde{K}'$-type) σ, the subspace $\mathcal{I}_\sigma^d$ of degree d homogeneous polynomials of the σ- isotypic subspace $\mathcal{I}_\sigma$ of $\mathcal{P}$ is $\tilde{K}$-($\tilde{K}'$-)invariant, and we may make the following

Definition 2.1 Let σ be a $\tilde{K}$-type or $\tilde{K}'$-type occurring in $\mathcal{P}$. The *degree of* σ is the degree of the polynomial of least degree in $\mathcal{I}_\sigma$.

There is a $\tilde{K} \times \tilde{K}'$-invariant subspace $\mathcal{H}$ of $\mathcal{P}$, the *space of joint harmonics*, with the following properties:

Theorem 2.2 **(Howe)**

(1) $\tilde{K}$ and $\tilde{K}'$ generate mutual commutants on $\mathcal{H}$, i.e.,

$$\mathcal{H} \cong \bigoplus_i \sigma_i \otimes \sigma_i', \tag{2.3}$$

where for all i, $\sigma_i \in \widehat{\tilde{K}}$ and $\sigma_i' \in \widehat{(\tilde{K}')}$, and the representations σ_i and σ_i' determine each other. Moreover, for all i, σ_i and σ_i' have the same degree.
(If $\sigma \otimes \sigma'$ is a summand in (2.3), we say that σ corresponds to σ' in $\mathcal{H}$.)

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(2) *Suppose $\pi \in \widehat{\tilde{G}}$, $\pi' \in \big(\tilde{G}'\big)\widehat{}$, and $\pi \leftrightarrow \pi'$ in the correspondence for the dual pair (G, G'). Let σ be a $\tilde{K}$-type occurring in π, and suppose that σ is of minimal degree among the $\tilde{K}$-types of π. Then σ occurs in $\mathcal{H}$. Let σ' be the $\tilde{K}'$-type which corresponds to σ in $\mathcal{H}$. Then σ' is a $\tilde{K}'$-type of minimal degree in π'.*

Now consider dual pairs of the form $\big(U(p,q), U(r,s)\big)$. If K and K' are maximal compact subgroups of $U(p,q)$ and $U(r,s)$ respectively, we will often refer to an irreducible representation of $\tilde{K}$ (or $\tilde{K}'$) as a *K-type for $\tilde{U}(p,q)$* (or $\tilde{U}(r,s)$).

For fixed p and q, the degree of a K-type for $\tilde{U}(p,q)$ will depend on r and s, as will the occurrence or non-occurrence of σ in the space of joint harmonics $\mathcal{H}$. Consequently, we will use the following

Notation 2.4 Let σ be a K-type for $\tilde{U}(p,q)$. (1) The *(r,s)-degree of* σ is the degree of σ (see above definition) for $U(p,q)$ a member of the dual pair $\big(U(p,q), U(r,s)\big)$.

(2) If σ occurs in the space of joint harmonics for the dual pair $\big(U(p,q), U(r,s)\big)$, we say that σ is *(r,s)-harmonic.*

We now describe the correspondence of K-types in $\mathcal{H}$ which is well known (see *e.g.* [L2], [P1]). We use small gothic letters with subscript 0 for real Lie algebras, and drop the subscript to denote their complexifications.

Choose Cartan subalgebras $\mathfrak{t}_0$ and $\mathfrak{t}_0'$ of $\mathfrak{k}_0 = \mathrm{Lie}(K)$ and $\mathfrak{k}_0' = \mathrm{Lie}(K')$ respectively. Highest weights of genuine K-types for $\tilde{U}(p,q)$ may be given by elements of $(i\mathfrak{t}_0)^*$ of the form $(a_1, a_2, \ldots, a_p; b_1, \ldots, b_q)$ with $a_1 \geq a_2 \geq \cdots \geq a_p$, $b_1 \geq \cdots \geq b_q$, and $a_i, b_j \in \mathbb{Z} + \frac{r-s}{2}$. Analogously for genuine K-types for $\tilde{U}(r,s)$.

Lemma 2.5 *The correspondence of K-types for $\tilde{U}(p,q)$ and $\tilde{U}(r,s)$ in the space of joint harmonics for the dual pair $\big(U(p,q), U(r,s)\big)$ is given as follows:*

(1) *If σ is a K-type for $\tilde{U}(p,q)$, then σ occurs in $\mathcal{H}$ if and only if the highest weight of σ is of the form*

$$\Big(\overbrace{\frac{r-s}{2}, \ldots, \frac{r-s}{2}}^{p}; \overbrace{\frac{s-r}{2}, \ldots, \frac{s-r}{2}}^{q}\Big) + (a_1, a_2, \ldots, a_k, 0, \ldots, 0, b_1, \ldots, b_l; c_1, c_2, \ldots, c_m, 0, \ldots, 0, d_1, \ldots, d_n),$$

with $k + n \leq r$ and $l + m \leq s$. Then $\sigma \leftrightarrow \sigma'$, where σ' is the K-type for $\tilde{U}(r,s)$ with highest weight

$$\Big(\overbrace{\frac{p-q}{2}, \ldots, \frac{p-q}{2}}^{r}; \overbrace{\frac{q-p}{2}, \ldots, \frac{q-p}{2}}^{s}\Big) + (a_1, a_2, \ldots, a_k, 0, \ldots, 0, d_1, \ldots, d_n; c_1, c_2, \ldots, c_m, 0, \ldots, 0, b_1, \ldots, b_l).$$

(2) *If σ is a K-type for $\tilde{U}(p,q)$ which occurs in the oscillator representation, with highest weight*

$$\Big(\underbrace{\frac{r-s}{2}, \ldots, \frac{r-s}{2}}_{p}; \underbrace{\frac{s-r}{2}, \ldots, \frac{s-r}{2}}_{q}\Big) + (x_1, x_2, \ldots, x_p; y_1, \ldots, y_q),$$

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then the (r,s)*-degree of* σ *is given by*

$$\sum_{i=1}^{p} |x_i| + \sum_{i=1}^{q} |y_i|.$$

Remark 2.6 It will be convenient to extend the notion of (r,s)-degree to all K-types, not only those which occur in $\mathcal{P}$, by formal definition using the above formula.

More Facts

Now we state some facts about the occurrence of discrete series representations, which are due to Jian-Shu Li [L2]. We parametrize genuine discrete series representations of $\tilde{U}(p,q)$ by their Harish-Chandra parameters $\lambda \in i\mathfrak{t}_0^*$, which are of the form

$$\lambda = (a_1, a_2, \dots, a_p; b_1, \dots, b_q)$$

with $a_i, b_j \in \mathbb{Z} + \frac{p+q+r+s+1}{2}$, *i.e.*, integers if $p+q$ and $r+s$ have different parities, and half integers if the parities are equal. Moreover, $a_1 > a_2 > \cdots > a_p$, $b_1 > \cdots > b_q$, and $a_i \neq b_j$ for all i, j. Analogously for discrete series of $\tilde{U}(r,s)$.

Theorem 2.7 *For* p, q, r, s *non-negative integers with* $r + s \geq p + q$*, let* $G = U(p,q)$*,* $G' = U(r,s)$*, and consider the dual pair* (G, G')*. Suppose that* π *is a genuine discrete series representation of* $\tilde{G}$ *with Harish-Chandra parameter* λ *and lowest* K*-type* σ*.*

(a) If σ *is* (r,s)*-harmonic, then* $\theta_{r,s}(\pi) \neq 0$*. Moreover,* $\theta_{r,s}(\pi)$ *is unitarizable and contains the* K*-type for* $\tilde{U}(r,s)$ *which corresponds to* σ *in* $\mathcal{H}$*.*

(b) Let $m = r + s - p - q$*, and suppose* λ *is given by*

$$\lambda = (a_1, \dots, a_x, b_1, \dots, b_y; c_1, \dots, c_z, d_1, \dots, d_w),$$

where $a_1 > a_2 > \cdots > a_x > 0 > b_1 > \cdots > b_y$*,* $c_1 > \cdots > c_z > 0 > d_1 > \cdots > d_w$*, and* $a_x, -b_1, c_z, -d_1 \geq \frac{m+1}{2}$*.*

If $r = x + w + m$ *and* $s = y + z$ *then* $\theta_{r,s}(\pi)$ *is the discrete series of* $\tilde{U}(r,s)$ *with Harish-Chandra parameter*

$$\lambda' = \Big(a_1, \dots, a_x, \frac{m-1}{2}, \frac{m-3}{2}, \dots, \frac{-m+1}{2}, d_1, \dots, d_w; c_1, \dots, c_z, b_1, \dots, b_y\Big).$$

If $r = x + w$ *and* $s = y + z + m$ *then* $\theta_{r,s}(\pi)$ *is the discrete series of* $\tilde{U}(r,s)$ *with Harish-Chandra parameter*

$$\lambda' = \Big(a_1, \dots, a_x, d_1, \dots, d_w; c_1, \dots, c_z, \frac{m-1}{2}, \frac{m-3}{2}, \dots, \frac{-m+1}{2}, b_1, \dots, b_y\Big).$$

The next result follows from the behavior of the oscillator representation with respect to tensor products [R]. See [P1] for details.

Lemma 2.8 *For nonnegative integers* p, q, r, s, r', s'*, with* $r + s \equiv r' + s' \pmod 2$*, let* $\omega_{p,q,r,s}$ *be the oscillator representation for the dual pair* $\big(U(p,q), U(r,s)\big)$*, and let* $\omega_{p,q,r',s'}$ *and* $\omega_{p,q,r+r',s+s'}$ *be defined analogously. Then*

(a) $\omega_{p,q,r,s} \otimes \omega_{p,q,r',s'}|_{\tilde{U}(r,s)\times\tilde{U}(r',s')} \cong \omega_{p,q,r+r',s+s'}|_{\tilde{U}(r,s)\times\tilde{U}(r',s')}$,
(b) $\omega_{p,q,r,s} \otimes \omega_{p,q,r',s'}|_{\Delta(\tilde{U}(p,q))}$ *factors to* $U(p,q)$ *and* $\omega_{p,q,r,s} \otimes \omega_{p,q,r',s'} \cong \omega_{p,q,r+r',s+s'}$ *as representations of* $U(p,q)$.

Although the groups $U(p,q)$ and $U(q,p)$ are naturally isomorphic, their Howe correspondences are different. The following result relates the two correspondences and is in [P1].

Proposition 2.9 *Let* $\pi \in \tilde{U}(p,q)\widehat{\ }_{\text{genuine}}$, $\pi' \in \tilde{U}(r,s)\widehat{\ }_{\text{genuine}}$, *and let* π^* *and* π'^* *be the contragredient representations of* π *and* π' *respectively. If* $\theta_{r,s}(\pi) = \pi'$ *then* $\theta_{s,r}(\pi^*) = \pi'^*$.

Corollary 2.10 *If* $\pi \in \tilde{U}(p,q)\widehat{\ }_{\text{genuine}}$ *and* $\pi' \in \tilde{U}(r,s)\widehat{\ }_{\text{genuine}}$ *then* $\pi \leftrightarrow \pi'$ *in the correspondence for the dual pair* $\big(U(p,q), U(r,s)\big)$ *if and only if* $\pi \leftrightarrow \pi'$ *in the correspondence for the dual pair* $\big(U(q,p), U(s,r)\big)$.

Proof This follows by applying Proposition 2.9 twice. ■

Now we describe the correspondence of infinitesimal characters for the dual pair $\big(U(p,q), U(r,s)\big)$ (see [Pr]). If $\pi \in \tilde{U}(p,q)\widehat{\ }_{\text{genuine}}$ and $\theta_{r,s}(\pi) = \pi'$, let λ and λ' be the infinitesimal characters of π and π' respectively. We refer to the correspondence $\lambda \leftrightarrow \lambda'$ as the correspondence of infinitesimal characters. The infinitesimal character of a representation of $\tilde{U}(p,q)$ may be given by an element of $\mathfrak{t}^* \cong \mathbb{C}^{p+q}$. Choose coordinates such that the infinitesimal character of the trivial representation of $\tilde{U}(p,q)$ is given by $(\frac{p+q-1}{2}, \frac{p+q-3}{2}, \dots, -\frac{p+q-1}{2})$. Similarly for $\tilde{U}(r,s)$.

Theorem 2.11 **(Przebinda)** *Assume* $p+q \leq r+s$, *and let* $m = r+s-p-q$. *The correspondence of infinitesimal characters for the dual pair* $\big(U(p,q), U(r,s)\big)$ *is given by*

$$\lambda \leftrightarrow (\lambda, \rho_m),$$

for $\lambda \in \mathbb{C}^{p+q}$ *and* $\rho_m = (\frac{m-1}{2}, \frac{m-3}{2}, \dots, \frac{-m+1}{2}) \in \mathbb{C}^m$.

3 First Occurrence for Discrete Series

The first steps in proving the first occurrence formula for a discrete series representation π will be to show that $m_+(\pi) + m_-(\pi) \geq 2p+2q+2$, and to narrow down the choices of Witt towers for which equality holds. It will turn out that we only have to look in neighboring towers, *i.e.*, in the m-th and $(m+2)$-th Witt towers for some integer m. Notice that such towers have opposite normalized discriminant. We will need the following lemma.

Lemma 3.1 *Let* t *be a positive integer, and let* $l \in \frac{1}{2}\mathbb{Z}$ *be such that* $2l \equiv t (\mathrm{mod}\, 2)$ *and* $\frac{t}{2} > l > -\frac{t}{2}$. *Let* $\chi_l = \det^l \in \tilde{U}(p,q)\widehat{\ }$, *and suppose that* $\theta_{k+t,k}(\chi_l) \neq 0$. *Then* $k \geq p+q$. *In particular,* $\theta_{r,s}(\mathbb{1}) \neq 0 \Rightarrow r = s$ *or* $r > p+q$ *and* $s > p+q$ *(stable range).*

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Proof If $\theta_{k+t,k}(\chi_l)\neq 0$, then the K-type with weight $(l,\dots,l;l,\dots,l)$ is $(k+t,k)$-harmonic.

$$(l,\dots,l;l,\dots,l)=\Big(\frac{t}{2},\dots,\frac{t}{2};\frac{-t}{2},\dots,\frac{-t}{2}\Big)+\Big(l-\frac{t}{2},\dots,l-\frac{t}{2};l+\frac{t}{2},\dots,l+\frac{t}{2}\Big).$$

By assumption, $l-\frac{t}{2}<0$ and $l+\frac{t}{2}>0$, so in the space of joint harmonics, $(l,\dots,l;l,\dots,l)$ corresponds to the K-type with highest weight

$$\Big(\frac{p-q}{2},\dots,\frac{p-q}{2};\frac{q-p}{2},\dots,\frac{q-p}{2}\Big)$$
$$+\Big(\overbrace{0,\dots,0}^{k+t};\overbrace{\underbrace{l+\frac{t}{2},\dots,l+\frac{t}{2}}_{q},0,\dots,0,\underbrace{l-\frac{t}{2},\dots,l-\frac{t}{2}}_{p}}^{k}\Big),$$

and we must have that $k\geq p+q$. ■

Proposition 3.2 *Let* $\pi\in\widehat{\tilde U(p,q)}_{\text{genuine}}$. *Suppose* r, s, r', *and* s' *are nonnegative integers such that* $r-s\neq r'-s'$. *If* $\theta_{r,s}(\pi)\neq 0$ *and* $\theta_{r',s'}(\pi)\neq 0$, *then*

$$r+s+r'+s'\geq 2p+2q+2. \tag{3.3}$$

If equality holds in (3.3), then $|r-s-(r'-s')|=2$.

Proof By Proposition 2.9, $\theta_{r',s'}(\pi)\neq 0\Rightarrow\theta_{s',r'}(\pi^*)\neq 0$. So we have

$$\begin{aligned}\theta_{r,s}(\pi)\neq 0\quad\text{and}\quad&\theta_{r',s'}(\pi)\neq 0\\ &\Rightarrow \mathrm{Hom}_{\tilde U(p,q)\times\tilde U(p,q)}(\omega_{p,q,r,s}\otimes\omega_{p,q,s',r'},\pi\otimes\pi^*)\neq 0\\ &\Rightarrow \mathrm{Hom}_{U(p,q)}(\omega_{p,q,r+s',s+r'},\pi\otimes\pi^*)\neq 0\quad\text{(by Lemma 2.8)}\\ &\Rightarrow \mathrm{Hom}_{U(p,q)}(\omega_{p,q,r+s',s+r'},\mathbb{1})\neq 0\\ &\Rightarrow \theta_{r+s',s+r'}(\mathbb{1})\neq 0.\end{aligned}$$

By Lemma 3.1, and since we are assuming that $r-s\neq r'-s'$, we have that $r+s'\geq p+q$, $r'+s\geq p+q$, and $r+s'\neq r'+s$. The trivial representation of $\tilde U(p,q)$ occurs only with groups of even rank, so that $r+s+r'+s'\geq 2p+2q+2$.

To prove the second part of the statement, notice that if $r+s+r'+s'=2p+2q+2$, then

$$\begin{aligned}r-s-r'+s'&=2p+2q+2-2s-2r'\leq 2r'+2s+2-2s-2r'=2,\quad\text{and}\\ r'-s'-r+s&=2p+2q+2-2s'-2r\leq 2s+2r'+2-2s'-2r=2.\end{aligned}$$
■

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If the conjecture is true, then for any π, the earliest occurrence has to be no later than a group of rank $p+q+1$. The next result confirms that this is indeed true if π is a discrete series representation.

Proposition 3.4 **(First Occurrence of Discrete Series)** *Let k be a nonnegative integer, and let $\pi \in \tilde{U}(p,q)\widehat{\ }_{\text{genuine}}$ be a discrete series representation with Harish-Chandra parameter*

(3.5a)
$$\lambda = \Bigl(a_1,\dots,a_x,\underbrace{\frac{k-1}{2},\frac{k-3}{2},\dots,\frac{-k+1}{2}}_{k},b_1,\dots,b_y;c_1,\dots,c_z,d_1,\dots,d_w\Bigr) \quad \textit{or}$$

$$(3.5b)\qquad \lambda = \Bigl(a_1,\dots,a_x,b_1,\dots,b_y;c_1,\dots,c_z,\underbrace{\frac{k-1}{2},\frac{k-3}{2},\dots,\frac{-k+1}{2}}_{k},d_1,\dots,d_w\Bigr)$$

with $a_x, -b_1, c_z, -d_1 \geq \frac{k+1}{2}$. Let $r = x+w$ and $s = y+z$.

(i) If $k = 0$ and $a_i, b_i, c_i, d_i \in \mathbb{Z}$ for all i, then $\theta_{r+1,s}(\pi) = \pi'$ and $\theta_{r,s+1}(\pi) = \pi''$, where π' and π'' are the discrete series representations with Harish-Chandra parameters

$$\lambda' = (a_1,\dots,a_x,0,d_1,\dots,d_w;c_1,\dots,c_z,b_1,\dots,b_y)$$

and

$$\lambda'' = (a_1,\dots,a_x,d_1,\dots,d_w;c_1,\dots,c_z,0,b_1,\dots,b_y)$$

respectively, and these are the earliest occurrences.

(ii) In all other cases, $\theta_{r,s}(\pi) = \pi'$, where π' is the discrete series representation of $\tilde{U}(r,s)$ with Harish-Chandra parameter

$$\lambda' = (a_1,\dots,a_x,d_1,\dots,d_w;c_1,\dots,c_z,b_1,\dots,b_y).$$

Moreover, if k is chosen to be maximal in (3.5) then $\theta_{r-1,s-1}(\pi) = 0$, i.e., this is the first occurrence.

Proof Part (i) follows from Theorem 2.7(b) and Proposition 3.2. For the first part of (ii), apply Theorem 2.7(b) to π'. We will prove the "moreover" part later independently, using Proposition 3.2, but we include a more direct proof here. Because of Corollary 2.10, it is sufficient to consider the Case (3.5a). Let σ be the lowest K-type (LKT) of π. We will show that σ is of minimal $(r-1,s-1)$-degree in π but is not $(r-1,s-1)$-harmonic. Theorem 2.2 then implies non-occurrence. Notice that the degrees for (r,s) and $(r-1,s-1)$ are equal. The highest weight Λ of σ is given by $\Lambda = \lambda + \rho_n - \rho_c$, where ρ_n and ρ_c are one half the sums of the noncompact and compact roots determined by λ respectively (see *e.g.* [Kn]). Define

$$\begin{aligned}
&\text{for} \quad 1 \leq i \leq x, \quad \alpha_i = \sharp\{j \leq z : a_i < c_j\},\\
&\text{for} \quad 1 \leq i \leq y, \quad \beta_i = \sharp\{j \leq w : b_i < d_j\},\\
&\text{for} \quad 1 \leq i \leq z, \quad \gamma_i = \sharp\{j \leq x : c_i < a_j\}, \quad \text{and}\\
&\text{for} \quad 1 \leq i \leq w, \quad \delta_i = \sharp\{j \leq y : d_i < b_j\}.
\end{aligned}$$

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Then a straightforward calculation yields that

$$\text{(3.6a)} \qquad \Lambda = \Big(\frac{r-s}{2},\dots,\frac{r-s}{2};\frac{s-r}{2},\dots,\frac{s-r}{2}\Big) + (A_1,\dots,A_x,Z_1,\dots,Z_k,B_1,\dots,B_y;C_1,\dots,C_z,D_1,\dots,D_w),$$

where

$$A_i = a_i - \alpha_i + z - x - \frac{k}{2} + i - \frac{1}{2} \geq 0 \quad (\text{since } a_i \geq x - i + \frac{k+1}{2} \text{ and } \alpha_i \leq z),$$

$$Z_i = 0,$$

$$\text{(3.6b)} \qquad B_i = b_i - \beta_i + \frac{k}{2} + i - \frac{1}{2} \leq 0,$$

$$C_i = c_i - \gamma_i + x - z + \frac{k}{2} + i - \frac{1}{2} \geq 0,$$

$$D_i = d_i - \delta_i - \frac{k}{2} + i - \frac{1}{2} \leq 0.$$

The $(r-1, s-1)$-degree of σ is therefore

$$\text{(3.7)} \qquad \sum_{i=1}^{x} A_i - \sum_{i=1}^{y} B_i + \sum_{i=1}^{z} C_i - \sum_{i=1}^{w} D_i.$$

Now let τ be an arbitrary K-type of π. Then the highest weight of τ is of the form

$$\Lambda + \sum_{\alpha_i \in \Delta^+} \xi_i \alpha_i,$$

where Δ^+ is the system of positive roots determined by λ, and the ξ_i are nonnegative integers (see *e.g.* [Kn]). An induction argument on the number of roots shows that if

$$\sum_{\alpha_i \in \Delta^+} \xi_i \alpha_i = (k_1,\dots,k_x,z_1,\dots,z_k,l_1,\dots,l_y;m_1,\dots,m_z,n_1,\dots,n_w)$$

then

$$\text{(3.8)} \qquad \sum_{i=1}^{x} k_i - \sum_{i=1}^{y} l_i + \sum_{i=1}^{z} m_i - \sum_{i=1}^{w} n_i \geq 0.$$

The (r,s)-degree of τ is

$$\begin{aligned} &\sum_{i=1}^{x} |A_i + k_i| + \sum_{i=1}^{k} |z_i| + \sum_{i=1}^{y} |B_i + l_i| + \sum_{i=1}^{z} |C_i + m_i| + \sum_{i=1}^{w} |D_i + n_i| \\ &\qquad \geq \sum_{i=1}^{x} (A_i + k_i) + \sum_{i=1}^{y} (-B_i - l_i) + \sum_{i=1}^{z} (C_i + m_i) + \sum_{i=1}^{w} (-D_i - n_i) \\ &\qquad \geq \sum_{i=1}^{x} A_i - \sum_{i=1}^{y} B_i + \sum_{i=1}^{z} C_i - \sum_{i=1}^{w} D_i \quad \text{by (3.8)}, \end{aligned}$$

which is the degree of σ. Hence σ is indeed of minimal degree in π.

Now if k in (3.5a) was chosen to be maximal, then we have that $a_x > \frac{k+1}{2}$ or $b_1 < \frac{-k-1}{2}$, so that

$$(3.9)\quad A_x > \frac{k+1}{2} - z + z - x - \frac{k}{2} + x - \frac{1}{2} = 0 \quad \text{or} \quad B_1 < \frac{-k-1}{2} + \frac{k}{2} + 1 - \frac{1}{2} = 0.$$

Also, $c_z, -d_1 \geq \frac{k+1}{2}$ implies that $C_z \geq k$ and $D_1 \leq -k$. If $k = 0$, then $c_z \geq 1$ or $d_1 \leq -1$ since otherwise we would have Case (3.5b) with k not maximal, so that

$$(3.10)\qquad C_z > 0 \quad \text{or} \quad D_1 < 0.$$

Moreover,

$$(3.11)\qquad A_x = 0 \Rightarrow a_x = \frac{k+1}{2} \Rightarrow c_z > \frac{k+1}{2} \Rightarrow C_z > 0,$$

and similarly

$$(3.12)\qquad B_1 = 0 \Rightarrow D_1 < 0.$$

It follows from (3.9)–(3.12) that

$$C_z > 0 \quad \text{and} \quad B_1 < 0, \quad \text{or} \quad A_x > 0 \quad \text{and} \quad D_1 < 0.$$

Lemma 2.5 now implies that σ is not $(r-1, s-1)$-harmonic. ■

Corollary 3.13 *Conjecture 1.2 is true for π as in Proposition 3.4(i).*

Theorem 3.14 *Let $\pi \in \widehat{\tilde{U}(p,q)}_{\text{genuine}}$ be a discrete series as in Proposition 3.4(ii).*

(a) If λ is given by (3.5a) then

$$(3.15\text{i})\qquad \theta_{r+k+2,s+k}(\pi) \neq 0 \Leftrightarrow a_x \geq \frac{k+3}{2} \quad \textit{and} \quad d_1 \leq -1;$$

$$(3.15\text{ii})\qquad \theta_{r+k,s+k+2}(\pi) \neq 0 \Leftrightarrow b_1 \leq -\frac{k+3}{2} \quad \textit{and} \quad c_z \geq 1.$$

(b) If λ is given by (3.5b) then

$$(3.15\text{iii})\qquad \theta_{r+k+2,s+k}(\pi) \neq 0 \Leftrightarrow a_x \geq 1 \quad \textit{and} \quad d_1 \leq -\frac{k+3}{2};$$

$$(3.15\text{iv})\qquad \theta_{r+k,s+k+2}(\pi) \neq 0 \Leftrightarrow b_1 \leq -1 \quad \textit{and} \quad c_z \geq \frac{k+3}{2}.$$

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Proof Recall that the Harish-Chandra parameters of discrete series representations of $\tilde{U}(p,q)$ which are contragredient representations are of the form

$$(x_1,\dots,x_p;y_1,\dots,y_q) \quad \text{and} \quad (-x_p,-x_{p-1},\dots,-x_1;-y_q,\dots,-y_1).$$

Because of Proposition 2.9 and Corollary 2.10, it is therefore enough to prove (3.15i). We will show that if $a_x \geq \frac{k+3}{2}$ and $d_1 \leq -1$ then the LKT of π is $(r+k+2,s+k)$-harmonic, so that by Theorem 2.7(a), $\theta_{r+k+2,s+k}(\pi) \neq 0$. The highest weight of the LKT σ of π (see (3.6)) is given by

$$\begin{aligned}(3.16\text{a}) \qquad &\Big(\frac{r+2-s}{2},\dots,\frac{r+2-s}{2};\frac{s-r-2}{2},\dots,\frac{s-r-2}{2}\Big)\\ &\quad + \big(A'_1,\dots,A'_x,-1,\dots,-1,B'_1,\dots,B'_y;C'_1,\dots,C'_z,D'_1,\dots,D'_w\big),\end{aligned}$$

where

$$\begin{aligned}(3.16\text{b}) \qquad A'_i &= a_i-\alpha_i+z-x-\frac{k}{2}+i-\frac{3}{2},\\ B'_i &= b_i-\beta_i+\frac{k}{2}+i-\frac{3}{2},\\ C'_i &= c_i-\gamma_i+x-z+\frac{k}{2}+i+\frac{1}{2},\\ D'_i &= d_i-\delta_i-\frac{k}{2}+i+\frac{1}{2},\end{aligned}$$

with $\alpha_i,\beta_i,\gamma_i,\delta_i$ as in the proof of Proposition 3.4.

It is easy to calculate that if $a_x \geq \frac{k+3}{2}$ then $A'_i \geq 0$ for all i, $B'_i \leq -1$ for all i, $C'_i \geq 1+k$ for all i, and if $d_1 \leq -1$ then $D'_i \leq 0$ for all i. Lemma 2.5 now implies that σ is $(r+k+2,s+k)$-harmonic.

Now suppose conversely that $a_x = \frac{k+1}{2}$. We need to show that $\theta_{r+k+2,s+k}(\pi)=0$. Consider the discrete series representation $\pi \otimes \det^{-1}$ of $\tilde{U}(p,q)$, which has Harish-Chandra parameter

$$\begin{aligned}\Big(&a_1-1,\dots,a_{x-1}-1,\frac{k-1}{2},\frac{k-3}{2},\dots,\frac{-k+1}{2},\frac{-k-1}{2},b_1-1,\dots,b_y-1;\\ &c_1-1,\dots,c_z-1,d_1-1,\dots,d_w-1\Big).\end{aligned}$$

Now $a_x = \frac{k+1}{2}$ implies that $c_z \geq \frac{k+3}{2}$, so that $c_z-1>0$, and Proposition 3.4 yields that $\theta_{r-1,s+1}(\pi\otimes\det^{-1}) \neq 0$. By Proposition 2.9, $\theta_{s+1,r-1}(\pi^*\otimes\det) \neq 0$. Now

$$\begin{aligned}\theta_{r+k+2,s+k}(\pi)\neq 0 &\Rightarrow \operatorname{Hom}_{\tilde{U}(p,q)\times\tilde{U}(p,q)}\big(\omega_{p,q,r+k+2,s+k}\otimes\omega_{p,q,s+1,r-1},\pi\otimes(\pi^*\otimes\det)\big)\neq 0\\ &\Rightarrow \operatorname{Hom}_{U(p,q)}(\omega_{p,q,r+s+k+3,r+s+k-1},\pi\otimes\pi^*\otimes\det)\neq 0 \quad \text{(by Lemma 2.8)}\\ &\Rightarrow \operatorname{Hom}_{U(p,q)}(\omega_{p,q,r+s+k+3,r+s+k-1},\det)\neq 0\\ &\Rightarrow \theta_{p+q+3,p+q-1}(\det)\neq 0 \quad (\text{since } p+q=r+s+k).\end{aligned}$$

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But by Lemma 3.1, $\theta_{p+q+3,p+q-1}(\det) = 0$, so we must have that $\theta_{r+k+2,s+k}(\pi) = 0$.

A similar argument shows that if $d_1 = -\frac{1}{2}$ (and $k = 0$) then $\theta_{r-1,s+1}(\pi \otimes \det) \neq 0$, and then $\theta_{r+2,s}(\pi) \neq 0$ would imply that $\theta_{p+q+3,p+q-1}(\det^{-1}) \neq 0$, which contradicts Lemma 3.1. So again, $\theta_{r+k+2,s+k}(\pi) = 0$. ■

Remark 3.17 The case $k = 0$, $a_x, -b_1, c_z, -d_1 \geq \frac{3}{2}$ in Theorem 3.14 follows from Theorem 2.7(b). In this case, $\theta_{r+2,s}(\pi)$ and $\theta_{r,s+2}(\pi)$ are the discrete series representations with Harish-Chandra parameters

$$\Big(a_1, \dots, a_x, \frac{1}{2}, -\frac{1}{2}, d_1, \dots, d_w; c_1, \dots, c_z, b_1, \dots, b_y\Big)$$

and

$$\Big(a_1, \dots, a_x, d_1, \dots, d_w; c_1, \dots, c_z, \frac{1}{2}, -\frac{1}{2}, b_1, \dots, b_y\Big)$$

respectively.

Notice that if k is chosen to be maximal in (3.5), then at least one of the conditions in Theorem 3.14 is satisfied, so that we have

Corollary 3.18 *Let $\pi \in \widehat{\tilde{U}(p,q)}_{\mathrm{genuine}}$ be a genuine discrete series representation. Then*

$$m_+(\pi) + m_-(\pi) = 2p + 2q + 2.$$

4 The Theta Lifts

It would be of interest to identify the Langlands parameters of all the representations which occur as the earliest theta lifts π' of a discrete series representation π of $\tilde{U}(p,q)$ as in Theorem 3.14. Using Theorems 2.11 and 2.7, we know their infinitesimal characters, that they are all unitary, and that they contain the K-type which corresponds to the LKT of π in the space of joint harmonics $\mathcal{H}$. For example, consider the discrete series π of $\tilde{U}(2,1)$ with Harish-Chandra parameter $(3, 0; -2)$. By Theorem 3.14, $\pi' = \theta_{5,1}(\pi) \neq 0$. This representation has (singular) infinitesimal character $(3, 1, 0, 0, -1, -2)$ and contains the K-type with highest weight $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}; -\frac{3}{2})$. In Example (2) at the end of this section, we identify this representation explicitly, by going through the short list of representations with this infinitesimal character and ruling out all but one of them. This is of course in general not possible, but there are candidates for these theta lifts which nicely fit all the requirements. After introducing some notation, we shall give their Langlands parameters. In light of Proposition 2.9 and Corollary 2.10, we may restrict our attention to Case (3.15i) of Theorem 3.14; all results for this case have obvious analogues for the other three cases, which we leave to the reader to formulate.

Definition 4.1 (a) For n a positive integer, let $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be an n-tuple of integers, and $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ an n-tuple of complex numbers. Then $\chi(\mu, \nu)$ is the character of $(\mathbb{C}^\times)^n$ given by

$$(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n}) \mapsto \prod_{i=1}^{n} r_i^{\nu_i} e^{i\mu_i\theta_i}.$$

(b) For $k \le \min\{p,q\}$ let $P = MN$ be a cuspidal parabolic subgroup of $\tilde{U}(p,q)$ with Levi factor $M \cong \tilde{U}(p-k,q-k) \times (\mathbb{C}^\times)^k$. For $\rho \in \tilde{U}(p-k,q-k)\widehat{}$ and $\chi = \chi(\mu,\nu)$ a character of $(\mathbb{C}^\times)^k$ we let (using normalized induction)

$$I(p,q,k,\rho,\mu,\nu) = \mathrm{Ind}_P^{\tilde{U}(p,q)}(\rho \otimes \chi \otimes \mathbb{1}).$$

(Notice that the composition series of $I(p,q,k,\rho,\mu,\nu)$ is uniquely defined.)

If ρ is a discrete series representation, we call $I(p,q,k,\rho,\mu,\nu)$ a generalized principal series representation (see [SV]).

A genuine limit of discrete series representation $\pi = \pi(\lambda,\Psi)$ of $\tilde{U}(p,q)$ (the $\det^{\frac{N}{2}}$-cover) is given by an element $\lambda \in i\mathfrak{t}_0^*$ of the form

$$\lambda = (\underbrace{x_1,\dots,x_1}_{k_1},\dots,\underbrace{x_m,\dots,x_m}_{k_m};\underbrace{x_1,\dots,x_1}_{l_1},\dots,\underbrace{x_m,\dots,x_m}_{l_m})$$

with $x_i \in \mathbb{Z} + \frac{p+q+N+1}{2}$ and $|k_i - l_i| \le 1$ for all i, and a system of positive roots $\Psi \subset \Delta(\mathfrak{g},\mathfrak{t})$ with the property that $\alpha \in \Psi \Rightarrow \langle \alpha,\lambda\rangle \ge 0$, and if α is a simple root in Ψ and $\langle\alpha,\lambda\rangle = 0$ then α is noncompact (see [V2], [P1]). With obvious notation, the compact roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ are

$$\Delta_c = \{e_i - e_j : 1 \le i,j \le p\} \cup \{f_i - f_j : 1 \le i,j \le q\},$$

and the noncompact roots are

$$\Delta_n = \{\pm(e_i - f_j) : 1 \le i \le p \text{ and } 1 \le j \le q\}.$$

We are now ready to describe our candidates for the theta lifts. Let $\pi \in \tilde{U}(p,q)\widehat{}_{\text{genuine}}$ be a discrete series representation with Harish-Chandra parameter given by (3.5a), with $\tilde{U}(p,q)$ the $\det^{\frac{p+q+k}{2}}$-cover. Assume that $a_x \ge \frac{k+3}{2}$ and $d_1 \le -1$. Let r and s be as in Proposition 3.4. Consider the induced representation $I(r+k+2, s+k, [\frac{k+1}{2}], \rho, \mu, \nu)$ given by the following data.

(4.2a) If k is odd then $\mu = (-1,\dots,-1)$; $\nu = (1,3,\dots,k)$, and $\rho = \rho(\lambda',\Psi')$ is the limit of discrete series of $\tilde{U}(r+\frac{k-1}{2}+2, s+\frac{k-1}{2})$ with $\lambda' = (a_1,\dots,a_x,\frac{k+1}{2},\frac{k-1}{2},\dots,1,0,d_1,\dots,d_w;c_1,\dots,c_z,-1,-2,\dots,\frac{-k+1}{2},b_1,\dots,b_y)$, and $f_z - e_{x+1} \in \Psi'$.

(4.2b) If k is even then $\mu = (-1,\dots,-1)$; $\nu = (2,4,\dots,k)$, and $\rho = \rho(\lambda',\Psi')$ is the limit of discrete series of $\tilde{U}(r+\frac{k}{2}+2, s+\frac{k}{2})$ with $\lambda' = (a_1,\dots,a_x,\frac{k+1}{2},\frac{k-1}{2},\dots,\frac{1}{2},-\frac{1}{2},d_1,\dots,d_w;c_1,\dots,c_z,-\frac{1}{2},-\frac{3}{2},\dots,\frac{-k+1}{2},b_1,\dots,b_y)$, and $f_z - e_{x+1} \in \Psi'$, $e_{x+\frac{k}{2}+2} - f_{z+1} \in \Psi'$.

Let π' be the unique (see [V2]) LKT constituent of $I(r+k+2, s+k, [\frac{k+1}{2}], \rho, \mu, \nu)$.

Conjecture 4.3 *Let π and π' be as above. Then $\pi' = \theta_{r+k+2,s+k}(\pi)$.*

We give some evidence for the truth of this conjecture.

Lemma 4.4 *Let π be a genuine discrete series representation of $\tilde{U}(p,q)$, and suppose $\theta_{r_0,s_0}(\pi) \neq 0$ is one of the first occurrences of π as in Proposition 3.4 or Theorem 3.14. Then the* LKT *σ of π is of minimal (r_0,s_0)-degree in π.*

Proof We already proved this for the earliest first occurrence in Proposition 3.4(ii), and the Case (i) is similar. So suppose we are in the setting of Theorem 3.14, the Harish-Chandra parameter of π is given by (3.5a) with k maximal, and we have $a_x \geq \frac{k+3}{2}$ and $d_1 \leq -1$, so that $\theta_{r+k+2,s+k}(\pi) \neq 0$. Then the highest weight Λ of σ is

$$\Big(\frac{r+2-s}{2},\dots,\frac{r+2-s}{2};\frac{s-r-2}{2},\dots,\frac{s-r-2}{2}\Big)$$
$$+ (A'_1,\dots,A'_x,-1,\dots,-1,B'_1,\dots,B'_y;C'_1,\dots,C'_z,D'_1,\dots,D'_w),$$

where $A'_i \geq 0$, $B'_i \leq -1$, $C'_i \geq 1$, and $D'_i \leq 0$ for all i, as in the proof of Theorem 3.14. The $(r+k+2,s+k)$-degree of σ is therefore

$$\sum_{i=1}^{x} A'_i + k - \sum_{i=1}^{y} B'_i + \sum_{i=1}^{z} C'_i - \sum_{i=1}^{w} D'_i.$$

Any other K-type τ of π has highest weight of the form

$$\Lambda + (k_1,\dots,k_x,z_1,\dots,z_k,l_1,\dots,l_y;m_1,\dots,m_z,n_1,\dots,n_w),$$

where

$$\sum_{i=1}^{x} k_i - \sum_{i=1}^{k} z_i - \sum_{i=1}^{y} l_i + \sum_{i=1}^{z} m_i - \sum_{i=1}^{w} n_i \geq 0 \tag{4.5}$$

(see the proof of Proposition 3.4). The $(r+k+2,s+k)$-degree of τ is

$$\begin{aligned}\sum_{i=1}^{x} |A'_i + k_i| &+ \sum_{i=1}^{k} |-1+z_i| + \sum_{i=1}^{y} |B'_i + l_i| + \sum_{i=1}^{z} |C'_i + m_i| + \sum_{i=1}^{w} |D'_i + n_i| \\ &\geq \sum_{i=1}^{x} (A'_i + k_i) + \sum_{i=1}^{k} (1-z_i) + \sum_{i=1}^{y} (-B'_i - l_i) + \sum_{i=1}^{z} (C'_i + m_i) + \sum_{i=1}^{w} (-D'_i - n_i) \\ &\geq \sum_{i=1}^{x} A'_i + k - \sum_{i=1}^{y} B'_i + \sum_{i=1}^{z} C'_i - \sum_{i=1}^{w} D'_i \quad \text{by (4.5)},\end{aligned}$$

which is the degree of σ. Hence σ is indeed of minimal degree in π. The other cases are similar. ■

Corollary 4.6 *Let π be as in Conjecture 4.3. Then the K-type which corresponds to the* LKT *of π in the space of joint harmonics for the dual pair $\big(U(p,q), U(r+k+2,s+k)\big)$ is of minimal (p,q)-degree in $\theta_{r+k+2,s+k}(\pi)$.*

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Proof This follows using Theorem 2.2. ∎

Proposition 4.7 *Let π and π' be as in Conjecture 4.3. Then*

(a) the infinitesimal characters of π' and $\theta_{r+k+2,s+k}(\pi)$ coincide; and
(b) the unique LKT *of π' is of minimal (p,q)-degree in π', and corresponds to the* LKT *σ of π in $\mathcal{H}$ for the dual pair $\big(U(p,q),U(r+k+2,s+k)\big)$.*

Proof The infinitesimal character of π' is given by

$$\begin{aligned}&\Big(\lambda',\Big(-\frac{1}{2}+\frac{1}{2},-\frac{1}{2}+\frac{3}{2},\dots,-\frac{1}{2}+\frac{k}{2};-\frac{1}{2}-\frac{1}{2},-\frac{1}{2}-\frac{3}{2},\dots,-\frac{1}{2}-\frac{k}{2}\Big)\Big)\in(\mathfrak{t}')^*\\ &\quad=\Big(\lambda',\Big(0,1,2,\dots,\frac{k-1}{2};-1,-2,\dots,-\frac{k+1}{2}\Big)\Big)\\ &\quad\sim(\lambda,\rho_{k+2})\quad\text{by } W(\mathfrak{g}',\mathfrak{t}')\end{aligned}$$

if k is odd, and

$$\begin{aligned}&\Big(\lambda',\Big(-\frac{1}{2}+1,-\frac{1}{2}+2,\dots,-\frac{1}{2}+\frac{k}{2};-\frac{1}{2}-1,-\frac{1}{2}-2,\dots,-\frac{1}{2}-\frac{k}{2}\Big)\Big)\\ &\quad=\Big(\lambda',\Big(\frac{1}{2},\frac{3}{2},\dots,\frac{k-1}{2};-\frac{3}{2},-\frac{5}{2},\dots,-\frac{k+1}{2}\Big)\Big)\\ &\quad\sim(\lambda,\rho_{k+2})\end{aligned}$$

if k is even. Part (a) follows using Theorem 2.11.

We prove part (b) for the case that k is odd; the other case is similar. First we compute the highest weight(s) of the LKT('s) of π'. The algorithm given in Section 3 of [P1] yields all LKT's of the standard module I_d induced from a certain discrete series, which contains the induced module (4.2a) as a summand, along with all generalized principal series similarly induced from the limits of discrete series with the same λ' (but different Ψ') as ρ. It will turn out that the number of these K-types is the same as the number of distinct summands, so that (4.2a) must have indeed a unique LKT σ'. The next task will be to pick out this K-type, which is also the LKT of π'.

The highest weights of the LKT's of I_d are those of the form

$$\lambda'+\mu+\rho(\mathfrak{u}\cap\mathfrak{p})-\rho(\mathfrak{u}\cap\mathfrak{k})+\delta_L, \tag{4.8}$$

where $\mu=d\chi|_{\mathfrak{t}}$, $\mathfrak{u}(r+k+2,s+k)=\mathfrak{k}+\mathfrak{p}$ is the Cartan decomposition, $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ is the theta stable parabolic subalgebra determined by $\lambda'+\mu$, $\rho(\cdot)$ denotes one half the sums of the respective roots, and δ_L is a fine weight (see [P1] and [V1]) for L. In our case, $\lambda'+\mu$ is given by

$$\begin{aligned}&\Big(a_1,\dots,a_x,\frac{k+1}{2},\frac{k-1}{2},\dots,1,0,\underbrace{-\frac{1}{2},\dots,-\frac{1}{2}}_{\frac{k+1}{2}},d_1,\dots,d_w;\\ &\qquad c_1,\dots,c_z,\underbrace{-\frac{1}{2},\dots,-\frac{1}{2}}_{\frac{k+1}{2}},-1,-2,\dots,\frac{-k+1}{2},b_1,\dots,b_y\Big).\end{aligned}$$

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Calculating the highest weights of the LKT's of I_d, we obtain

$$\Big(A_1,\dots,A_x,\underbrace{Y_1,Y,\dots,Y}_{\frac{k+3}{2}},\underbrace{Z,\dots,Z}_{\frac{k+1}{2}},D_1,\dots,D_w;$$

$$C_1,\dots,C_z,\underbrace{Z',\dots,Z'}_{\frac{k+1}{2}},\underbrace{Y',\dots,Y'}_{\frac{k-1}{2}},B_1,\dots,B_y\Big)+\delta_L,$$

where (with α_i, β_i, γ_i, and δ_i defined as in the proof of Proposition 3.4)

$$A_i = a_i - \alpha_i + i + \frac{z+y-x-w}{2} - \frac{3}{2};$$

$$Y_1 = \begin{cases} \frac{x+y-z-w+k}{2} + \frac{1}{2} & \text{if } c_z = \frac{k+1}{2}, \\ \frac{x+y-z-w+k}{2} & \text{if } c_z > \frac{k+1}{2}; \end{cases}$$

$$Y = \frac{x+y-z-w+k}{2};$$

$$Z = \frac{x+y-z-w+k}{2};$$

$$D_i = d_i - \delta_i + i + \frac{x+y-z-w}{2} + \frac{1}{2};$$

$$C_i = c_i - \gamma_i + i + \frac{x-y-z+w}{2} + \frac{1}{2} \quad \text{for } i \le z-1;$$

$$C_z = \begin{cases} c_z - \gamma_z + z + \frac{x-y-z+w}{2} & \text{if } c_z = \frac{k+1}{2}, \\ c_z - \gamma_z + z + \frac{x-y-z+w}{2} + \frac{1}{2} & \text{if } c_z > \frac{k+1}{2}; \end{cases}$$

$$Z' = \frac{-x-y+z+w-k}{2} - 1;$$

$$Y' = \frac{-x-y+z+w-k}{2} - 1;$$

$$B_i = b_i - \beta_i + i + \frac{-x-y+z+w}{2} - \frac{3}{2}.$$

The fine weights δ_L are

(4.9) $$\Big(0,\dots,0,\underset{x+1}{\pm\frac{1}{2}},0,\dots,0;0,\dots,0,\underset{z}{\mp\frac{1}{2}},0,\dots,0\Big) \quad \text{if } c_z = \frac{k+1}{2},$$

and 0 if $c_z > \frac{k+1}{2}$. Consequently, I_d has one LKT if $c_z > \frac{k+1}{2}$ (which then must be σ'), and two if $c_z = \frac{k+1}{2}$, corresponding to the two limit of discrete series representations $\rho = \rho(\lambda', \Psi')$ and $\tilde{\rho} = \tilde{\rho}(\lambda', \widetilde{\Psi'})$, where $\widetilde{\Psi'}$ is the system of positive roots containing $e_{x+1} - f_z$. The highest weight Λ' of the LKT of ρ may be easily computed using the Blattner formula,

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and we find that if ξ is the weight obtained by choosing

$$\delta_L = \begin{cases} \Big(0,\dots,0,\underset{x+1}{-\tfrac12},0,\dots,0;0,\dots,0,\underset{z}{+\tfrac12},0,\dots,0\Big) & \text{if } c_z = \frac{k+1}{2},\\ 0 & \text{otherwise,}\end{cases}$$

then ξ restricts to Λ' on $\mathfrak{u}(1)^{r+\frac{k-1}{2}+2} \oplus \mathfrak{u}(1)^{s+\frac{k-1}{2}}$. Consequently, the K-type (for $\tilde{U}(r+k+2,s+k)$) with highest weight ξ contains the LKT of ρ, and Frobenius reciprocity now implies that the induced representation (4.2a) contains this K-type. It follows that σ' has highest weight ξ.

Recall that $p = x+k+y$ and $q = z+w$. Therefore,

$$\begin{aligned}\xi = {}& \Big(\frac{p-q}{2},\dots,\frac{p-q}{2};\frac{q-p}{2},\dots,\frac{q-p}{2}\Big)\\ &+ (A_1',\dots,A_x',\underbrace{0,\dots,0}_{k+2},D_1',\dots,D_w';C_1',\dots,C_z',\underbrace{-1,\dots,-1}_{k},B_1',\dots,B_w'),\end{aligned}$$

where the A_i', B_i', C_i', and D_i' are given by (3.16b). A quick check with Lemma 2.5 confirms that σ' corresponds to σ in $\mathcal{H}$. It only remains to show that σ' is of minimal (p,q)-degree in π'. It follows from Theorem 6.3.12 of [V1] and Theorem 10.44 of [KV] (see Lemma 5.1.1 of [P1]) that the highest weight of any K-type of I_d (and hence of π') is of the form

$$\delta + \sum_\alpha n_\alpha \alpha,$$

with δ the highest weight of a LKT of I_d, n_α nonnegative integers, and the sum running over roots in $\Delta(\mathfrak{l}) \cup \Delta(\mathfrak{u})$. Such a sum is of the form

$$\begin{aligned}&(k_1,\dots,k_x,\xi_1,\dots,\xi_{\frac{k+3}{2}},\zeta_1,\dots,\zeta_{\frac{k+1}{2}},n_1,\dots,n_w;\\ &m_1,\dots,m_z,\varphi_1,\dots,\varphi_{\frac{k+1}{2}},\psi_1,\dots,\psi_{\frac{k-1}{2}},l_1,\dots,l_y),\end{aligned} \tag{4.10}$$

with (check by induction on the number of roots in the sum)

$$\sum_{i=1}^{x} k_i + \sum_{i=1}^{\frac{k+3}{2}} \xi_i - \sum_{i=1}^{\frac{k+1}{2}} \zeta_i - \sum_{i=1}^{w} n_i + \sum_{i=1}^{z} m_i - \sum_{i=1}^{\frac{k+1}{2}} \varphi_i - \sum_{i=1}^{\frac{k-1}{2}} \psi_i - \sum_{i=1}^{y} l_i \geq 0. \tag{4.11}$$

The degree of σ' is

$$\sum_{i=1}^{x} A_i' - \sum_{i=1}^{w} D_i' + \sum_{i=1}^{z} C_i' + k - \sum_{i=1}^{y} B_i'.$$

The degree of a K-type of the above form with $\delta = \xi$ is

$$\begin{aligned}&\sum_{i=1}^{x} |A_i'+k_i| + \sum_{i=1}^{\frac{k+3}{2}} |\xi_i| + \sum_{i=1}^{\frac{k+1}{2}} |\zeta_i| + \sum_{i=1}^{w} |D_i'+n_i| + \sum_{i=1}^{z} |C_i'+m_i|\\ &\quad + \sum_{i=1}^{\frac{k+1}{2}} |-1+\varphi_i| + \sum_{i=1}^{\frac{k-1}{2}} |-1+\psi_i| + \sum_{i=1}^{y} |B_i'+l_i|\end{aligned}$$

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$$\geq \sum_{i=1}^{x}(A_i' + k_i) + \sum_{i=1}^{\frac{k+3}{2}} \xi_i + \sum_{i=1}^{\frac{k+1}{2}}(-\zeta_i) + \sum_{i=1}^{w}(-D_i' - n_i) + \sum_{i=1}^{z}(C_i' + m_i)$$
$$+ \sum_{i=1}^{\frac{k+1}{2}}(1-\varphi_i) + \sum_{i=1}^{\frac{k-1}{2}}(1-\psi_i) + \sum_{i=1}^{y}(-B_i' - l_i)$$
$$\geq \sum_{i=1}^{x} A_i' + \sum_{i=1}^{w}(-D_i') + \sum_{i=1}^{z} C_i' + \sum_{i=1}^{\frac{k+1}{2}} 1 + \sum_{i=1}^{\frac{k-1}{2}} 1 + \sum_{i=1}^{y}(-B_i') \quad \text{(by (4.11))},$$

which is the degree of σ'. ■

Remark 4.12 Notice that if $k \geq 1$ then π' is not tempered.

Examples (1) In the case where, in the setting of Conjecture 4.3, the infinitesimal character of $\theta_{r+k+2,s+k}(\pi)$ is regular (that is the case in Remark 3.17), Conjecture 4.3 follows from Theorem 2.7.

In some specific low-rank cases, Conjecture 4.3 may be proved by looking at all representations of $\tilde{U}(r+k+2, s+k)$ with the correct infinitesimal character, and excluding all except π' as a possible theta lift of π:

(2) Let $p = 2, q = 1, \pi = \pi(\lambda)$ with $\lambda = (3, 0; -2)$. Then $r = 2, s = 0$, and $k = 1$. By Theorem 3.14, $\theta_{5,1}(\pi) \neq 0$, and it has infinitesimal character $\varpi = (3, 1, 0, 0, -1, -2)$ (by Theorem 2.11). The LKT σ of π has highest weight

$$\Lambda = (3, 1; -3) = (2, 2; -2) + (1, -1; -1),$$

hence has $(5, 1)$-degree 3 and corresponds in $\mathcal{H}$ to the K-type σ_0 with highest weight $\zeta_0 = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}; -\frac{3}{2})$. The Vogan norm (see Definition 5.4.18 of [V1]) of σ_0 is $\|\sigma_0\| = \frac{123}{2}$. The group $\tilde{U}(5, 1)$ (the connected cover) has six (genuine) representations π_1 through π_6 with infinitesimal character ϖ: π_1 and π_2 are limit of discrete series representations $\pi_i(\lambda_i, \Psi_i)$ with $\lambda_i = (3, 1, 0, -1, -2; 0)$. The LKT σ_1 of π_1 has highest weight $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2})$ and is of $(2, 1)$-degree 3, which is the degree of σ, but easily seen not to be $(2, 1)$-harmonic (see Lemma 2.5), so $\theta_{5,1}(\pi) \neq \pi_1$. The LKT σ_2 of π_2 has highest weight $(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{1}{2})$, and $\|\sigma_2\| = \frac{127}{2} > \|\sigma_0\|$, so π_2 does not contain σ_0, and therefore $\theta_{5,1}(\pi) \neq \pi_2$.

The representations π_3 through π_6 occur as the LKT constituents of $I(5, 1, 1, \xi_i, \mu_i, \nu_i)$, where $\xi_i \in \tilde{U}(4)\widehat{}$ is the representation with infinitesimal character λ_i, and

$$\lambda_3 = (3, 1, 0, -1), \quad \mu_3 = -2,\ \nu_3 = 2;$$
$$\lambda_4 = (3, 1, 0, -2), \quad \mu_4 = -1,\ \nu_4 = 1;$$
$$\lambda_5 = (3, 0, -1, -2), \quad \mu_5 = 1,\ \nu_5 = 1;$$
$$\lambda_6 = (1, 0, -1, -2), \quad \mu_6 = 3,\ \nu_6 = 3.$$

The representation π' of Conjecture 4.3 is easily identified as π_4.

The highest weight of the LKT σ_3 of π_3 is $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; -\frac{5}{2})$, so σ_3 is of $(2,1)$-degree 3 and corresponds to the K-type with highest weight $(3,0;-2)$ in $\mathcal{H}$. This K-type is easily seen (using the Blattner formula or the Vogan norm) not to occur in π, so that $\theta_{5,1}(\pi) \neq \pi_3$.

The unique LKT σ_5 of π_5 has highest weight $(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{3}{2})$ and Vogan norm $\frac{123}{2} = \|\sigma_0\|$, so that σ_0 is not a K-type which occurs in π_5. Similarly, π_6 has a unique LKT σ_6 with highest weight

$$\Big(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{7}{2}\Big) = \zeta_0 - (2,1,1,1,0;-5).$$

Using Lemma 5.1.1 of [P1], it is easy to check that σ_0 does not occur in π_6.

So we have proved that $\theta_{5,1}(\pi) = \pi_4$, confirming the conjecture in this case.

(3) Let $p = 1$, $q = 2$, and $\lambda = (-\frac{1}{2}; \frac{1}{2}, -\frac{5}{2})$, and $\pi = \pi(\lambda)$. By Theorem 3.14, $\theta_{3,2}(\pi) \neq 0$. In this case, there are 14 representations of $\tilde{U}(3,2)$ with the correct infinitesimal character, 13 of which may be ruled out as the theta lift of π using the methods of the previous example and the following facts:

(i) If ρ is a representation of $\tilde{U}(3,2)$ which occurs in the correspondence with $\tilde{U}(1,2)$, then by persistence, ρ also lifts to a representation of $\tilde{U}(2,3)$, and this correspondence is known explicitly (see [P1]).

(ii) If ρ corresponds to π then ρ does not contain any K-types of $(1,2)$-degree less than the $(3,2)$-degree of the LKT of π.

In this way, we can show that $\theta_{3,2}(\pi)$ is the limit of discrete series representation $\rho(\lambda', \Psi')$ with $\lambda' = (\frac{1}{2}, -\frac{1}{2}, -\frac{5}{2}; \frac{1}{2}, -\frac{1}{2})$ and $e_2 - f_2, f_1 - e_1 \in \Psi'$, confirming the conjecture in this case as well.

5 Irreducible Principal Series

Using Corollary 3.18, parabolic induction and the induction principle ([K], [AB], [P1]), we prove Conjecture 1.2 for a large collection of representations, namely those which are irreducible generalized principal series representations (see Definition 4.1(b), and [SV] for conditions of irreducibility). Let us first recall the results from [P1] and [AB] that we need.

Theorem 5.1 **(Induction Principle for** $U(p,q)$**, [P1])** *For* $i = 1, 2$, *let* $\pi_i \in \tilde{U}(p_i, q_i)\widehat{\ }$, $\sigma_i \in \mathrm{GL}(k_i, \mathbb{C})\widehat{\ }$ *(the admissible duals), and suppose that* $\pi_1 \leftrightarrow \pi_2$ *and* $\sigma_1 \leftrightarrow \sigma_2$ *in the correspondences for the dual pairs* $\big(U(p_1,q_1), U(p_2,q_2)\big)$ *and* $\big(\mathrm{GL}(k_1,\mathbb{C}), \mathrm{GL}(k_2,\mathbb{C})\big)$ *respectively. Let* χ_1 *and* χ_2 *be the characters of* $\mathrm{GL}(k_1,\mathbb{C})$ *and* $\mathrm{GL}(k_2,\mathbb{C})$ *given by*

$$\chi_1(g_1) = |\det(g_1)|^{p_2+q_2+k_2-p_1-q_1-k_1}, \quad \textit{and}$$
$$\chi_2(g_2) = |\det(g_2)|^{p_1+q_1+k_1-p_2-q_2-k_2}, \quad \textit{for } g_i \in \mathrm{GL}(k_i,\mathbb{C}).$$

Let ω *be the oscillator representation for the dual pair*

$$(G_1, G_2) = \big(\tilde{U}(p_1+k_1, q_1+k_1), \tilde{U}(p_2+k_2, q_2+k_2)\big).$$

Then there are parabolic subgroups $P_i = M_i N_i$ *of* G_i *with Levi factors* $M_i \cong \tilde{U}(p_i,q_i) \times \mathrm{GL}(k_i,\mathbb{C})$, *and a nonzero* $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, K_1 \times K_2)$*-map of the associated Harish-Chandra modules*

$$\omega \to \mathrm{Ind}_{P_1}^{G_1}(\pi_1 \otimes \sigma_1 \otimes \chi_1 \otimes \mathbb{1}) \otimes \mathrm{Ind}_{P_2}^{G_2}(\pi_2 \otimes \sigma_2 \otimes \chi_2 \otimes \mathbb{1}).$$

Remark 5.2 In the case where the induced representations are irreducible, this means that they correspond in the Howe correspondence for the dual pair $\big(G_1, G_2\big)$.

Recall from [D] that the irreducible admissible representations of $\mathrm{GL}(k,\mathbb{C})$ may be parametrized by pairs $(\mu,\nu) \in \mathbb{Z}^k \times \mathbb{C}^k$. Given such a pair, let $\rho_{\mu,\nu}$ be the unique irreducible quotient of

$$\mathrm{Ind}_{MN}^{\mathrm{GL}(k,\mathbb{C})}\big(\chi(\mu,\nu)\otimes \mathbb{1}\big),$$

where MN is a parabolic subgroup of $\mathrm{GL}(k,\mathbb{C})$ with Levi factor $M \cong (\mathbb{C}^\times)^k$ and N such that $\mathrm{Re}\{\langle \nu,\alpha\rangle\} \geq 0$ for all $\alpha \in \Delta(\mathfrak{n})$.

Theorem 5.3 **(Adams, Barbasch [AB])** *In the correspondence for the dual pair* $\big(\mathrm{GL}(k,\mathbb{C}), \mathrm{GL}(k,\mathbb{C})\big)$, *every irreducible admissible representation of* $\mathrm{GL}(k,\mathbb{C})$ *occurs. Explicitly, the correspondence is given by*

$$\rho_{\mu,\nu} \leftrightarrow \rho_{\mu,-\nu}.$$

Note 5.4 In describing the correspondence above, we take into consideration the fact that the embedding of the dual pair $\big(\mathrm{GL}(k_1,\mathbb{C}), \mathrm{GL}(k_2,\mathbb{C})\big)$ in the setting of Theorem 5.1 differs from the embedding in [AB] (see [P1]).

We are now ready to prove the following:

Theorem 5.5 *Let* $\pi = I(p,q,k,\sigma,\mu,\nu)$ *be a genuine irreducible generalized principal series representation of* $\tilde{U}(p,q)$. *If* $\theta_{r,s}(\sigma) \neq 0$ *then* $\theta_{r+k,s+k}(\pi) \neq 0$.

Proof If $\pi = I(p,q,k,\sigma,\mu,\nu)$ is irreducible, then $\pi = \mathrm{Ind}_{M'N'}^{\tilde{U}(p,q)}(\sigma \otimes \rho_{\mu,\nu} \otimes \mathbb{1})$, for any parabolic subgroup $M'N'$ of $\tilde{U}(p,q)$ with Levi factor $M' \cong \tilde{U}(p-k,q-k) \times \mathrm{GL}(k,\mathbb{C})$. Notice also that if $\rho_{\mu',\nu'}$ is a representation of $\mathrm{GL}(k,\mathbb{C})$ and $t \in \mathbb{C}$, then $\rho_{\mu',\nu'} \otimes |\det|^t \cong \rho_{\mu,\nu'+\nu_t}$, where $\nu_t = (t,\ldots,t)$. Suppose $\theta_{r,s}(\sigma) = \sigma'$. Then by Theorems 5.1 and 5.3, $\theta_{r+k,s+k}(\pi)$ is a constituent of $I(r+k,s+k,k,\sigma',\mu,-\nu)$. In particular, it is nonzero, and the theorem is proved. ∎

Corollary 5.6 *If* $\pi \in \widehat{\tilde{U}(p,q)}_{\text{genuine}}$ *is an irreducible generalized principal series representation, then* $m_+(\pi) + m_-(\pi) = 2p + 2q + 2$.

Proof This follows from Proposition 3.2 and Theorem 5.5 using the facts that by Corollary 3.18, $m_+(\sigma) + m_-(\sigma) = 2p + 2q + 2 - 4k$, and for all r and s, $U(r,s)$ and $U(r+k,s+k)$ are in the same Witt tower. ∎

6 One More Theorem

In order to complete the display of our evidence in support of Conjecture 1.2, we restate the relevant result from [P2].

Theorem 6.1 *Let* $\pi \in \widehat{\tilde{U}(p,q)}_{\text{genuine}}$ *(the* $\det^{\frac{p+q+1}{2}}$*-cover), and realize* π *as the Langlands subquotient of an induced representation of the form* $I(p,q,k,\sigma,\mu,\nu)$ *(see [V2], [P1]). Recall* *that* $\sigma = \sigma(\lambda, \Psi)$ *is a genuine limit of discrete series representation of* $\tilde{U}(p-k, q-k)$. *If* λ *is given by*

$$\lambda = (x_1, x_2, \dots, x_{p-k}; y_1, \dots, y_{q-k}) \quad \textit{with} \quad x_i \neq 0 \neq y_j \quad \textit{for all } i, j,$$

then there are unique integers r *and* s *with* $r+s = p+q+1$ *such that* $\theta_{r,s}(\pi) \neq 0$ *and* $\theta_{r+1,s-1}(\pi) \neq 0$. *Consequently, Conjecture 1.2 is true for* π.

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