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ON PP-ENDOMORPHISM RINGS

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ABSTRACT. A characterization is given of when all kernels (respectively images) of endomorphisms of a module are direct summands, a necessary condition being that the endomorphism ring itself is a left (respectively right) PP-ring. This result generalizes theorems of Small, Lenzing and Colby-Rutter and shows that R is left hereditary if and only if the endomorphism ring of every injective left module is a right PP-ring.

If R is a ring and ${}_RM$ is a module, it is well known that end ${}_RM$ is a regular ring (in the sense of von Neumann) if and only if both $\ker \alpha$ and $\operatorname{im} \alpha$ are direct summands of ${}_RM$ for all α in $S = \operatorname{end} {}_RM$. In this note, each of these conditions is characterized separately, and one of the requirements is that S be a left (respectively right) PP-ring. Here a ring S is called a left PP-*ring* if every principal left ideal is projective, or equivalently if, for all $b \in S$, $\ell_S(b) = Se$ for some $e^2 = e \in S$. (Left and right annihilators are denoted by $\ell(X)$ and $r(X)$.) Right PP-rings are defined analogously, and these rings seem to have been first introduced by Hattori [2]. Throughout the paper, all rings are associative with unity, all modules are unitary and endomorphisms are written opposite the scalars.

If M_S is a right S-module and $b \in S$ then, as Hattori [2] observed, the inclusions

$$M\ell_S(b) \subseteq \ell_M(b) \text{ and } Mb \subseteq \ell_M[r_S(b)]$$

are, respectively, equalities if and only if the respective natural sequences

$$0 \longrightarrow M \otimes Sb \longrightarrow M \otimes S \text{ and } \hom(S, M) \longrightarrow \hom(bS, M) \longrightarrow 0$$

are exact. We will call M_S *P-flat*, respectively *P-injective*, when this condition is satisfied. (Hattori calls M_S "torsion free", respectively "divisible".) Hattori proved that the following are equivalent: (1) S is regular; (2) every M_S is P-flat; (3) every M_S is P-injective.

The first main result is the following:

THEOREM 1. *The following are equivalent for a faithful module M_S:*
(1) For all $b \in S$, $\ell_M(b) = Me$ for some $e^2 = e \in S$.
(2) M_S is P-flat and S is a left PP-*ring.*

PROOF. (1) $\Rightarrow$ (2). If $b \in S$, let $\ell_M(b) = Me$ for some $e^2 = e \in S$. Then $Meb = 0$ so $eb = 0$ because M_S is faithful. Hence $\ell_M(b) = Me \subseteq M\ell_S(b)$ so M_S is P-flat. Clearly, $Se \subseteq \ell_S(b)$; if $sb = 0$ then $ms \in \ell_M(b) = Me$ for each $m \in M$. Hence $mse = ms$ for all $m \in M$ so $s \in Se$ (again because M_S is faithful). Thus $\ell_S(b) = Se$ so S is a left PP-ring.

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(2) $\Rightarrow$ (1). Let $b \in S$. Since S is a left PP-ring, we have $\ell_S(b) = Se$, $e^2 = e \in S$. Hence the fact that M_S is P-flat implies $\ell_M(b) = M\ell_S(b) = MSe = Me$. ∎

Note that M_S need not be faithful for (2) $\Rightarrow$ (1).

Our main interest in Theorem 1 is when $M = {}_RM$ is an R-module for some ring R and $S = \text{end}\,{}_RM$. Write ${}_RN \mid {}_RM$ to signify that ${}_RN$ is a direct summand of ${}_RM$. Then Theorem 1 specializes as follows:

THEOREM 1′. *Given a left module ${}_RM$, write $S = \text{end}\,{}_RM$. Then the following conditions are equivalent:*

(1) $\ker\beta \mid {}_RM$ *for all* $\beta \in S$.

(2) M_S *is P-flat and S is a left* PP-*ring.*

In order to apply Theorem 1′, we need conditions on ${}_RM$ which imply that M_S is P-flat where $S = \text{end}\,{}_RM$. The module ${}_RM$ is called *a selfgenerator* if it generates each of its images, that is, $Rm = M \cdot \text{hom}_R(M, Rm)$ for all $m \in M$. Thus, every generator (and hence every free module) is a selfgenerator. Clearly, ${}_RM$ is a selfgenerator if $Rm \mid M$ for all $m \in M$. Hence every semisimple module is a selfgenerator as is every regular module [5].

LEMMA 1. *If ${}_RM$ is a selfgenerator then M_S is P-flat where $S = \text{end}\,{}_RM$.*

PROOF. Let $m\beta = 0$, $m \in M$, $\beta \in S$. Since ${}_RM$ is a selfgenerator, let $m = \Sigma m_i\alpha_i$, $m_i \in M$, $\alpha_i \in \text{hom}_R(M, Rm)$. Then $M\alpha_i\beta \subseteq (Rm)\beta = 0$ so $\alpha_i \in \ell_S(\beta)$ for each i. Hence $m \in M\ell_S(\beta)$, as required. ∎

Now Theorem 1′ gives immediately

PROPOSITION 2. *Let $S = \text{end}\,{}_RM$ and assume ${}_RM$ is a selfgenerator. Then S is a left* PP-*ring if and only if* $\ker\beta \mid M$ *for all* $\beta \in S$.

If $n \geq 1$ we call a module *n-hereditary* if every n-generated submodule is projective. A ring R is *left n-hereditary* if ${}_RR$ is n-hereditary.

LEMMA 2. *If R is left n-hereditary, then ${}_RF$ is n-hereditary for all f.g. free modules ${}_RF$.*

PROOF. If $\{f_1, \ldots, f_k\}$ is a basis of ${}_RF$, induct on $k \geq 1$. It is clear if $k = 1$. If $k > 1$, let $G = Rf_1 \oplus \cdots \oplus Rf_{k-1}$. Suppose ${}_RM \subseteq F$ is n-generated. Then

$$P = \frac{M}{M \cap G} \cong \frac{M + G}{G} \subseteq \frac{F}{G} \cong {}_RR$$

so $M/(M \cap G)$ is projective (being n-generated). Hence $M \cong (M \cap G) \oplus P$, so $M \cap G$ is also n-generated. It is thus projective by induction, and we are done. ∎

A similar argument shows that every free module over a hereditary ring is again hereditary. This, with Lemma 1, gives the following known results.

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COROLLARY. *(1)* Colby-Rutter [1]*: A ring R is left hereditary if and only if* end ${}_RF$ *is a left* PP*-ring for all free modules* ${}_RF$.

(2) Lenzing [3]*: If* $n \geq 1$*, a ring R is left n-hereditary if and only if* $M_n(R)$ *is a left* PP*-ring.*

(3) Small [4]*: A ring R is left semihereditary if and only if* $M_n(R)$ *is a left* PP*-ring for all* $n \geq 1$.

PROOF. If R is left hereditary (n-hereditary) so is each free module ${}_RF$. If $\beta \in S =$ end ${}_RF$, then $F\beta$ is projective in both cases. Since F_S is P-flat, end ${}_RF$ is a left PP-ring by Proposition 2. Conversely, if $L \subseteq R$ is a (n-generated) left ideal and ${}_RF \longrightarrow L \longrightarrow 0$ where ${}_RF$ is free (respectively ${}_RF = R^n$), choose $L' \subseteq F$ with $L' \cong L$. Then there exists $\beta: F \longrightarrow L' \longrightarrow 0$ so $\ker \beta \mid F$ by Proposition 2. Thus L' (and hence L) is projective. ∎

We now prove the "dual" to Theorem 1.

THEOREM 2. *The following conditions are equivalent for a faithful module* M_S*:*

(1) For all $b \in S$, $Mb = Me$ *for some* $e^2 = e \in S$.

(2) M_S *is P-injective and S is a right* PP*-ring.*

PROOF (1) ⇒ (2). Given $b \in S$, let $Mb = Me$, $e^2 = e \in S$, so that $b(1 - e) = 0$ (because M_S is faithful). If $m \in \ell_M[r_S(b)]$, this means $m(1 - e) = 0$, so $m \in Me = Mb$. Hence M_S is P-injective and it suffices to show $r_S(b) = (1 - e)S$. We have already shown that $1 - e \in r_S(b)$. If $bs = 0$, $s \in S$, then $Mes = Mbs = 0$ so $es = 0$ (again because M_S is faithful). Hence $s = (1 - e)s \in (1 - e)S$, as required.

(2) ⇒ (1). Let $b \in S$ and write $r_S(b) = fS, f^2 = f \in S$. Hence $bf = 0$ so $b = b(1 - f)$ and $Mb \subseteq M(1 - f)$. But $M(1 - f) \subseteq \ell_M[fS] = \ell_M[r_S(b)] = Mb$ because M_S is P-injective. Hence (1) follows with $e = 1 - f$. ∎

As in Theorem 1, the proof of (2) ⇒ (1) does not require that M_S is faithful.

If we specialize to the case where $M = {}_RM$ and $S =$ end ${}_RM$, Theorem 2 becomes

THEOREM 2′. *Given a left module* ${}_RM$*, write* $S =$ end ${}_RM$*. Then the following conditions are equivalent:*

(1) $M\beta \mid M$ *for all* $\beta \in S$.

(2) M_S *is P-injective and S is a right* PP*-ring.*

If we take $R = M = S$ in Theorem 2′ we obtain

COROLLARY. *A ring S is regular if and only if* S_S *is P-injective and S is a right* PP*-ring.*

Thus, for example, a right selfinjective, right PP-ring is regular.

It follows from Theorem 1′ (and its Corollary (1)) that a ring R is left hereditary if and only if end ${}_RP$ is a left PP-ring for all projective modules ${}_RP$. Theorem 2′ gives the "dual".

PROPOSITION 3. *A ring R is left hereditary if and only if $S = \text{end}\,_RM$ is a right* PP-*ring for every injective module $_RM$.*

PROOF. Given the condition, let $\beta\colon {}_RM \longrightarrow {}_RN$ be epic and define $\gamma \in \text{end}\,[M \oplus E(N)]$ by $(m,x)\gamma = (0, x\beta)$. Then $N \cong \text{im}\,\gamma$ is injective by hypothesis. The rest follows from Theorem 2′. ∎

Call a module $_RM$ a *selfcogenerator* if it cogenerates each of its cokernels, that is, $0 \neq x \in M/M\beta$, $\beta \in \text{end}\,_RM$, implies that $x\lambda \neq 0$ for some $\lambda \in \text{hom}_R(M/M\beta, M)$. Clearly cogenerators have this property as do modules $_RM$ in which $M\beta \mid M$ for all $\beta \in \text{end}\,_RM$. We have the "duals" of Lemma 1 and Proposition 2.

LEMMA 3. *If $_RM$ is a selfcogenerator then M_S is P-injective where $S = \text{end}\,_RM$.*

PROOF. Given $\beta \in S$, let $m \in \ell_M[r_S(\beta)]$ and assume $m \notin M\beta$. Then $(m + M\beta)\lambda \neq 0$ for some $\lambda\colon M/M\beta \longrightarrow M$, so $m\alpha \neq 0$ where $\alpha \in S$ is defined by $m\alpha = (m + M\beta)\lambda$. But $\beta\alpha = 0$ so this is a contradiction. ∎

PROPOSITION 4. *If $_RM$ is a selfcogenerator then $S = \text{end}\,_RM$ is a right* PP-*ring if and only if $M\beta \mid M$ for all $\beta \in S$.*

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