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On the structure of Stone lattices

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C.C. Chen and G. Grätzer have shown that a Stone lattice is determined by a triple (C, D, ϕ) where C is a boolean algebra, D is a distributive lattice with 1 and ϕ is an e-homomorphism from C into $\mathcal{D}(D)$, the lattice of dual ideals of D.

It is shown here that any Stone lattice is, up to an isomorphism, a subdirect product of its centre $C(L)$ and a special Stone lattice $M(L)$. Special Stone lattices are characterised, in the terminology of the Chen-Grätzer triple, by the fact that the e-homomorphism ϕ is one to one.

In this paper we characterise a special Stone lattice L as a triple (H, C, D_0) where H is a distributive lattice with 0 and 1, C is a boolean e-subalgebra of the centre of H and D_0 is a sublattice of H with 0 such that

$$d \in D_0 \ \& \ c \in C \Rightarrow d \wedge c \in D_0 ,$$

and which separates the elements of C in the sense that for any $c_1 \neq c_2$ in C there is a d in D_0 with $d \leq c_1$ but $d \nleq c_2$.

It then turns out that C is $C(L)$ and D_0 is the dual of $D(L)$.

1. Introduction

We adopt, without further explanation, the notation and terminology of Chen and Grätzer. The motivation for the results given below comes from

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PROPOSITION 1. *Let* M *be a Stone lattice and let* $\theta : B \to C(M)$ *be an e-homomorphism from the boolean algebra* B *onto the centre of* M *which preserves complementation. Let* L *be the set of ordered pairs* $\langle x, b\rangle$ *, with* x *in* M *,* b *in* B *and* $b\theta = x^{**}$ *. Then* L *is a subdirect product of* M *and* B *, it is a Stone lattice, its centre* $C(L)$ *is isomorphic to* B *and* $D(L) = D(M) \times \{1\}$ *.*

Proof. It is easily verified that

$$b_i\theta = x_i^{**} \ , \quad i = 1, 2 \Rightarrow (b_1 \wedge b_2)\theta = (x_1 \wedge x_2)^{**} \ \ \& \\ (b_1 \vee b_2)\theta = (x_1 \vee x_2)^{**} \ .$$

It follows that L is a sublattice of $M \times B$ and, since θ is onto $C(M)$, L is a subdirect product of M and B . L is distributive, it has a least element $\langle 0, 0\rangle$ and a greatest element $\langle 1, 1\rangle$. Since θ preserves complementation

$$\langle x, b\rangle \in L \Rightarrow \langle x^*, b'\rangle \in L$$

where the prime denotes complementation in B . Since

$$\langle x, b\rangle \wedge \langle y, c\rangle = \langle 0, 0\rangle \Rightarrow \langle y, c\rangle \le \langle x^*, b'\rangle \ ,$$

L is pseudocomplemented by

$$\langle x, b\rangle^* = \langle x^*, b'\rangle \ .$$

Since

$$\langle x, b\rangle^* \vee \langle x, b\rangle^{**} = \langle x^* \vee x^{**}, b \vee b^*\rangle = \langle 1, 1\rangle \ ,$$

L is a Stone lattice. Clearly

$$D(L) = \{\langle d, 1\rangle : d \in D(M)\} = D(M) \times \{1\}$$

and

$$C(L) = \{\langle x, b\rangle : x \in C(M),\ b \in B,\ b\theta = x\} \ .$$

The correspondence $\langle x, b\rangle \to b$ and $b \to \langle b\theta, b\rangle$ are mutually inverse isomorphisms between $C(L)$ and B .

This proposition suggests the possibility that any Stone lattice L is representable, up to an isomorphism, as a subdirect product of another suitably chosen Stone lattice M and a boolean algebra B isomorphic to the centre of L , in the manner described in the proposition. In what follows we show that this is so and we demonstrate it in the following way. We introduce a $*$-congruence τ on the Stone lattice L (i.e. a lattice congruence which preserves pseudocomplements). We say that a Stone lattice is special when the $*$-congruence τ on it is the identity relation. L/τ is a special Stone lattice and τ_0, the restriction of τ to $C(L)$, is a congruence on $C(L)$ which preserves complements, further $C(L/\tau) = C(L)/\tau_0$. The mapping $C(L) \to C(L/\tau)$ is an e-homomorphism which preserves complementation and L is isomorphic to a subdirect product of L/τ and $C(L)$ in the manner described in Proposition 1 with L/τ playing the role of M.

The $*$-congruence τ is defined explicitly by writing $x\tau y$ to mean

(i) $\forall d \in D(L)$, $d \geq x^* \iff d \geq y^*$, and

(ii) $x \vee x^* = y \vee y^*$.

Note that (ii) is always true when x and y are in $C(L)$.

Through the paper L denotes a fixed but arbitrary Stone lattice, unless the contrary is explicitly stated.

2. The congruence τ_0 on $C(L)$

For a in $C(L)$ write

$$G_a = \{d : d \in D(L),\ d \geq a^*\} .$$

Then $a\tau_0 b$ if and only if $G_a = G_b$. Then τ_0 is an equivalence relation on $C(L)$, we write $a\tau_0$ for the τ_0-equivalence class in $C(L)$ determined by a, $C(L)/\tau_0$ for the set of τ_0-equivalence classes and $\tau_0^\natural$ for the natural mapping $C(L) \to C(L)/\tau_0$.

In fact τ_0 is a congruence on $C(L)$ which preserves complements. That τ_0 preserves join and meets follows from the fact that

$$G_{a\wedge b} = G_a \wedge G_b ,$$
$$G_{a\vee b} = G_a \vee G_b ,$$

where the lattice operations on the right are those of the lattice

$\mathcal{D}(D(L))$. From this it follows that $C(L)/\tau_0$ is a lattice with ordering

$$a\tau_0 \leq b\tau_0 \text{ if and only if } G_a \subseteq G_b ,$$

and lattice operations

$$a\tau_0 \wedge b\tau_0 = (a \wedge b)\tau_0 , \quad a\tau_0 \vee b\tau_0 = (a \vee b)\tau_0 .$$

Note that

$$0\tau_0 = \{a : a \in C(L), G_a = \{1\}\}$$

is the least element of $C(L)/\tau_0$ and

$$1\tau_0 = \{a : a \in C(L), G_a = D(L)\}$$

is its greatest element.

Next we observe that G_a is in the centre of $\mathcal{D}(D(L))$, its complement being G_{a^*} , so that

$$G_a \wedge G_b = G_0 \Leftrightarrow G_b \subseteq G_{a^*} .$$

Thus

$$a\tau_0 \wedge b\tau_0 = 0\tau_0 \Leftrightarrow b\tau_0 \leq a^*\tau_0 \Leftrightarrow a\tau_0 \leq b^*\tau_0 .$$

In particular

$$a\tau_0 = b\tau_0 \Rightarrow b^*\tau_0 \leq a^*\tau_0 \leq b^*\tau_0$$

so that

$$a\tau_0 b \Leftrightarrow a^*\tau_0 b^* .$$

Thus one can define $(a\tau_0)^* = a^*\tau_0$. One verifies that $C(L)/\tau_0$ is a boolean algebra with complementation $a\tau_0 \to (a\tau_0)^*$.

We note that the congruence τ_0 is just that associated with the structure map ϕ^L of Chen and Grätzer; this is an e-homomorphism from $C(L)$ into $\mathcal{D}(D(L))$. We summarize the results above in

LEMMA 2.1. *The map* $\tau_0^{\natural} : C(L) \to C(L)/\tau_0$ *is an* e-*homomorphism from* *the boolean algebra* $C(L)$ *onto the boolean algebra* $C(L)/\tau_0$ *which preserves complementation.*

We conclude this section with

LEMMA 2.2. *If* $a\tau_0 b$ *then* $d \vee a = d \vee b$ *for every* d *in* $D(L)$.

Proof. $d \vee a$ is in $G_{a^*} = G_{b^*}$. Thus $d \vee a \geq d \vee b$, by symmetry $d \vee b \geq d \vee a$.

3. τ is a *-congruence on L

We start by providing some background motivation for the introduction of the *-congruence τ. For a in $C(L)$ let

$$F_a = \{x : x \in L, x^{**} = a\},$$

then as noted by Chen and Grätzer, the correspondences

$$x \to x \vee a^* \text{ and } d \to d \wedge a$$

are mutually inverse isomorphisms between F_a and G_a. It follows that

LEMMA 3.1. *If* $a\tau_0 b$ *then the correspondences*

$$x \to (x \vee a^*) \wedge b \text{ and } y \to (y \vee b^*) \wedge a$$

are mutually inverse isomorphisms between the lattices F_a *and* F_b.

Elements of F_a and F_b $(a\tau_0 b)$ which correspond under these isomorphisms will be said (provisionally) to be τ_0-similar; more explicitly, elements x, y in L are τ_0-similar when

(i) $x^{**}\tau_0 y^{**}$;

(ii) $(x \vee x^*) \wedge y^{**} = y$ and $(y \vee y^*) \wedge x^{**} = x$.

Then we have

LEMMA 3.2. *Elements* x, y *in* L *are* τ_0*-similar if and only if they are* τ*-equivalent.*

Proof. τ-equivalence clearly implies τ_0-similarity. To prove the converse note that, since $y \vee y^*$ is dense, $x^{**}\tau_0 y^{**}$ implies that $y \vee y^* \geq x^*$. From (ii) above we obtain

$$x \vee x^* = (y \vee y^* \vee x^*) \wedge (x^* \vee x^{**}) = y \vee y^* .$$

We mention the obvious.

COROLLARY.

1. $a, b \in C(L)$ & $a\tau_0 b \Rightarrow a\tau b$
2. $a \in C(L)$, $x \in L$ & $a\tau x \Rightarrow x \in C(L)$ & $a\tau_0 x$;
3. $0\tau = 0\tau_0$, $1\tau = 1\tau_0$.

We proceed now to show that τ is a *-congruence on L. That τ preserves meets and pseudocomplements is easy to prove and we dispose of it in

LEMMA 3.3.

1. If $x\tau y$ then $x^*\tau y^*$;

2. if $x_i\tau y_i$, $i = 1, 2$ then $(x_1 \wedge x_2)\tau(y_1 \wedge y_2)$.

Proof. Firstly

$$x\tau y \Rightarrow x^{**}\tau_0 y^{**} \Rightarrow x^*\tau_0 y^* \Rightarrow x^*\tau y^* .$$

Secondly, when $x_i\tau y_i$,

$$(x_1 \wedge x_2)^{**} = (x_1^{**} \wedge x_2^{**})\ (y_1^{**} \wedge y_2^{**}) = (y_1 \wedge y_2)^{**}$$

and, using Lemma 2.2 at the third line below,

$$\begin{aligned}(x_1 \wedge x_2) \vee (x_1 \wedge x_2)^* &= \{(x_1 \vee x_1^*) \wedge (x_2 \vee x_2^*)\} \vee (x_1 \wedge x_2)^* \\ &= \{(y_1 \vee y_1^*) \wedge (y_2 \vee y_2^*)\} \vee (x_1 \wedge x_2)^* \\ &= \{(y_1 \vee y_1^*) \wedge (y_2 \vee y_2^*)\} \vee (y_1 \wedge y_2)^* \\ &= (y_1 \wedge y_2) \vee (y_1 \wedge y_2)^* .\end{aligned}$$

This proves the lemma.

To show that τ preserves lattice joins is a little more difficult; we do so through a number of contributory results.

LEMMA 3.4. *If $x\tau y$ and d is in $D(L)$ then*

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$$d \vee x = d \vee y .$$

Proof. By Lemma 2.2

$$x^{**}\tau_0 y^{**} \Rightarrow d \vee x^{**} = d \vee y^{**} ,$$
$$x\tau y \Rightarrow d \vee x \vee x^* = d \vee y \vee y^* .$$

Then when $x\tau y$ one has

$$(d \vee x^{**}) \wedge (d \vee x \vee x^*) = (d \vee y^{**}) \wedge (d \vee y \vee y^*)$$

that is,

$$d \vee x = d \vee y .$$

This is the desired result.

LEMMA 3.5. *Write* $d_1 = x_1 \vee x_1^*$, $d_2 = x_2 \vee x_2^*$; *then*

$$x_1 \vee x_2 \vee (x_1 \vee x_2)^* = (d_1 \wedge d_2) \vee [(d_1 \vee d_2) \wedge x_1^{**} \wedge x_2^{**}] .$$

Proof. Routine computation of the right-hand side.

We are now able to establish

LEMMA 3.6. *If* $x_i \tau y_i$, $i = 1, 2$; *then* $(x_1 \vee x_2)\tau(y_1 \vee y_2)$.

Proof. Clearly $(x_1 \vee x_2)^{**}\tau_0(y_1 \vee y_2)^{**}$. It is then only necessary to show that

$$x_1 \vee x_2 \vee (x_1 \vee x_2)^* = y_1 \vee y_2 \vee (y_1 \vee y_2)^* .$$

But this equation follows easily from Lemmas 3.4 and 3.5 since $(x_1^{**} \wedge x_2^{**})\tau(y_1^{**} \wedge y_2^{**})$ and τ preserves meets, so that

$$[(d_1 \vee d_2) \wedge x_1^{**} \wedge x_2^{**}]\tau[(d_1 \vee d_2) \wedge y_1^{**} \wedge y_2^{**}] .$$

Finally $d_i = x_i \vee x_i^* = y_i \vee y_i^*$ and so we obtain the desired result.

Collecting the above results we have

THEOREM 1. *L/τ is a special Stone lattice with pseudocomplementation $(x\tau)^* = x^*\tau$. $C(L/\tau) = C(L)/\tau_0$ and $D(L/\tau)$ is isomorphic to $D(L)$. Further $\tau_0 : C(L) \to C(L/\tau)$ is an e-homomorphism of $C(L)$ onto $C(L/\tau)$ which preserves complementation.*

That L/τ is a special Stone lattice is easily verified and the only part of the theorem which calls for proof is the statement about $D(L/\tau)$; to establish its isomorphism with $D(L)$ we need

LEMMA 3.7.

(i) if d_1, d_2 *are in* $D(L)$ *and* $d_1 \tau d_2$ *then* $d_1 = d_2$,

(ii) if d *is in* $D(L)$ *then* $d\tau$ *is in* $D(L/\tau)$,

(iii) if $x\tau$ *is in* $D(L/\tau)$ *then there is exactly one* d *in* $D(L)$ *with* $x\tau = d\tau$.

Proof. If d_1, d_2 are in $D(L)$ and $d_1 \tau d_2$ then

$$d_1 = d_1 \vee d_1^* = d_2 \vee d_2^* = d_2 .$$

If d is in $D(L)$ then

$$(d\tau)^* = d^*\tau = 0\tau ,$$

so $d\tau$ is in $D(L/\tau)$.

Finally if $x\tau$ is in $D(L/\tau)$ then $(x\tau)^* = 0\tau$ so

$$(x \vee x^*)\tau = x\tau \vee x^*\tau = x\tau$$

and $x \vee x^*$ is dense.

It follows from Lemma 3.7 that

$$D(L/\tau) = \{d\tau : d \in D(L)\}$$

and the isomorphism between $D(L/\tau)$ and $D(L)$ is an obvious consequence.

4. L is a subdirect product of L/τ and $C(L)$

We prove

THEOREM 2. *Let* L *be a Stone lattice. Then* L *is* $*$*-isomorphic with the subdirect product of* L/τ *and* $C(L)$ *consisting of all ordered pairs* $\langle t, a\rangle$ *with* t *in* L/τ *, and* a *in* $C(L)$ *and* $a\tau_0 = t^{**}$.

Proof. The theorem is proved by a routine verification that ξ, η defined by

$$x\xi = \langle x\tau, x^{**}\rangle , \quad x \in L ,$$

$$\langle t, a\rangle\eta = t \wedge a , \quad a\tau_0^{\eta} = t^{**} ,$$

are mutually inverse $*$-isomorphisms between the two Stone lattices in question.

Thus any Stone lattice L determines a triple $\big(C(L), M(L), \theta\big)$ where $M(L) = L/\tau$ is a special Stone lattice and θ is an e-homomorphism form $C(L)$ onto $C\big(M(L)\big)$ which preserves complementation. Conversely, if (B, M, θ) is a triple where

(i) B is a boolean algebra,

(ii) M is a special Stone lattice,

(iii) θ is an e-homomorphism form B onto $C(M)$ which preserves complementation,

then the Stone lattice L constructed in Proposition 1 is such that $B = C(L)$ and $M \simeq L/\tau$. This is the content of

PROPOSITION 2. *If M in Proposition* 1 *is a special Stone lattice then L/τ is $*$-isomorphic to M .*

Proof. Returning to the notation of Proposition 1 one finds that

$$G_{\langle x^{**},\, b\rangle} = G_{x^{**}}X\{1\}$$

and it follows that, if M is a special Stone lattice,

$$\langle x^{**}, b\rangle \tau_0 \langle y^{**}, c\rangle \iff x^{**} = y^{**} \ .$$

Thus

$$\begin{aligned} \langle x, b\rangle \tau \langle y, c\rangle \ &\Rightarrow \ x^{**} = y^{**} \ \& \ x \vee x^* = y \vee y^* \\ &\Rightarrow \ x = (x \vee x^*) \wedge x^{**} = (y \vee y^*) \wedge y^{**} = y \ . \end{aligned}$$

Hence, for $\langle x, b\rangle$ in L , one has

$$\langle x, b\rangle \tau = \{\langle x, c\rangle \ : \ c\theta = b\theta\} \ .$$

It is now routine to verify that the correspondence $x \to \langle x, b\rangle \tau$ is a $*$-isomorphism between M and L/τ .

The preceeding results reduce the study of arbitrary Stone lattices to that of special Stone lattices. As noted in the summary, special Stone lattices are characterised by the fact that the structure map in their Chen-Grätzer triple is one to one. It follows at once that a special Stone lattice M is characterised by the distributive lattice $D(M)$ and

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a boolean e-subalgebra of the centre of $\mathcal{D}(D(M))$, namely the image of the lattice $C(M)$ under the structure map ϕ. However, rather than pursue a direct analogy with the approach of Chen and Grätzer we embark on a slightly more oblique one in the next section. This second approach is based on the observation that the sublattice of $\mathcal{D}(D(M))$ consisting of the principal dual ideals of $D(M)$ is isomorphic with the dual of D. We can then characterise a special Stone lattice through the distributive lattice $\mathcal{D}(D(M))$, a boolean e-subalgebra of its centre and $P(D(M))$, the sublattice of principal dual ideals of $D(M)$.

5. The construction of Stone lattices

We prove the following

THEOREM 3. *Let H be a distributive lattice with* 0 *and* 1. *Let C be a boolean e-subalgebra of H and let D_0 be a sublattice of H which contains* 0 *and which has the property that*

$$d \in D_0 \ \&\ c \in C \Rightarrow d \wedge c \in D_0 .$$

Let L be the set of ordered pairs $\langle d, c\rangle$ with d in D_0, c in C and $d \leq c$. Order L by the prescription $\langle d_1, c_1\rangle \leq \langle d_2, c_2\rangle$ if and only if $c_1 \leq c_2$ and $d_2 \wedge c_1 \leq d_1$.

Then $(L, \leq)$ is a lattice which is a Stone lattice pseudocomplemented by

$$\langle d, c\rangle^* = \langle 0, c'\rangle ,$$

the prime denoting complementation in C. Its centre is isomorphic to C and its lattice of dense elements is isomorphic to the dual of D_0. Moreover $(L, \leq, ^*)$ is a special Stone lattice if and only if D_0 separates the elements of C, in the sense that for any $c_1 \neq c_2$ in C there is d in D_0 with $d \leq c_1$ but $d \not\leq c_2$.

Proof. $\leq$ is clearly reflexive and antisymmetric on L. To establish transitivity we note

$$\langle d_1, c_1\rangle \le \langle d_2, c_2\rangle \le \langle d_3, c_3\rangle \Rightarrow c_1 \le c_2 \le c_3 \ \& \ c_1 \wedge d_2 \le d_1 \ \& \ c_2 \wedge d_3 \le d_2$$
$$\Rightarrow c_1 \le d_3 \ \& \ c_1 \wedge d_3 = c_1 \wedge c_2 \wedge d_3 \le c_1 \wedge d_2 \le d_1 .$$

To establish the lattice property we show that

(1) $\langle d_1, c_1\rangle \wedge \langle d_2, c_2\rangle = \langle (d_1 \wedge c_2) \vee (d_2 \wedge c_1), c_1 \wedge c_2\rangle$,

(2) $\langle d_1, c_1\rangle \vee \langle d_2, c_2\rangle = \langle (d_1 \wedge c_2') \vee (d_2 \wedge c_1') \vee (d_1 \wedge d_2), c_1 \vee c_2\rangle$.

To prove (1), one verifies firstly that its right-hand side is in L and is a common lower bound to $\langle d_1, c_1\rangle$, $\langle d_2, c_2\rangle$; if $\langle d_3, c_3\rangle$ in L is any other lower bound one has $d_3 \le c_3 \le c_1 \wedge c_2$ and $c_3 \wedge (d_1 \vee d_2) \le d_3$. Then

$$c_3 \le \{(d_1 \vee d_2) \wedge c_1 \wedge c_2\} \le d_3 ,$$

so

$$\langle d_3, c_3\rangle \le \langle (d_1 \vee d_2) \wedge c_1 \wedge c_2, c_1 \wedge c_2\rangle .$$

Since

$$(d_1 \vee d_2) \wedge c_1 \wedge c_2 = (d_1 \wedge c_2) \vee (d_2 \wedge c_1) ,$$

we obtain the desired result.

To establish (2) one verifies firstly that its right-hand side is in L and is a common upper bound to $\langle d_1, c_1\rangle$, $\langle d_2, c_2\rangle$. On the other hand if $\langle d_3, c_3\rangle$ in L is any other upper bound then $d_3 \vee c_1 \vee c_2 \le c_3$ and $d_3 \wedge c_i \le d_i$, $i = 1, 2$. Whence

$$d_3 \le (d_1 \vee c_1') \wedge (d_2 \vee c_2') .$$

From

$$\langle (d_1 \vee c_1') \wedge (d_2 \vee c_2') \wedge (c_1 \vee c_2), c_1 \vee c_2\rangle \le \langle d_3 \wedge (c_1 \vee c_2), c_1 \vee c_2\rangle \le \langle d_3, c_3\rangle$$

and

$$(d_1 \vee c_1') \wedge (d_2 \vee c_2') \wedge (c_1 \vee c_2) = (d_1 \wedge c_2') \vee (d_2 \wedge c_1') \vee (d_1 \wedge d_2)$$

we obtain the desired result.

It is routine, though tedious, to verify that L is distributive. Straight-forward calculations establish that, $\langle 0, 0\rangle$ is the least element in L , $\langle 0, 1\rangle$ is the greatest element,

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$$\langle d, c\rangle^* = \langle 0, c'\rangle ,$$

and that L is a Stone lattice and finally that

$$C(L) = \{\langle 0, c\rangle : c \in C\}$$
$$D(L) = \{\langle d, 1\rangle : d \in D_0\} .$$

To establish the conditions under which L is a special Stone lattice observe that

$$\langle d, 1\rangle \geq \langle 0, c'\rangle \iff d \wedge c' = 0 .$$

Since $d \wedge c$ and $d \wedge c'$ belong to D_0

$$d = (d \wedge c) \vee (d \wedge c')$$

and we deduce that

$$\langle 0, c_1\rangle \tau_0 \langle 0, c_2\rangle \iff \text{"}\forall d \in D_0,\ d \leq c_1 \iff d \leq c_2\text{"} .$$

But if L is a special Stone lattice τ_0 is the identity relation on $C(L)$ and we have the property stated in the theorem. Conversely if that property holds then τ_0 is the identity relation on $C(L)$. If then $\langle d_1, c_1\rangle \tau \langle d_2, c_2\rangle$ we must have $c_1 = c_2 = c$, say, and

$$\langle d_1, 1\rangle = \langle d_1, c\rangle \vee \langle d_1, c\rangle^* = \langle d_2, c\rangle \vee \langle d_2, c\rangle^* = \langle d_2, 1\rangle$$

and so τ is the identity relation on L, that is L is a special Stone lattice.

We show now that any special Stone lattice can be constructed in the way described in Theorem 3.

THEOREM 4. *Let L' be a special Stone lattice. Take $H = \mathcal{D}\big(D(L')\big)D_0 = P\big(D(L')\big)$, the set of principal dual ideals of $D(L')$, and let C be a boolean e-subalgebra of $C\Big[\mathcal{D}\big(D(L')\big)\Big]$. Then L' and L are isomorphic as Stone lattices.*

Proof. We firstly verify that L is a special Stone lattice. This follows from the fact that the intersection of any principal dual ideal of a distributive lattice with an arbitrary dual ideal in the centre of the lattice of its dual ideals is again a principal dual ideal. Thus

$$d \in P(D(L')) \;\&\; c \in C\left[D(D(L'))\right] \Rightarrow d \wedge c \in P(D(L')) .$$

That D_0 separates C is obvious since L' is special.

The theorem is proved by verifying that the correspondence

$$x \to \langle [x \vee x^*), G_{x^{**}} \rangle$$

from L' onto L is the desired isomorphism.

Reference

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