Ergod. Th. & Dynam. Sys., (2024), **44**, 1468–1472
doi:10.1017/etds.2023.45

A note on preimage entropy

TAO WANG

MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University, Changsha, Hunan 410081, P. R. China
(e-mail: twang@hunnu.edu.cn)

(*Received* 6 *December* 2022 *and accepted in revised form* 22 *May* 2023)

Abstract. Cheng and Newhouse (*Ergod. Th. & Dynam. Sys.* **25** (2005), 1091–1113) proved a variational principle for topological preimage entropy $h_{\mathrm{pre}}(f)$:

$$h_{\mathrm{pre}}(f) = \sup_{\mu\in\mathcal{M}(X,f)} h_{\mathrm{pre},\mu}(f).$$

Unfortunately, we show in this note that this variational principle is not true.

Key words: preimage entropy, variational principle
2020 Mathematics Subject Classification: 37D35, 37A35 (Primary)

1. *Introduction*

Let (X, f) be a topological dynamical system (t.d.s. for short), that is, (X, d) is a compact metric space and $f : X \to X$ is a continuous self-map. Preimage entropies were introduced and studied by Langevin and Przytycki [6], Hurley [5], Nitecki and Przytycki [7], and Fiebig, Fiebig and Nitecki [3]. These quantities give relevant information of how 'non-invertible' a system is. Among these entropy-like invariants, there are two kinds of pointwise preimage entropies:

$$h_m(f) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} s(n, \epsilon, f^{-n}(x)),$$

$$h_p(f) = \sup_{x\in X} \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(n, \epsilon, f^{-n}(x)),$$

where $s(n, \epsilon, Z)$ (or $s(n, \epsilon, Z, f)$) denotes the largest cardinality of any (n, ϵ)-separated set of $Z \subset X$. An important question is: can one introduce the counterpart of $h_m(f)$ or $h_p(f)$ from the measure-theoretic point of view, and obtain a variational principle relating them?

The first progress on this research was made by Cheng and Newhouse [1]. They defined a new notion of topological preimage entropy:

$$h_{\mathrm{pre}}(f) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X, k\geq n} s(n,\epsilon, f^{-k}(x)).$$

On the measure-theoretic side, they defined a corresponding measure-theoretic preimage entropy:

$$h_{\mathrm{pre},\mu}(f) = \sup_{\alpha} h_{\mathrm{pre},\mu}(f,\alpha),$$

where α ranges over all finite partitions of X,

$$h_{\mathrm{pre},\mu}(f,\alpha) = h_{\mu}(f,\alpha|\mathcal{B}^{-}) = \limsup_{n\to\infty}\frac{1}{n}H_{\mu}(\alpha_0^{n-1}|\mathcal{B}^{-}),$$

and $\mathcal{B}^{-}$ is the infinite past σ-algebra $\bigcap_{n\geq 0} f^{-n}\mathcal{B}$ related to the Borel σ-algebra $\mathcal{B}$. In addition, they stated a variational principle:

$$h_{\mathrm{pre}}(f) = \sup_{\mu\in\mathcal{M}(X,f)} h_{\mathrm{pre},\mu}(f), \tag{1.1}$$

where $\mathcal{M}(X, f)$ denotes the set of all f-invariant Borel probability measures on X.

Recently, Wu and Zhu [9] developed a variational principle for $h_m(f)$ under the condition of uniform separation of preimages. They introduced a new version of pointwise metric preimage entropy:

$$h_{m,\mu}(f) = \sup_{\alpha} h_{m,\mu}(f,\alpha),$$

where α ranges over all finite partitions of X and

$$h_{m,\mu}(f,\alpha) = \limsup_{n\to\infty}\frac{1}{n}H_{\mu}(\alpha_0^{n-1}|f^{-n}\mathcal{B}).$$

For f with uniform separation of preimages, the authors [9] established the following variational principle relating $h_{m,\mu}(f)$ and $h_m(f)$:

$$h_m(f) = \sup_{\mu\in\mathcal{M}(X,f)} h_{m,\mu}(f).$$

In fact, it was shown in [10, Proposition 3.1] that

$$h_{m,\mu}(f) = h_{\mathrm{pre},\mu}(f) \quad \text{for any } \mu\in\mathcal{M}(X, f).$$

For related definitions of topological and measure-theoretic entropies, we refer to the books [2, 4, 8].

In this note, we shall give an example to show that

$$h_{\mathrm{pre}}(f) > \sup_{\mu\in\mathcal{M}(X,f)} h_{\mathrm{pre},\mu}(f).$$

So the variational principle in equation (1.1) is not true.

2. *Main result*

In this section, we will state and prove our main result.

LEMMA 2.1. *Let $A = \{0, 1, 2\}$ and endow $A^{\mathbb{N}\times\mathbb{N}}$ with the product topology of the discrete topology on A. Denote by $f : A^{\mathbb{N}\times\mathbb{N}} \to A^{\mathbb{N}\times\mathbb{N}}$ the left shift map on rows; that is,*

$$(fx)_{m,i} = x_{m,i+1} \quad \text{for } m, i \in \mathbb{N}.$$

For each array $x = (x_{m,i})_{m,i\geq 0}$, denote by $i_0(x)$ the minimal $i \geq 0$ such that $x_{0,i} = 0$. If such an i does not exist, then we set $i_0(x) = \infty$. Let $X \subset A^{\mathbb{N}\times\mathbb{N}}$ consist of arrays such that:

(1) *for all $i \geq i_0(x)$ and all $m \geq 0$, we have $x_{m,i} = 0$;*
(2) *for all $0 \leq i < i_0(x)$ and all $m \geq 0$, we have $x_{m,i} \in \{1, 2\}$ and if both $m \geq 1$ and $i \geq 1$, then $x_{m,i} = x_{m-1,i-1}$.*

For the t.d.s (X, f), we have $h_{\text{pre}}(f) \geq \log 2$ and $h_{\text{pre},\mu}(f) = 0$ for any $\mu \in \mathcal{M}(X, f)$.

Proof. For $0 \leq n \leq \infty$, let A_n denote the set of points $x \in X$ with $i_0(x) = n$ and $\mathbf{0}$ denote the array consisting of just zeros. Then we have the following observations.

(1) $A_0 = \{\mathbf{0}\}$ and the element $\mathbf{0}$ has infinitely many preimages.
(2) Any element $x \in X \setminus A_0$ has exactly one preimage.
(3) (A_∞, f) is an invertible subsystem.

Let $\epsilon_0 > 0$ be so small that $x, y \in X$ with $x_{0,0} \neq y_{0,0}$ implies $d(x, y) \geq \epsilon_0$. Note that if we just observe the zero-row of $f^{-n}(\mathbf{0})$, we will see elements starting with any block of any length $0 \leq k \leq n$ over 1, 2 (followed by zeros). So we have

$$s(n, \epsilon_0, f^{-n}(\mathbf{0})) \geq \sum_{k=0}^{n} 2^k = 2^{n+1} - 1.$$

Hence,

$$\begin{aligned} h_{\text{pre}}(f) &= \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X, k\geq n} s(n, \epsilon, f^{-k}(x)) \\ &\geq \limsup_{n\to\infty} \frac{1}{n} \log s(n, \epsilon_0, f^{-n}(\mathbf{0})) \\ &\geq \log 2. \end{aligned}$$

Now we pass to evaluating the measure-theoretic preimage entropy. Notice that for each $0 < n < \infty$, the set A_n is visited by any orbit at most once implying that $\mu(A_n) = 0$ for any $\mu \in \mathcal{M}(X, f)$. So, any invariant measure μ is supported by $A_0 \cup A_\infty$. Fix $\mu \in \mathcal{M}(X, f)$. Without loss of generality, we may assume that $\mu(A_0) > 0$ and $\mu(A_\infty) > 0$. Consider the conditional measures

$$\mu_{A_0}(\cdot) = \frac{\mu(\cdot \cap A_0)}{\mu_{A_0}} \quad \text{and} \quad \mu_{A_\infty}(\cdot) = \frac{\mu(\cdot \cap A_\infty)}{\mu_{A_\infty}}.$$

It is easy to verify that both μ_{A_0} and μ_{A_∞} are invariant and $\mu = \mu(A_0)\mu_{A_0} + \mu(A_\infty)\mu_{A_\infty}$. By the affinity of measurable conditional entropy (see, for example, [2, Theorem 2.5.1], [1, Theorem 2.3] or [9, Proposition 2.12]), we have

https://doi.org/10.1017/etds.2023.45 Published online by Cambridge University Press

$$\begin{aligned} h_{\text{pre},\mu}(f) &= \mu(A_0)h_{\text{pre},\mu_{A_0}}(f) + \mu(A_\infty)h_{\text{pre},\mu_{A_\infty}}(f) \\ &\le \mu(A_0)h_{\mu_{A_0}}(f) + \mu(A_\infty)h_{\text{pre},\mu_{A_\infty}}(f) \\ &= 0. \end{aligned}$$

□

By Lemma 2.1, we can get our main result.

THEOREM 2.2. *There exists a t.d.s. (X, f) such that*

$$0 = \sup_{\mu \in \mathcal{M}(X,f)} h_{\text{pre},\mu}(f) < \log 2 \le h_{\text{pre}}(f).$$

Thus, the Cheng–Newhouse variational principle in equation (1.1) fails.

3. *Another definition of preimage entropy*

In [1], the authors show that $h_{\text{pre}}(f)$ can also be defined as

$$h_{\text{pre}}(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X, k \ge 1} s(n, \epsilon, f^{-k}x). \tag{3.1}$$

This result is based on their variational principle in equation (1.1). Now we shall give a topological proof of equation (3.1). In fact, it is a consequence of the following result.

For $Z \subset X$, let $r(n, \epsilon, Z)$ denote the smallest cardinality of any (n, ϵ)-spanning set of $Z \subset X$. It is clear that the above topological notions of entropies defined by separated sets can also be defined by spanning sets.

THEOREM 3.1. *Let $f : X \to X$ be a continuous map. Then,*

$$\begin{aligned} h_{\text{pre}}(f) &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} s(n, \epsilon, P_x) \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} r(n, \epsilon, P_x), \end{aligned}$$

where

$$P_x = \bigcup_{j \ge 0} f^{-j} f^j x.$$

Proof. Fix $y \in X$, $n \in \mathbb{N}$ and $k \ge n$. If $f^{-k}y \neq \emptyset$, then pick $x \in f^{-k}y$. So,

$$r(n, \epsilon, f^{-k}y) = r(n, \epsilon, f^{-k}f^k x) \le r(n, \epsilon, P_x),$$

which implies

$$h_{\text{pre}}(f) \le \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} r(n, \epsilon, P_x).$$

Next, we show the remaining inequality. Fix $s > h_{\text{pre}}(f)$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$r(n, \epsilon, f^{-k}x) \le e^{sn}$$

for all $x \in X$, $n \ge N$ and $k \ge n$.

https://doi.org/10.1017/etds.2023.45 Published online by Cambridge University Press

Fix $x \in X$, $n \geq N$. For $k \geq n$, let $E_k \subset X$ be an (n, ϵ)-spanning set of $f^{-k} f^k x$ with $\#E_k = r(n, \epsilon, f^{-k} f^k x) \leq e^{sn}$. Let $K(X)$ be the space of non-empty closed subsets of X equipped with the Hausdorff metric. Then we have $E_k \in K(X)$. As X is compact, $K(X)$ is also compact. So there exists a subsequence $\{k_j\}_{j\geq 1}$ such that $E_{k_j} \to E (j \to \infty)$. Then we have $\#E \leq e^{sn}$.

We claim that

$$P_x \subset \bigcup_{z \in E} B_n(z, 2\epsilon).$$

To see this, pick $y \in P_x$. Then there exists $J \geq n$ such that for any $j \geq J$, one has

$$y \in f^{-k_j} f^{k_j} x \subset \bigcup_{z \in E_{k_j}} B_n(z, \epsilon).$$

Furthermore, we can pick $z_{k_j} \in E_{k_j}$ to get

$$d_n(y, z_{k_j}) < \epsilon \quad \text{for all } j \geq J.$$

Without loss of generality, we assume that $\lim_{j\to\infty} z_{k_j} = z$. Then it is easy to see that $z \in E$ and

$$d_n(y, z) \leq \epsilon.$$

So the claim is true. Hence, we have $r(n, 2\epsilon, P_x) \leq \#E \leq e^{sn}$, from which one can get

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} r(n, \epsilon, P_x) \leq s.$$

By the choice of s, we obtain the reversed inequality. □

Acknowledgements. I would like to thank the referee for valuable suggestions that greatly improved the manuscript. This work was supported by National Nature Science Foundation of China (Grant No. 12001192)

REFERENCES

[1] W. C. Cheng and S. Newhouse. Pre-image entropy. *Ergod. Th. & Dynam. Sys.* **25** (2005), 1091–1113.
[2] T. Downarowicz. *Entropy in Dynamical Systems (New Mathematical Monographs, 18)*. Cambridge University Press, Cambridge, 2011.
[3] D. Fiebig, U. Fiebig and Z. Nitecki. Entropy and preimage sets. *Ergod. Th. & Dynam. Sys.* **23** (2003), 1785–1806.
[4] E. Glasner. *Ergodic Theory via Joinings (Mathematical Surveys and Monographs, 101)*. American Mathematical Society, Providence, RI, 2003.
[5] M. Hurley. On topological entropy of maps. *Ergod. Th. & Dynam. Sys.* **15** (1995), 557–568.
[6] R. Langevin and F. Przytycki. Entropie de limage inverse dune application. *Bull. Soc. Math. France* **120** (1992), 237–250.
[7] Z. Nitecki and F. Przytycki. Preimage entropy for mappings. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **9** (1999), 1815–1843.
[8] P. Walters. *An Introduction to Ergodic Theory*. Springer, New York, 1982.
[9] W. Wu and Y. Zhu. On preimage entropy, folding entropy and stable entropy. *Ergod. Th. & Dynam. Sys.* **41** (2020), 1217–1249.
[10] W. Wu and Y. Zhu. Entropy via preimage structure. *Adv. Math.* **406** (2022), 108483.

https://doi.org/10.1017/etds.2023.45 Published online by Cambridge University Press