ANZIAM J. 42(2001), 372–386

AN APPLICATION OF NEWTON'S METHOD TO DIFFERENTIAL AND INTEGRAL EQUATIONS

J. M. GUTIÉRREZ[1] and M. A. HERNÁNDEZ[1]

(Received 7 November 1995; revised 18 October 1998)

Abstract

Newton's method is applied to an operator that satisfies stronger conditions than those of Kantorovich. Convergence and error estimates are compared in the two situations. As an application, we obtain information on the existence and uniqueness of a solution for differential and integral equations.

1. Introduction

Consider the equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator between two Banach spaces X and Y. Newton's well-known method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad n \geq 0, \tag{2}$$

allows us to find a solution of (1) or, at least, to obtain information on the existence and uniqueness of such a solution. Working in Banach spaces also allows us to apply our results to different problems such as scalar equations, differential equations or integral equations.

In [3] the convergence of Newton's method is studied under stronger conditions than those of Kantorovich ([6, 7, 9]). When these hold, the results on the existence and uniqueness of the solution of (1) are different to those of Kantorovich and are sometimes better, as we show later. In this paper we apply these results to the case of differential and integral equations. Finally, a method is developed to find regions where the solution is located and where it is unique.

[1]Department of Mathematics and Computation, University of La Rioja, C/ Luis de Ulloa s/n, 26004, Logroño, Spain.

https://doi.org/10.1017/S1446181100012001 Published online by Cambridge University Press

2. Newton's method under strong-type Kantorovich conditions

From now on, we assume that the operator F defined in (1) is twice Fréchet differentiable in

$$\Omega_0 = \overline{B(x_0, r_0)} = \{x \in X;\ \|x - x_0\| \le r_0\}$$

(let $B(x_0, r_0)$ be the respective open ball). Let us assume too that there exists $\Gamma_0 = F'(x_0)^{-1} \in \mathscr{L}(Y, X)$ (the space of bounded linear operators from Y to X) and that F'' satisfies the Lipschitz condition

$$\left\|\Gamma_0\left[F''(x) - F''(x_0)\right]\right\| \le k\|x - x_0\|, \quad x \in \Omega_0. \tag{3}$$

Most authors study the sequence (2) under the so-called Kantorovich conditions, that is,

(I) $\left\|\Gamma_0 F(x_0)\right\| \le a$,

(II) $\left\|\Gamma_0 F''(x)\right\| \le M, x \in \Omega_0$,

(III) $aM \le 1/2$ and

$$\frac{1 - \sqrt{1 - 2aM}}{M} \le r_0,$$

or even slightly weaker conditions. In [1, 11] and [13], condition (II) is relaxed to

(II′) $\left\|\Gamma_0\left[F'(x) - F'(y)\right]\right\| \le M\|x - y\|, x, y \in \Omega_0$,

or to

(II″) $\left\|\Gamma_0\left[F'(x) - F'(y)\right]\right\| \le M\|x - y\|^p, x, y \in \Omega_0, p \in [0, 1]$.

The next theorem guarantees the convergence of Newton's method under stronger conditions. The details of the proof can be seen in [3], where the majorizing function that helps us to prove the convergence of (2) is established. In this case that function is a third-degree polynomial, instead of the classic second-degree polynomial given by Kantorovich. Results on uniqueness of solution and error estimates are also given.

THEOREM 2.1. *Let us assume that F satisfies*

$$\left\|\Gamma_0 F(x_0)\right\| \le a, \qquad \left\|\Gamma_0 F''(x_0)\right\| \le b$$

and (3). *Let us suppose too that the polynomial*

$$p(t) = (k/6)t^3 + (b/2)t^2 - t + a \tag{4}$$

has two positive roots $r_1 \le r_2$, that is, $p(m) \le 0$, where

$$m = \frac{2}{b + \sqrt{b^2 + 2k}}.$$

https://doi.org/10.1017/S1446181100012001 Published online by Cambridge University Press

Then, if $r_1 \leq r_0$, the sequence $\{x_n\}$ defined in (2) *converges to x^*, the solution of* (1) *in $\overline{B(x_0, r_1)}$. Further, the solution is unique in $B(x_0, r_2)$ for $r_1 < r_2$ or in $\overline{B(x_0, r_1)}$ for $r_1 = r_2$. Finally, if $\{t_n\}$ denotes the sequence of iterates in Newton's method to solve $p(t) = 0$, starting at $t_0 = 0$, we have*

$$\|x^* - x_n\| \leq r_1 - t_n.$$

If $r_1 < r_2$, we have the error estimates

$$(r_2 - r_1)\frac{\alpha^{2^n}}{r - \alpha^{2^n}} \leq r_1 - t_n \leq (r_2 - r_1)\frac{\theta^{2^n}}{R - \theta^{2^n}}$$

depending on the roots of p, where $-r_3$ is the negative root of p, and

$$0 < r = \frac{r_3 - r_2}{r_3 - r_1} < 1, \qquad 0 < R = 1 - \frac{r_2 - r_1}{r_3 + r_1} < 1,$$
$$0 < \alpha = r\frac{r_1}{r_2} < 1, \qquad 0 < \theta = R\frac{r_1}{r_2} < 1.$$

If $r_1 = r_2$, then

$$r_1\left(\frac{r_3 - r_1}{2r_3 - r_1}\right)^n \leq r_1 - t_n \leq \frac{r_1}{2^n}.$$

PROOF. Here we give only a sketch of the proof. For details, see [3]. For each $x \in X$, the linear operator on X defined by

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x) : X \to X$$

is the derivative of the operator

$$G(x) = x - F'(x)^{-1}F(x)$$

that defines Newton's method [4].

Under the previous assumptions it can be shown that

$$\|L_F(x)\| \leq \frac{(b + kt)(a - t + (1/2)\,bt^2 + (k/6)\,t^3)}{\left[1 - (k/2)\,t^2 - bt\right]^2},$$

where $x \in [x_n, x_{n+1}]$, $t \in [t_n, t_{n+1}]$ are such that $x = x_n + \tau(x_{n+1} - x_n)$, $t = t_n + \tau(t_{n+1} - t_n)$.

Observe that the expression on the right-hand side of the equation above is

$$L_p(t) = \frac{p(t)p''(t)}{p'(t)^2},$$

where p is the polynomial defined in (4).

So, we have for $n \in \mathbb{N}$,

$$\|x_{n+1} - x_n\| = \|G(x_n) - G(x_{n-1})\| = \left\| \int_{x_{n-1}}^{x_n} L_F(x)\,dx \right\|$$
$$\leq \int_{t_{n-1}}^{t_n} L_p(t)\,dt = t_{n+1} - t_n.$$

Therefore $\{t_n\}$ is a majorizing sequence of $\{x_n\}$ and the classic theory on majorizing sequences ([6, 10]) allows us to conclude the result. To obtain the error expresions for the sequence $\{t_n\}$ we use the technique developed by Ostrowski [7] for a second-degree polynomial.

REMARKS. • An equivalent condition to $p(m) \leq 0$ is (see [14])

$$a \leq \frac{b^2 + 4k - b\sqrt{b^2 + 2k}}{3k(b + \sqrt{b^2 + 2k})}.$$

In [5] it was stated that $p(m) \leq 0$ holds provided that one of the conditions:

$$a - \frac{1}{2b} + \frac{k}{6b^3} \leq 0, \qquad 6ab^3 + 9a^2k^2 + 18abk \leq 3b^2 + 8k$$

or

$$3ak^2 + 3bk + b^3 \leq [b^2 + 2k]^{3/2} \tag{5}$$

is satisfied.

• We can give a similar result to Theorem 2.1 (see [3]) assuming, instead of (3),

$$\left\|\Gamma_0\left[F''(x) - F''(x_0)\right]\right\| \leq k\|x - x_0\|^p, \quad x \in \Omega_0,\ p \geq 0.$$

In this case the function "test" that allows us to prove convergence is

$$f(t) = a - t + \frac{b}{2}t^2 + \frac{k}{(p+1)(p+2)}t^{p+2}, \quad t \geq 0.$$

The following example [5] contains a function and a starting point that do not satisfy the Kantorovich conditions but do, however, satisfy the conditions of the previous theorem. The convergence of the sequence (2) cannot then be established from the Kantorovich theorem but can be established from Theorem 2.1.

EXAMPLE 1. Let $X = [-1, 1]$, $Y = \mathbb{R}$ and let $f : X \to Y$ be the polynomial

$$f(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{3},$$

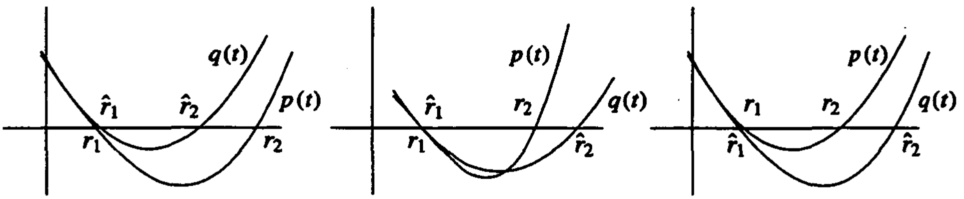

FIGURE 1. Location of the roots of p and q.

with $x_0 = 0$.

With the notation of Theorem 2.1 we have $a = 2/5$, $M = 8/5$, $b = 2/5$ and $k = 6/5$. Consequently, $aM = 16/25 > 1/2$ and the Kantorovich condition fails.

However,

$$3ak^2 + 3bk + b^3 = \frac{404}{125} < \frac{512}{125} = (b^2 + 2k)^{3/2},$$

so (5) is fulfilled, $p(t)$ has two positive roots and we can use Theorem 2.1 to prove the convergence of Newton's method.

Sometimes the convergence of (2) can be established using either the Kantorovich theorem or Theorem 2.1. Then we wonder which result gives us the most accurate information about the solutions of (1). Under the assumptions of Theorem 2.1 we may determine the solutions of (1) in terms of the roots of the polynomial (3). Under the Kantorovich assumptions the information is obtained from the polynomial

$$q(t) = \frac{M}{2}t^2 - t + a. \tag{6}$$

Let us denote by $\hat r_1 \le \hat r_2$ the roots of q. Then

$$p(\hat r_j) = \frac{\hat r_j^2}{2}\left(\frac{k}{3}\hat r_j - (M - b)\right), \quad j = 1, 2.$$

Observe that

$$p(\hat r_1) \le 0 \iff k\left(1 - \sqrt{1 - 2aM}\right) \le 3M(M - b),$$

$$p(\hat r_2) \le 0 \iff k\left(1 + \sqrt{1 - 2aM}\right) \le 3M(M - b).$$

We consider three distinct situations (see Figure 1):

Case 1. Suppose $k\left(1 + \sqrt{1 - 2aM}\right) \le 3M(M - b)$. Then $r_1 \le \hat r_1$, $\hat r_2 \le r_2$ and, consequently, the solution x^* is located in $\overline{B(x_0, r_1)}$ and is unique in $B(x_0, r_2)$.

Case 2. Suppose $k\left(1 - \sqrt{1 - 2aM}\right) \le 3M(M - b) < k\left(1 + \sqrt{1 - 2aM}\right)$. In this situation $r_1 \le \hat r_1$, $r_2 \le \hat r_2$, and the solution x^* is located in $\overline{B(x_0, r_1)}$ and is unique in $B(x_0, \hat r_2)$.

Case 3. Suppose $3M(M-b) \le k\left(1-\sqrt{1-2aM}\right)$. Now we have $\hat{r}_1 \le r_1, r_2 \le \hat{r}_2$, thus x^* is located in $\overline{B(x_0, \hat{r}_1)}$ and is unique in $B(x_0, \hat{r}_2)$.

3. Application to differential and integral equations

Now we are going to study some differential and integral equations applying the results of the previous section. In particular, we obtain results on the existence and uniqueness of solutions for these equations.

In this way let us consider, for instance, the Banach space X of functions differentiable in $[0, \tau]$ and vanishing at $t = 0$, where

$$X = \{y \in C^1[0, \tau];\ y(0) = 0\}$$

and is equipped with the norm

$$\|y\| = \max_{t\in[0,\tau]} |y(t)| + \lambda \max_{t\in[0,\tau]} |y'(t)|,$$

where $\lambda > 0$ is a coefficient to be determined later.

We are going to study the differential equation

$$\begin{cases} y'(t) - \phi(t, y(t)) = 0, \\ y(0) = 0. \end{cases} \tag{7}$$

Suppose $\phi(t, u)$ is continuous and has a continuous second derivative in u in the region

$$0 \le t \le \tau, \quad |u - y_0(t)| \le \delta, \quad y_0 \in X.$$

Let $\Omega = \{y \in X; \|y - y_0\| \le \delta\}$ and let $F : \Omega \subseteq X \to C[0, \tau]$ be the operator defined by

$$F(y)(t) = y'(t) - \phi(t, y(t)), \quad y \in \Omega.$$

The problem of solving (7) is equivalent to finding a solution of $F(y) = 0$.

For $y_0 \in \Omega$, we have

$$F'(y_0)y(t) = y'(t) - \phi'_u(t, y_0(t))y(t),$$
$$F''(y_0)xy(t) = -\phi''_{u^2}(t, y_0(t))x(t)y(t),$$

for $x, y \in \Omega$.

Put $\Gamma_0 = F'(y_0)^{-1}$. Our goal is to find an upper bound for $\|\Gamma_0\|$. In order to do that, let $x = \Gamma_0(y)$. Then x is a solution of the differential equation

$$x' = y(t) + \phi'_u(t, y_0(t))x,$$

https://doi.org/10.1017/S1446181100012001 Published online by Cambridge University Press

that is, by [2] and [12],

$$x(t) = \psi(t) \int_0^t \frac{y(s)}{\psi(s)}\, ds,$$

where $\psi(t) = \exp\left(\int_0^t \phi'_u(s, y_0(s))\, ds\right)$.

Notice that

$$\max_{t\in[0,\tau]} |x(t)| \le \tau \frac{\max_{t\in[0,\tau]} |\psi(t)|}{\min_{t\in[0,\tau]} |\psi(t)|} \|y\|$$

and

$$\max_{t\in[0,\tau]} |x'(t)| \le \max_{t\in[0,\tau]} |\phi'_u(t, y_0(t))| \max_{t\in[0,\tau]} |x(t)| + \|y\| \le \theta \|y\|,$$

where

$$\theta = 1 + \tau \max_{t\in[0,\tau]} |\phi'_u(t, y_0(t))| \frac{\max_{t\in[0,\tau]} |\psi(t)|}{\min_{t\in[0,\tau]} |\psi(t)|}.$$

Consequently,

$$\|\Gamma_0\| \le \lambda\theta + \tau \frac{\max_{t\in[0,\tau]} |\psi(t)|}{\min_{t\in[0,\tau]} |\psi(t)|}. \tag{8}$$

This bound enables us to apply Theorem 2.1 and obtain information on the existence and uniqueness of solutions for (7).

THEOREM 3.1. *Let $y_0 \in \Omega$ satisfy*

(i) $|y'_0(t) - \phi(t, y_0(t))| \le a'$, $t \in [0, \tau]$,
(ii) $|\phi'_u(t, y_0(t))| \le M_1$, $t \in [0, \tau]$,
(iii) $|\phi''_{u^2}(t, y_0(t))| \le b'$, $t \in [0, \tau]$,
(iv) $|\phi''_{u^2}(t, u) - \phi''_{u^2}(t, v)| \le k'|u - v|$, $t \in [0, \tau]$, $|u - y_0(t)| \le \delta$, $|v - y_0(t)| \le \delta$.

Put $\theta_1 = \tau e^{2\tau M_1}$, $\theta_2 = 1 + M_1\theta_1$, $B = \theta_1 + \lambda\theta_2$, $a = a'B$, $b = b'B$ and $k = k'B$. If the polynomial

$$p(t) = (k/6)\, t^3 + (b/2)\, t^2 - t + a$$

has two positive roots $r_1 \le r_2$ and $m \le \delta$ (where m is the minimum of p), the differential equation (7) *has a solution y^* defined in* $[0, \tau]$. *Further*

$$|y^*(t) - y_0(t)| \le r_1, \quad t \in [0, \tau]$$

and the solution is unique in $\|u - y_0\| < r_2$.

PROOF. Observe that

$$\max_{t\in[0,\tau]} |\psi(t)| \le e^{\tau M_1}, \quad \min_{t\in[0,\tau]} |\psi(t)| \ge e^{-\tau M_1}$$

and, from (8), $\|\Gamma_0\| \le \theta_1 + \lambda\theta_2 = B$. Then

$$\|\Gamma_0 F(x_0)\| \le \|\Gamma_0\| \|F(x_0)\| \le Ba' = a$$

and, in a similar way,

$$\|\Gamma_0 F''(x_0)\| \le b, \quad \|\Gamma_0 [F''(y) - F''(y_0)]\| \le k\|y - y_0\|.$$

So the result follows from Theorem 2.1.

Notice that if we take $B = \theta_1$ and the polynomial p has two different positive roots, then we can guarantee the existence of a solution for (7) by taking λ small enough.

We now consider the problem of applying Newton's method to an integral equation

$$x(s) = \int_0^1 K(s, t, x(t))\, dt, \quad s \in [0, 1], \tag{9}$$

where the kernel $K(s, t, u)$ is continuous in all its arguments and has continuous derivatives of all orders required. In the function space X we introduce the operator F such that $F(x) = y$, where

$$y(s) = x(s) - \int_0^1 K(s, t, x(t))\, dt.$$

We study the equation $F(x) = 0$. Let x_0 be the initial approximation. Assume that X and K are such that $F'(x_0)$ can be obtained by differentiating under the integral sign, that is, that $z = F'(x_0)x$ means that

$$z(s) = x(s) - \int_0^1 K'_u(s, t, x_0(t))x(t)\, dt. \tag{10}$$

It is well-known ([6, 8]), that $\Gamma_0 = F'(x_0)^{-1}$ has the form $w = \Gamma_0(y)$, where

$$w(s) = y(s) + \int_0^1 G(s, t)y(t)\, dt,$$

and $G(s, t)$ is the resolvent of (10) for the kernel $K(s, t) = K'_u(s, t, x_0(t))$. Then, if

$$\int_0^1 |G(s, t)|\, dt \le B', \quad s \in [0, 1],$$

we have $\|\Gamma_0\| \le 1 + B' = B$.

From this bound and taking Theorem 2.1 into account, we can establish results on the existence and uniqueness of a solution for (9).

THEOREM 3.2. *Suppose K is continuous in all its arguments and has continuous derivatives of all orders required and assume that*

(i) $\left|\int_0^1 K(s,t,x_0(t))\,dt - x_0(s)\right| \le a'$, $s \in [0,1]$,
(ii) $\left|\int_0^1 K''_{u^2}(s,t,x_0(t))\,dt\right| \le b'$, $s \in [0,1]$,
(iii) $\left|\int_0^1 [K''_{u^2}(s,t,u) - K''_{u^2}(s,t,v)]\,dt\right| \le k'|u-v|$, *for* s, u *and* v *satisfying* $s \in [0,1]$, $|u - x_0(s)| \le \delta$, $|v - x_0(s)| \le \delta$.

Let $a = a'B$, $b = b'B$ *and* $k = k'B$. *If the polynomial*

$$p(t) = (k/6)\,t^3 + (b/2)\,t^2 - t + a$$

has two positive roots $r_1 \le r_2$ *and if* $m \le \delta$ *(where* m *is the minimum of* p*), then the integral equation* (9) *has a solution* x^* *defined in* $[0,1]$. *Moreover,*

$$\left|x^*(t) - x_0(t)\right| \le r_1, \quad t \in [0,1]$$

and the solution is unique in $\{u \in X;\ \|u - y_0\| < r_2\}$.

EXAMPLE 2. Let us illustrate the above theorem with an example. Consider the initial value problem

$$\begin{cases} y'(t) = t + \sin y(t), \\ y(0) = 0. \end{cases} \tag{11}$$

Solving such a problem is equivalent to finding a solution of the equation $F : X \to C[0,1]$, where $X = \{y \in C^1[0,1];\ y(0) = 0\}$ and

$$F(y)(t) = y'(t) - t - \sin y(t).$$

We define a norm in X by setting

$$\|y\| = \max_{t\in[0,1]} |y(t)| + \lambda \max_{t\in[0,1]} |y'(t)|,$$

where $\lambda > 0$ will be determined later, and we consider the max-norm in the space $C[0,1]$.

Taking $y_0(t) = 0$ as an initial approximation, it follows that there exists $\Gamma_0 = F'(x_0)^{-1}$, defined by

$$\Gamma_0 y(t) = e^t \int_0^t e^{-s} y(s)\,ds.$$

Then

$$\left\|\Gamma_0 F(y_0)\right\| = \max_{t\in[0,1]} \left\|t + 1 - e^t\right\| + \lambda \max_{t\in[0,1]} \left\|1 - e^t\right\| = e - 2 + \lambda(e-1) \equiv a$$

and

$$\left\|\Gamma_0 F''(y_0)\right\| = 0 \equiv b.$$

Finally, for $y \in X$, we have

$$\left\|\Gamma_0 F''(y)\right\| \leq e - 1 + \lambda e \equiv M,$$
$$\left\|\Gamma_0 \left[F''(y) - F''(y_0)\right]\right\| \leq (e - 1 + \lambda e)\|y - y_0\|$$

and therefore $k = e - 1 + \lambda e$. Let a_0, b_0, M_0 and k_0 be the limits when $\lambda \to 0$ of a, b, M and k respectively, that is,

$$a_0 = e - 2, \quad b_0 = 0, \quad M_0 = k_0 = e - 1.$$

Analysing the polynomials

$$q_0(t) = \frac{M_0}{2} t^2 - t + a_0, \quad p_0(t) = \frac{k_0}{6} t^3 + \frac{b_0}{2} t^2 - t + a_0,$$

we see that q_0 has no positive roots (and the polynomial q arising from the Kantorovich theorem has none either) but, nevertheless, p_0 has positive roots. We can write the polynomial p defined in (4) in the form

$$p(t) = p_0(t) + \lambda e \left(1 + \frac{t^3}{6}\right).$$

Then taking λ small enough, we can affirm that p has two positive roots close to the roots of p_0: $r_1 = 1.0465$, $r_2 = 1.111$. So we can guarantee that the initial value problem (11) has a solution y^* defined in [0, 1] and satisfying

$$|y^*(t) - y_0(t)| \leq r_1, \quad t \in [0, 1].$$

Further, the solution is unique in $\{u \in X;\ \|u - y_0\| < r_2\}$.

4. Other consequences

In this section we analyse some consequences of Theorem 2.1. First, let us suppose, instead of the existence of Γ_0, the existence of an operator close to Γ_0.

THEOREM 4.1. *Suppose there exists a linear operator $\Gamma \in \mathscr{L}(Y, X)$ having a continuous inverse, and suppose the following conditions are satisfied:*

(1) $\|\Gamma F(x_0)\| \leq a'$,
(2) $\|\Gamma F'(x_0) - I\| \leq c < 1$,
(3) $\|\Gamma F''(x_0)\| \leq b'$,
(4) $\|\Gamma[F''(x) - F''(x_0)]\| \leq k'\|x - x_0\|$, $x \in \Omega_0$.

https://doi.org/10.1017/S1446181100012001 Published online by Cambridge University Press

Then, provided that the polynomial

$$\overline{p}(t) = \frac{k'}{6}t^3 + \frac{b'}{2}t^2 - (1-c)t + a' \tag{12}$$

has two positive roots, $r_1' \le r_2'$ and $r_1' \le r_0$, (1) has a solution $x^* \in \overline{B(x_0, r_1')}$ and this solution is unique in $B(x_0, r_2')$ if $r_1' < r_2'$ or in $\overline{B(x_0, r_1')}$ if $r_1' = r_2'$.

PROOF. From (3) and taking into account Banach's theorem on the existence of inverse operators (see [6]), we have that there exists $U = [\Gamma F'(x_0)]^{-1}$ and that

$$\|U\| = \left\| [\Gamma F'(x_0)]^{-1} \right\| \le \frac{1}{1-c}.$$

So there exists $\Gamma_0 = F'(x_0)^{-1} = U\Gamma$ and therefore

$$\|\Gamma_0 F(x_0)\| = \|U\Gamma F(x_0)\| \le \frac{a'}{1-c},$$
$$\left\|\Gamma_0 F''(x_0)\right\| = \left\|U\Gamma F''(x_0)\right\| \le \frac{b'}{1-c}.$$

Finally, for $x \in X$,

$$\left\|\Gamma_0 \left[F''(x) - F''(x_0)\right]\right\| = \left\|U\Gamma \left[F''(x) - F''(x_0)\right]\right\| \le \frac{k'}{1-c}\|x - x_0\|.$$

The result follows from applying Theorem 2.1 to the polynomial

$$\frac{k'}{6(1-c)}t^3 + \frac{b'}{2(1-c)}t^2 - t + \frac{a'}{1-c}$$

or, equivalently, the polynomial given by (12).

Sometimes we can replace the given equation $F(x) = 0$ by a simpler equation that is close to it, and is, in general, easier to solve. Following Kantorovich [6] we obtain conditions under which it is possible to judge the solubility of the given equation from a solution of the approximate equation in the form

$$F(x) \equiv \Pi(x) + \mu R(x) = 0, \tag{13}$$

where μ is a linear operator belonging to $\mathscr{L}(Y, X)$. In particular, μ may be a numerical coefficient.

Let x_0 be a solution of the simplified equation $\Pi(x) = 0$. We have that x_0 may be a good choice as an initial approximation to a solution of (13).

COROLLARY 4.2. *Suppose that*

(1) $\Pi(x_0) = 0$,
(2) *There exists* $\Gamma = \Pi'(x_0)^{-1}$ *and* $\|\Gamma\| \le B$,

(3) $\|R(x_0)\| \le \eta$, $\|R'(x_0)\| \le \alpha$,

(4) $\|\Pi''(x_0)\| \le \beta_1$, $\|R''(x_0)\| \le \beta_2$,

(5) $\|\Pi''(x) - \Pi''(x_0)\| \le \kappa_1 \|x - y\|$, $\|R''(x) - R''(x_0)\| \le \kappa_2 \|x - x_0\|$, $x \in \Omega_0$.

Then, provided that $\|\mu\| B\alpha < 1$ *and the polynomial*

$$\tilde{p}(t) = \frac{B(\kappa_1 + \kappa_2\|\mu\|)}{6} t^3 + \frac{B(\beta_1 + \beta_2\|\mu\|)}{2} t^2 - (1 - \alpha B\|\mu\|)\, t + B\eta\|\mu\|$$

has positive roots $r_1'' \le r_2''$, (13) *has a solution* $x^* \in \overline{B(x_0, r_1'')}$, *and this solution is unique in* $B(x_0, r_2'')$ *if* $r_1'' < r_2''$ *or in* $\overline{B(x_0, r_1'')}$ *if* $r_1'' = r_2''$.

PROOF. The results follow immediately from taking $\Gamma = \Pi'(x_0)^{-1}$ in Theorem 4.1.

As an application we study an integral equation in the form

$$x(t) = \int_0^1 K(s, t, x(s))\, ds, \tag{14}$$

where the kernel $K(s, t, u)$ is continuous in all its arguments and has continuous derivatives of all orders required.

To find an initial approximation x_0 close enough to the solution, let us consider, instead of the kernel $K(s, t, u)$, a simpler kernel in the form

$$H(s, t, u) = \sum_{k=1}^{m} h_k(s, u) w_k(t),$$

where $\{w_k\}_{k=1}^m$ are linearly independent functions. The solution of an integral equation

$$x(t) = \int_0^1 H(s, t, x(s))\, ds$$

arising from this kind of kernel (a degenerate kernel) is (see [8])

$$x_0(t) = \sum_{k=1}^{m} A_k w_k(t),$$

where A_k, $k = 1, \dots, m$, are solutions of the system

$$A_j = \int_0^1 h_j(s, x_0(s))\, ds = \int_0^1 h_j\left(s, \sum_{k=1}^{m} A_k w_k(s)\right) ds, \quad j = 1, \dots, m.$$

Consider x_0 so obtained as the initial approximation to (14). Finding a solution of (14) is equivalent to solving

$$F(x) \equiv \Pi(x) + \mu R(x) = 0,$$

where

$$\Pi(x)(t) = x(t) - \int_0^1 H(s, t, x(s))\, ds,$$

$$R(x)(t) = \int_0^1 [H(s, t, x(s)) - K(s, t, x(s))]\, ds, \qquad \mu = 1.$$

Then if the conditions of Corollary 4.2 are satisfied, we obtain information on the existence and uniqueness of a solution for (14). Let us illustrate these comments with an example.

EXAMPLE 3. Let $X = C[0, 1]$ be equipped with the max-norm. Consider the integral equation

$$x(t) = 1 - \frac{7}{6}t + \int_0^1 st \sin x(s)\, ds, \quad 0 \le t \le 1, \ x \in C[0, 1]. \tag{15}$$

In this case $K(s, t, u) = st \sin u$. Let us take as approximate kernel

$$H(s, t, u) = stu.$$

We choose as initial approximation of (15) the solution of the integral equation

$$x(t) = 1 - \frac{7}{6}t + \int_0^1 H(s, t, x(s))\, ds = 1 - \frac{7}{6}t + \int_0^1 stx(s)\, ds,$$

that is,

$$x_0(t) = 1 - t, \quad 0 \le t \le 1.$$

To find $\Pi'(x_0)^{-1}$ we solve the integral equation

$$x(t) = f(t) + \int_0^1 H'(s, t, x_0(s))x(s)\, ds = f(t) + \int_0^1 stx(s)\, ds,$$

which has the solution

$$x(t) = f(t) + \frac{3}{2}\int_0^1 stf(s)\, ds.$$

So $\Gamma = \Pi'(x_0)^{-1}$ is defined by

$$\Gamma y(t) = y(t) + \frac{3}{2}\int_0^1 stf(s)\, ds, \quad y \in C[0, 1].$$

In this situation, we have

$$\|\Gamma\| \leq 7/4 \equiv B,$$

$$|R(x_0)(t)| = \left|\int_0^1 [H(s,t,x_0(s) - K(s,t,x_0(s))]\,ds\right|$$
$$\leq \int_0^1 st|1-s-\sin(1-s)|\,ds \leq \int_0^1 \frac{s^4}{6}\,ds = \frac{1}{30}.$$

Then

$$\|R(x_0)\| \leq 1/30 \equiv \eta,$$

$$\left|R'(x_0)(t)\right| = \left|\int_0^1 [H'_u(s,t,x_0(s) - K'_u(s,t,x_0(s))]\,ds\right|$$
$$\leq \int_0^1 st|1-\cos(1-s)|\,ds \leq \frac{1}{2}\int_0^1 s(1-s)^2\,ds = \frac{1}{24}$$

and therefore

$$\|R'(x_0)\| \leq 1/24 \equiv \alpha.$$

With the notation of Corollary 4.2, it is easy to check that $\beta_1 = \kappa_1 = 0$. Further,

$$\left|R''(x_0)(t)\right| = \left|\int_0^1 [H''_{u^2}(s,t,x_0(s) - K''_{u^2}(s,t,x_0(s))]\,ds\right|$$
$$= \int_0^1 st\sin(1-s)\,ds \leq \int_0^1 s(1-s)\,ds = \frac{1}{6}$$

and $\beta_2 = 1/6$. Finally, applying the mean-value theorem, we deduce that $\kappa_2 = 1/2$.

On the other hand, if we consider the Kantorovich conditions, we have to obtain an upper bound for $R''(u)$, for all u in the function space considered:

$$\left|R''(u)(t)\right| = \left|\int_0^1 st\sin u(s)\,ds\right| \leq \frac{1}{2} \equiv M.$$

So we can obtain information on the solution of (15) by analysing the roots of the polynomials defined in (4) and (6). In this case they are

$$p(t) = \frac{7}{48}\left(t^3 + t^2 - \frac{89}{14}t + \frac{2}{5}\right),$$
$$q(t) = \frac{7}{48}\left(3t^2 - \frac{89}{14}t + \frac{2}{5}\right).$$

The polynomial p has two positive roots: $r_1 = 0.0636$ and $r_2 = 2.0318$. The roots of q are $\hat{r}_1 = 0.0645$ and $\hat{r}_2 = 2.0541$. Notice that Case 2 in Section 1 holds and (15) has a solution $x^* \in C[0, 1]$ satisfying

$$|x^*(t) - x_0(t)| \leq r_1, \quad t \in [0, 1].$$

Furthermore, this solution is unique in $\{u \in X;\ \|u - y_0\| < r_2\}$.

Acknowledgement

This work was supported in part by DGES grant PB-98-0198 and grant API-99/B14 from the University of La Rioja.

References

[1] I. K. Argyros, "Newton-like methods under mild differentiability conditions with error analysis", *Bull. Austr. Math. Soc.* **37** (1988) 131–147.
[2] M. Braun, *Differential equations and their applications* (Springer, 1983).
[3] J. M. Gutiérrez, "A new semilocal convergence theorem for Newton's method", *J. Comput. Appl. Math.* **79** (1997) 131–145.
[4] J. M. Gutiérrez, M. A. Hernández and M. A. Salanova, "Accessibility of solutions by Newton's method", *Inter. J. Computer Math.* **57** (1995) 239–247.
[5] Z. Huang, "A note on the Kantorovich theorem for Newton iteration", *J. Comput. Appl. Math.* **47** (1993) 211–217.
[6] L. V. Kantorovich and G. P. Akilov, *Functional analysis* (Pergamon Press, 1982).
[7] A. M. Ostrowski, *Solution of equations in Euclidean and Banach spaces* (Academic Press, 1943).
[8] D. Porter and D. S. G. Stirling, *Integral equations, a practical treatment, from spectral theory to applications* (Cambridge, 1990).
[9] L. B. Rall, *Computational solution of nonlinear operator equations* (Robert E. Krieger Publishing Company, 1979).
[10] W. C. Rheinboldt, "A unified convergence theory for a class of iterative processes", *SIAM J. Numer. Anal.* **5** (1968) 42–63.
[11] J. Rockne, "Newton's method under mild differentiability conditions with error analysis", *Numer. Math.* **18** (1972) 401–412.
[12] F. Simmons, *Differential equations with applications and historical notes* (McGraw-Hill, 1972).
[13] T. Yamamoto, "A convergence theorem for Newton-like methods in Banach spaces", *Numer. Math.* **51** (1987) 545–557.
[14] T. Yamamoto, "On the method of tangent hyperbolas in Banach spaces", *J. Comput. Appl. Math.* **21** (1988) 75–86.