THE FACTORS OF A SQUARE-FREE INTEGER

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This note is concerned with the number $C(n)$ of ordered nontrivial factorizations of an integer n in the special case where n is square free. If $F(m)$ denotes $C(p_1 \ldots p_m)$, where p_i are distinct primes, it is shown that

$$(1) \qquad \limsup_{m\to\infty} \left| \frac{2F(m)}{m!} - \frac{1}{m+1} \right|^{1/m} = \frac{1}{(\alpha^2 + 4\pi^2)^{1/2}}, \quad \alpha = \log 2,$$

and that $2F(m)/m! = 1/\alpha^{m+1} + o(1/(2\pi)^m)$.

In the general case, Long [1] has shown that $C(n)$ is equal to the number of elements in the set of all pairs of complementing sets of order $n > 1$. He established the formula $2C(n) = \Sigma_{d|n} C(d)$ and also pointed out that $C(n)$ is equal to the number of perfect partitions of $n - 1$. This result is due to MacMahon [2, p. 108] who gave another formula [3, pp. 843-844] for $C(n)$. Neither of these results seems to be useful in estimating $F(m)/m!$ as $m \to \infty$.

The number $f(m,k)$ of ordered nontrivial factorizations of $p_1 \ldots p_m$ with exactly k factors is precisely the number of ways in which m objects, no two alike, can be distributed in k boxes, no two alike, with the additional requirement that there be no empty box. This problem is well known and references are, for example, Niven [4, pp. 110-111] and Riordan [5, pp. 42-43, 91]. It is known that

$$(2) \qquad f(m,k) = \sum_{r=0}^{k} (-1)^r \, C(k,r) \, (k-r)^m ,$$

where $C(k, r)$ are binomial coefficients. In fact, $f(m, k) = k! \; S(m, k)$, where $S(m,k)$ are Stirling numbers of the second kind, and it is not difficult to find a generating function for $f(m, k)$. Consider

$$\begin{aligned}(e^z - 1)^k &= \sum_{r=0}^{k} (-1)^r \, C(k,r) \, e^{(k-r)z} \\ &= \sum_{r=0}^{k} (-1)^r \, C(k,r) \sum_{m=0}^{\infty} (k-r)^m (z^m/m!) \\ &= \sum_{m=0}^{\infty} \sum_{r=0}^{k} (-1)^r \, C(k,r) \, (k-r)^m (z^m/m!) .\end{aligned}$$

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The inner sum is zero when $m < k$ and it follows from (2) that

$$(e^z - 1)^k = \sum_{m=k}^{\infty} f(m, k) (z^m/m!) , \qquad k \geqq 1 .$$

The total number of ordered nontrivial factorizations of $p_1 \ldots p_m$ is $F(m) = \sum_{k=1}^{m} f(m, k)$. Thus

$$\sum_{k=0}^{\infty} (e^z - 1)^k = 1 + \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} f(m, k) (z^m/m!)$$

$$= 1 + \sum_{m=1}^{\infty} \sum_{k=1}^{m} f(m, k) (z^m/m!) = 1 + \sum_{m=1}^{\infty} F(m) (z^m/m!) .$$

If $|e^z - 1| < 1$ the series on the left converges to $1/(2 - e^z)$. It is easy to see, however, that the singularity of this function nearest the origin is $z = \alpha = \log 2$. Then, with the definition $F(0) = 1$,

$$\text{(3)} \qquad G(z) \equiv 1/(2 - e^z) = \sum_{m=0}^{\infty} F(m) (z^m/m!), \qquad |z| < \alpha .$$

It follows from (3) that $\limsup_{m \to \infty} |F(m)/m!|^{1/m} = 1/\alpha$, but this does not even lead to the asymptotic result $2F(m) \sim m!/\alpha^{m+1}$. Also (3) yields a formula for $F(m)$ since

$$2G(z) = \frac{1}{1 - (e^z/2)} = \sum_{j=0}^{\infty} e^{jz}/2^j = \sum_{j=0}^{\infty} 1/2^j \sum_{m=0}^{\infty} j^m (z^m/m!)$$

Then from (3), $2F(m) = \sum_{j=0}^{\infty} j^m/2^j$, but again this does not seem to be helpful.

The function $H(z) \equiv 2G(z) - 1/(\alpha - z)$ has the power series expansion

$$\text{(4)} \qquad H(z) = \sum_{m=0}^{\infty} \left(\frac{2F(m)}{m!} - \frac{1}{\alpha^{m+1}} \right) z^m , \qquad |z| < \alpha .$$

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But now $z = \alpha$ is not a singularity of $H(z)$ and the expansion is valid up to the singularities nearest the origin which are $z = \alpha \pm 2\pi i$. Thus the series in (4) converges for $|z| < (\alpha^2 + 4\pi^2)^{1/2}$ and the result (1) follows. In particular the series converges for $z = 2\pi$ so that

$$\lim_{m\to\infty} \left(\frac{2F(m)}{m!} - \frac{1}{\alpha^{m+1}} \right) (2\pi)^m = 0$$

and $2F(m)/m! = 1/\alpha^{m+1} + o(1/(2\pi)^m)$.

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