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AN APPROXIMATION THEOREM FOR SOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS

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Abstract The main result establishes that a weak solution of degenerate elliptic equations can be approximated by a sequence of solutions of non-degenerate elliptic equations. To this end we prove an approximation theorem for A_p-weights.

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1. Introduction

Let L be a degenerate elliptic operator in divergence form

$$Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x)), \quad D_j = \frac{\partial}{\partial x_j}, \tag{1.1}$$

and the coefficients a_{ij} are measurable, real-valued functions defined on a bounded open set $\Omega \subset \mathbb{R}^n$, satisfying the degenerate ellipticity condition

$$|\xi|^2\omega(x) \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant v(x)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega, \tag{1.2}$$

$$a_{ij}(x) = a_{ji}(x), \tag{1.3}$$

where ω and v are weight functions (that is, $\omega, v \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\omega(x) > 0$ and $v(x) > 0$ a.e. in $\mathbb{R}^n$).

The main purpose of this paper (see Lemma 4.13 and Theorem 4.14) is to establish that a weak solution $u \in H^{1,2}(\Omega, \omega, v)$ for the Dirichlet problem

$$\left.\begin{aligned} Lu = g - \sum_{j=1}^{n} D_j f_j, \quad \text{in } \Omega,\\ u - \psi \in H_0^{1,2}(\Omega, \omega, v), \end{aligned}\right\} \tag{P}$$

can be approximated by a sequence of solutions of non-degenerate elliptic equations.

The first step is to prove a general approximation theorem for A_p-weights ($p \geqslant 1$) (see Lemma 4.13), and to this end we will need the definition of dyadic cubes (Definition 4.10) and the Jones Factorization Theorem (Theorem 4.11). Lemma 4.13 is the key point for Theorem 4.14, and the crucial point consists of showing that a weak limit of a sequence of solutions of approximate problems is in fact a solution of the original problem.

The plan of the paper is the following: § 2 is devoted to introducing and studying the 'weighted Sobolev spaces' $H^{k,p}(\Omega,\omega)$ and $W^{k,p}(\Omega,\omega)$, and the existence of a solution to the Dirichlet problem. In § 3 we will present a short proof to the approximation theorem in the case $\omega = v$ (the proof is like the one given in Theorem 3.14 of [**6**]). Finally, in § 4 we generalize the results in the case $\omega \neq v$.

We make the following basic assumption on the weights ω and v.

The weighted Sobolev inequality (WSI). There is an index $q = 2\sigma$, $\sigma > 1$, such that for every ball B and every $f \in \mathrm{Lip}_0(B)$ (i.e. $f \in \mathrm{Lip}(B)$ whose support is contained in the interior of B),

$$\left(\frac{1}{v(B)}\int_B |f|^q v\,\mathrm{d}x\right)^{1/q} \leqslant CR_B\left(\frac{1}{\omega(B)}\int_B |\nabla f|^2\omega\,\mathrm{d}x\right)^{1/2}, \tag{1.4}$$

with the constant C independent of f and B, R_B is the radius of B and the symbol ∇ indicates the gradient, $v(B) = \int_B v(x)\,\mathrm{d}x$ and $\omega(B) = \int_B \omega(x)\,\mathrm{d}x$ (see [**2**]).

Thus, we can write

$$\left(\int_B |f|^q v\,\mathrm{d}x\right)^{1/q} \leqslant C_{B,\omega,v}\left(\int_B |\nabla f|^2\omega\,\mathrm{d}x\right)^{1/2},$$

where $C_{B,\omega,v}$ is called the Sobolev constant and

$$C_{B,\omega,v} = C\frac{[v(B)]^{1/q}R_B}{[\omega(B)]^{1/2}}. \tag{1.5}$$

For instance, the WSI holds if ω and v are as in Theorem 4.8, Chapter X, of [**11**], or if ω and v are as in Theorem 1.5 of [**1**]. In case $\omega = v$, see Theorem 1.2 of [**5**] or Theorem 15.23 of [**9**].

2. Definitions and basic results

In this section, we present a brief discussion of the function spaces $H^{k,p}(\Omega,\omega)$, $W^{k,p}(\Omega,\omega)$ and their basic properties, and we prove the existence and uniqueness for the Dirichlet problem.

Throughout this paper we assume that $\Omega \subset \mathbb{R}^n$ is a fixed bounded open set.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ and let ω be a weight function. We shall denote by $L^p(\Omega,\omega)$, $1 \leqslant p < \infty$, the Banach space of all measurable functions, f, defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p\omega(x)\,\mathrm{d}x\right)^{1/p} < \infty.$$

Definition 2.2. The weight ω belongs to the Muckenhoupt class, $\omega \in A_p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, if there is a constant $\boldsymbol{C}$ (called A_p-constant, $\boldsymbol{C} = C(\omega, p)$) such that

$$\left(\frac{1}{|Q|}\int_Q \omega \, \mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega^{-1/(p-1)} \, \mathrm{d}x\right)^{p-1} \leqslant \boldsymbol{C}, \quad \text{when } 1 < p < \infty,$$

and

$$\left(\frac{1}{|Q|}\int_Q \omega \, \mathrm{d}x\right) \leqslant \boldsymbol{C} \operatorname*{ess\,inf}_Q \omega, \quad \text{when } p = 1,$$

for all cubes Q in $\mathbb{R}^n$, $\boldsymbol{C}$ is independent of Q, where $|Q|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$ (see [**10**]).

We have that $A_1 \subset A_q \subset A_p$ for all $1 < q \leqslant p$.

Remark 2.3. If $\omega \in A_p$, then ω is a doubling weight (that is, $\omega(B(x, 2r)) \leqslant C\omega(B(x, r))$, where $C = 2^{np}C(\omega, p)$) (see Corollary 15.7 in [**9**]).

Proposition 2.4. *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$.*

(a) *If $f \in L^p(\Omega, \omega)$ and $\omega^{-p'/p} \in L_1(\Omega)$, then $f \in L_1(\Omega)$ (with $1/p + 1/p' = 1$).*

(b) *If $f_m \to f$ in $L^p(\Omega, \omega)$ and $\omega^{-p'/p} \in L_1(\Omega)$, then $f_m \to f$ in $L_1(\Omega)$.*

Proof. It is an immediate consequence of Hölder's inequality.
Note that, if $w \in A_p$, then we have $\omega^{-p'/p} \in L_1(\Omega)$. □

Definition 2.5. Let $\Omega \subset \mathbb{R}^n$, $1 \leqslant p < \infty$ and let ω be a weight function.

(a) The space $H^{k,p}(\Omega, \omega)$, $k \in \mathbb{N}$, is defined as the closure of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\|u\|_{k,p} = \left(\int_\Omega |u(x)|^p \omega(x) \, \mathrm{d}x + \sum_{1 \leqslant |\alpha| \leqslant k} \int_\Omega |D^\alpha u(x)|^p \omega(x) \, \mathrm{d}x\right)^{1/p}.$$

(b) The space $H_0^{k,p}(\Omega, \omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{0,k,p} = \left(\sum_{1 \leqslant |\alpha| \leqslant k} \int_\Omega |D^\alpha u(x)|^p \omega(x) \, \mathrm{d}x\right)^{1/p}.$$

The spaces $H^{k,2}(\Omega, \omega)$ and $H_0^{k,2}(\Omega, \omega)$ are Hilbert spaces.

(c) The dual space of $H^{1,2}(\Omega, \omega)$ is the space

$$\begin{aligned}(H^{1,2}(\Omega, \omega))^* &= H^{-1,2}(\Omega, \omega) \\ &= \bigg\{ g - \operatorname{div} \boldsymbol{f} : \boldsymbol{f} = (f_1, \dots, f_n), \\ &\qquad \text{with } \frac{g}{\omega} \text{ and } \frac{f_i}{\omega} \in L^2(\Omega, \omega),\ i = 1, \dots, n \bigg\}.\end{aligned}$$

If $T = g - \operatorname{div} \boldsymbol{f} \in H^{-1,2}(\Omega,\omega)$, then (see [**4**])

$$\langle T, u\rangle = \int_\Omega g(x)u(x)\,\mathrm{d}x + \sum_{j=1}^{n}\int_\Omega f_j(x)D_ju(x)\,\mathrm{d}x, \quad \forall u \in H^{1,2}(\Omega,\omega).$$

Definition 2.6. Let $\Omega \subset \mathbb{R}^n$ and let ω be a weight function. The space $W^{k,p}(\Omega,\omega)$ is defined by

$$W^{k,p}(\Omega,\omega) = \{u \in L^p(\Omega,\omega) : D^\alpha u \in L^p(\Omega,\omega),\ |\alpha| \leqslant k\}$$

with norm

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{0\leqslant|\alpha|\leqslant k}\int_\Omega |D^\alpha u|^p\omega\,\mathrm{d}x\right)^{1/p}.$$

The spaces $W^{k,p}(\Omega,\omega)$ are Banach spaces. The Banach spaces $W_0^{k,p}(\Omega,\omega)$ arise by taking the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega,\omega)$.

Theorem 2.7 (the Muckenhoupt Theorem). *Let ω be a weight in $\mathbb{R}^n$ and let*

$$[M(f)](x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y$$

be the Hardy–Littlewood maximal function. Then, for $p > 1$,

$$M : L^p(\mathbb{R}^n,\omega) \to L^p(\mathbb{R}^n,\omega)$$

is continuous (that is, $\|Mf\|_{L^p(\mathbb{R}^n,\omega)} \leqslant C_M\|f\|_{L^p(\mathbb{R}^n,\omega)}$) if and only if $\omega \in A_p$.

The constant C_M depends only on n, p and the A_p-constant $C(\omega,p)$ of ω.

Proof. See [**7**, Chapter IV, Corollary 4.3]. □

Theorem 2.8. *If $\omega \in A_p$, $1 < p < \infty$, then $H^{k,p}(\Omega,\omega) = W^{k,p}(\Omega,\omega)$.*

Proof. See [**3**, Proposition 3.5]. □

Definition 2.9. We say that an element $u \in H^{1,2}(\Omega,\omega)$ is a weak solution of

$$Lu = g - \sum_{j=1}^{n} D_jf_j, \quad \text{with } \frac{g}{\omega}, \frac{f_j}{\omega} \in L^2(\Omega,\omega),$$

if

$$\int_\Omega a_{ij}(x)D_iu(x)D_j\varphi(x)\,\mathrm{d}x = \sum_{j=1}^{n}\int_\Omega f_j(x)D_j\varphi(x)\,\mathrm{d}x + \int_\Omega g(x)\varphi(x)\,\mathrm{d}x, \quad \forall\varphi \in H_0^{1,2}(\Omega,\omega).$$

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Theorem 2.10. *Let L be the operator (1.1) and with coefficients a_{ij} satisfying $a_{ij} = a_{ji}$ and the degenerate ellipticity condition*

$$\lambda|\xi|^2\omega(x) \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \Lambda\omega(x)|\xi|^2, \quad \forall\xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega, \tag{2.1}$$

where λ and Λ are positive constants. If $\psi \in H^{1,2}(\Omega,\omega)$, $\omega \in A_2$, $(g/\omega) \in L^2(\Omega,\omega)$ and $(f_j/\omega) \in L^2(\Omega,\omega)$, then the Dirichlet problem

$$\left.\begin{aligned} Lu = g - \sum_{j=1}^{n} D_j f_j, \quad \text{in } \Omega,\\ u - \psi \in H_0^{1,2}(\Omega,\omega), \end{aligned}\right\} \tag{D1}$$

has a unique solution $u \in H^{1,2}(\Omega,\omega)$ and

$$\|u\|_{H^{1,2}(\Omega,\omega)} \leqslant C\left(\left\|\frac{g}{\omega}\right\|_{L^2(\Omega,\omega)} + \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)} + \|\psi\|_{H^{1,2}(\Omega,\omega)}\right).$$

Proof. It is a consequence of the Lax–Milgram Theorem and the proof follows the lines of Theorem 2.2 of [**5**] and Theorem 8.3 of [**8**], by using Theorem 1.2 of [**5**] (the WSI in the case $\omega = v$): if $\omega \in A_p$, $1 < p < \infty$, there exist constants C_Ω and δ positive such that for all $u \in C_0^\infty(\Omega)$ and all σ satisfying $1 \leqslant \sigma \leqslant (n/(n-1)) + \delta$,

$$\|u\|_{L^{p\sigma}(\Omega,\omega)} \leqslant C_\Omega \||\nabla u|\|_{L^p(\Omega,\omega)} \tag{2.2}$$

(σ as in the WSI). □

3. The approximation theorem in the case where $\omega = v$

The following lemma can be proved in exactly the same way as Lemma 2.1 in [**6**]. Our lemma provides a general approximation theorem for A_p weights ($1 \leqslant p < \infty$) by means of weights which are bounded away from 0 and infinity and whose A_p-constants depend only on the A_p-constant of ω.

Lemma 3.1. *Let $\alpha, \beta > 1$ be given and let $\omega \in A_p$, $p \geqslant 1$, with A_p-constant $C(\omega,p)$ and let $a_{ij} = a_{ji}$ be measurable, real-valued functions satisfying*

$$\lambda\omega(x)|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \Lambda\omega(x)|\xi|^2 \tag{3.1}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega \subset \mathbb{R}^n$. Then there exist weights $\omega_{\alpha\beta} \geqslant 0$ a.e. and measurable real-valued functions $a_{ij}^{\alpha\beta}$ such that the following conditions are met.

(i) *$c_1(1/\beta) \leqslant \omega_{\alpha\beta} \leqslant c_2\alpha$ in Ω, where c_1 and c_2 depend only on ω and Ω.*

(ii) *There exist weights $\tilde{\omega}_1$ and $\tilde{\omega}_2$ such that $\tilde{\omega}_1 \leqslant \omega_{\alpha\beta} \leqslant \tilde{\omega}_2$, where $\tilde{\omega}_i \in A_p$ and $C(\tilde{\omega}_i,p)$ depends only on $C(\omega,p)$ $(i = 1,2)$.*

(iii) $\omega_{\alpha\beta} \in A_p$, *with constant* $C(\omega_{\alpha\beta}, p)$ *depending only on* $C(\omega, p)$ *uniformly on* α *and* β.

(iv) *There exists a closed set* $F_{\alpha\beta}$ *such that* $\omega_{\alpha\beta} \equiv \omega$ *in* $F_{\alpha\beta}$ *and* $\omega_{\alpha\beta} \sim \tilde{\omega}_1 \sim \tilde{\omega}_2$ *in* $F_{\alpha\beta}$ *with equivalence constants depending on* α *and* β *(i.e. there are positive constants* $c_{\alpha\beta}$ *and* $C_{\alpha\beta}$ *such that* $c_{\alpha\beta}\tilde{\omega}_i \leqslant \omega_{\alpha\beta} \leqslant C_{\alpha\beta}\tilde{\omega}_i$, $i = 1, 2$*). Moreover,* $F_{\alpha\beta} \subset F_{\alpha'\beta'}$, *if* $\alpha \leqslant \alpha'$, $\beta \leqslant \beta'$, *and the complement of* $\bigcup_{\alpha,\beta\geqslant 1} F_{\alpha\beta}$ *has zero measure.*

(v) $\omega_{\alpha\beta} \to \omega$ a.e. *in* $\mathbb{R}^n$ *as* $\alpha, \beta \to \infty$.

(vi)
$$\lambda\omega_{\alpha\beta}(x)|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x)\xi_i\xi_j \leqslant \Lambda\omega_{\alpha\beta}(x)|\xi|^2,$$
$$\sum_{i,j=1}^{n} |a_{ij}^{\alpha\beta}(x)| \leqslant C\omega_{\alpha\beta}(x)$$
for every $\xi \in \mathbb{R}^n$ *and a.e.* $x \in \Omega$, *with constant* C *independent of* α, β.

Proof. The proof of this lemma follows the lines of Lemma 2.1 in [**6**]. □

Let us now state the main result of this section. The proof is like the one given in Theorem 3.14 of [**6**], and we present a short proof.

Theorem 3.2. *Assume* $\omega \in A_2$.

(I) *With the same notation and hypotheses as in Theorem 2.10. Then the solution* $u \in H^{1,2}(\Omega, \omega)$ *of the problem*
$$\left.\begin{aligned} Lu = g - \sum_{j=1}^{n} D_j f_j, \quad \text{in } \Omega, \\ u - \psi \in H_0^{1,2}(\Omega, \omega), \end{aligned}\right\} \tag{D1}$$
is the weak limit in $H^{1,2}(\Omega, \tilde{\omega}_1)$ *of a sequence of solutions* $u_m \in H^{1,2}(\Omega, \omega_m)$ *of the problems*
$$\left.\begin{aligned} L_m u_m = g_m - \sum D_j f_{jm}, \\ u_m - \psi \in H_0^{1,2}(\Omega, \omega_m), \end{aligned}\right\} \tag{P$_m$}$$
where $\omega_m = \omega_{mm}$, $L_m u_m = -\sum D_j(a_{ij}^{mm}(x) D_i u_m)$, $f_{jm} = f_j(\omega/\omega_m)^{-1/2}$ *and* $g_m = g(\omega/\omega_m)^{-1/2}$ *(*ω_{mm}, a_{ij}^{mm} *and* $\tilde{\omega}_1$ *are as in Lemma 3.1).*

(II) *Moreover, if* $u \in H_0^{1,2}(\Omega, \omega)$ *is the solution of the equation* $Lu = -\sum D_j f_j$, *with* $f_j/\omega \in L^p(\Omega, \omega)$, $p > 2n - \varepsilon'$ *(choosing* ε' *appropriately), then* u *is the uniform limit of* $(u_m)_{m\in N}$ *in any compact subset of* Ω, *where* $u_m \in H_0^{1,2}(\Omega, \omega_m)$ *is a solution of the problem*
$$L_m u_m = -\sum_{j=1}^{n} D_j f_{jm}, \quad \text{with } f_{jm} = f_j\left(\frac{\omega}{\omega_m}\right)^{(1-p)/p}.$$

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Proof. We shall denote $A^m(x) = (a_{ij}^{mm}(x))_{i,j=1,\dots,n}$.
(I) First, we note that

$$\left\|\frac{g_m}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} = \left\|\frac{g}{\omega}\right\|_{L^2(\Omega,\omega)} \quad \text{and} \quad \left\|\frac{f_{jm}}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} = \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)}.$$

Consequently, we have $T_m = g_m - \sum D_j f_{jm} \in H^{-1,2}(\Omega,\omega_m)$. Using the fact that u_m is a solution of the problem (P_m), by Theorem 2.10 we obtain (because $\omega_m \in A_2$)

$$\begin{aligned}\|u_m\|_{H^{1,2}(\Omega,\omega_m)} &\leqslant C\biggl(\left\|\frac{g_m}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} + \sum_{j=1}^n \left\|\frac{f_{jm}}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} + \|\psi\|_{H^{1,2}(\Omega,\omega)}\biggr)\\ &= C\biggl(\left\|\frac{g}{\omega}\right\|_{L^2(\Omega,\omega)} + \sum_{j=1}^n \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)} + \|\psi\|_{H^{1,2}(\Omega,\omega)}\biggr) = C_1, \end{aligned} \tag{3.2}$$

where the constant C is independent of m. By Lemma 3.1 (ii) $\tilde\omega_1 \leqslant \omega_m$, we obtain

$$\begin{aligned}\|u_m\|_{H^{1,2}(\Omega,\tilde\omega_1)} &\leqslant \|u_m\|_{H^{1,2}(\Omega,\omega_m)}\\ &\leqslant C\biggl(\left\|\frac{g}{\omega}\right\|_{L^2(\Omega,\omega)} + \sum_{j=1}^n \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)} + \|\psi\|_{H^{1,2}(\Omega,\omega)}\biggr) = C_1. \end{aligned} \tag{3.3}$$

Consequently, $\{u_m\}$ is a bounded sequence in $H^{1,2}(\Omega,\tilde\omega_1)$. Therefore there is a subsequence, again denoted by $\{u_m\}$, and $\tilde u \in H^{1,2}(\Omega,\tilde\omega_1)$ such that $u_m \to \tilde u$ weakly in $L^2(\Omega,\tilde\omega_1)$ and $\nabla u_m \to \nabla\tilde u$ weakly in $L^2(\Omega,\tilde\omega_1)$ (see Theorem 1.31 of [**9**]). We have that $\tilde u \in H^{1,2}(\Omega,\omega)$ (the proof proceeds as in Theorem 3.14 of [**6**]).

We need to show that $\tilde u$ is a weak solution of the equation $L\tilde u = g - \sum D_j f_j$, that is

$$\int_\Omega a_{ij}(x) D_i\tilde u(x) D_j\varphi(x)\,\mathrm{d}x = \int_\Omega g(x)\varphi(x)\,\mathrm{d}x + \int_\Omega f_j(x) D_j\varphi(x)\,\mathrm{d}x, \quad \forall\varphi \in H_0^{1,2}(\Omega,\omega).$$

Using the fact that u_m is a solution of (P_m), we have $L_m u_m = g_m - \sum D_j f_{jm}$, that is

$$\begin{aligned}&\int_\Omega a_{ij}^{mm}(x) D_i u_m(x) D_j\varphi(x)\,\mathrm{d}x\\ &\qquad = \int_\Omega g_m(x)\varphi(x)\,\mathrm{d}x + \int_\Omega f_{jm}(x) D_j\varphi(x)\,\mathrm{d}x, \quad \forall\varphi \in H_0^{1,2}(\Omega,\omega_m).\end{aligned}$$

Moreover, over F_k (for $m \geqslant k$) we have the following properties:

(i) $\omega = \omega_m$,

(ii) $g = g_m$ and $f_j = f_{jm}$; and

(iii) $a_{ij}^{mm}(x) = a_{ij}(x)$.

If $\varphi \in H_0^{1,2}(\Omega,\omega)$, we get

$$\begin{gathered} G : H^{1,2}(\Omega,\tilde\omega_1) \to \mathbb{R},\\ G(u) = \int_\Omega a_{ij}(x) D_i u(x) D_j\varphi(x)\chi_{F_k}(x)\,\mathrm{d}x \end{gathered}$$

is a bounded linear functional.

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Using this fact and the properties (i)–(iii) we obtain

$$\begin{aligned}\int_{F_k} a_{ij}(x)D_i\tilde{u}(x)D_j\varphi(x)\,\mathrm{d}x &= \lim_{m\to\infty}\int_{F_k} a_{ij}^{mm}(x)D_iu_m(x)D_j\varphi(x)\,\mathrm{d}x \\ &= \lim_{m\to\infty}\biggl(\int_\Omega a_{ij}^{mm}(x)D_iu_m(x)D_j\varphi(x)\,\mathrm{d}x \\ &\qquad\qquad - \int_{\Omega\cap F_k^c} a_{ij}^{mm}(x)D_iu_m(x)D_j\varphi(x)\,\mathrm{d}x\biggr) \\ &= \lim_{m\to\infty}\biggl(\int_\Omega g_m(x)\varphi(x)\,\mathrm{d}x + \int_\Omega f_{jm}(x)D_j\varphi(x)\,\mathrm{d}x \\ &\qquad\qquad - \int_{\Omega\cap F_k^c} a_{ij}^{mm}(x)D_iu_m(x)D_j\varphi(x)\,\mathrm{d}x\biggr) \end{aligned} \tag{3.4}$$

(recall that u_m is a solution of (P_m)). We have, by the Lebesgue-dominated convergence theorem and $\tilde{\omega}_2 \in A_2$,

$$\int_\Omega g_m\varphi\,\mathrm{d}x \to \int_\Omega g\varphi\,\mathrm{d}x \quad \text{and} \quad \int_\Omega f_{jm}D_j\varphi\,\mathrm{d}x \to \int_\Omega f_jD_j\varphi\,\mathrm{d}x. \tag{3.5}$$

Using $\sum |a_{ij}^{mm}(x)| \leqslant C\omega_m(x)$, with C independent of m, we obtain

$$\begin{aligned}\biggl|\int_{\Omega\cap F_k^c} a_{ij}^{mm}D_iu_mD_j\varphi\,\mathrm{d}x\biggr| &\leqslant \int_{\Omega\cap F_k^c} |a_{ij}^{mm}|\,|D_iu_m|\,|D_j\varphi|\,\mathrm{d}x \\ &\leqslant \int_{\Omega\cap F_k^c} C\omega_m|D_iu_m|\,|D_j\varphi|\,\mathrm{d}x \quad (C \text{ is independent of } m) \\ &= C\int_{\Omega\cap F_k^c} |D_iu_m|\omega_m^{1/2}|D_j\varphi|\omega_m^{1/2}\,\mathrm{d}x \\ &\leqslant C\biggl(\int_{\Omega\cap F_k^c}|D_iu_m|^2\omega_m\,\mathrm{d}x\biggr)^{1/2}\biggl(\int_{\Omega\cap F_k^c}|D_j\varphi|^2\omega_m\,\mathrm{d}x\biggr)^{1/2} \\ &\leqslant CC_1C_\varphi\biggl(\int_{F_k^c\cap\Omega}\omega_m\,\mathrm{d}x\biggr)^{1/2} \quad (\text{by } (3.2)) \\ &= CC_1C_\varphi[\omega_m(F_k^c\cap\Omega)]^{1/2}.\end{aligned}$$

By Theorem 2.9, Chapter IV, of [**7**], there exist constants $\delta > 0$, $C > 0$ such that, if $\bar{\Omega} \subset Q_o$ (Q_o is a fixed cube), then

$$\omega_m(F_k^c\cap\Omega) \leqslant \tilde{\omega}_2(F_k^c\cap\Omega) \leqslant C\tilde{\omega}_2(Q_o)\biggl(\frac{|F_k^c|}{|Q_o|}\biggr)^\delta.$$

Using Lemma 3.1, we know that $|F_k^c| \to 0$ when $k \to \infty$. Then

$$\omega_m(F_k^c\cap\Omega) \to 0 \quad \text{when } k \to \infty,$$

and we obtain

$$\lim_{k\to\infty}\int_{\Omega\cap F_k^c} a_{ij}^{mm}(x)D_iu_m(x)D_j\varphi(x)\,\mathrm{d}x = 0. \tag{3.6}$$

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Therefore, by (3.5) and (3.6) we conclude

$$\int_{\Omega} a_{ij}(x) D_i \tilde{u}(x) D_j \varphi(x)\,\mathrm{d}x = \int_{\Omega} g(x)\varphi(x)\,\mathrm{d}x + \int_{\Omega} f_j(x) D_j \varphi(x)\,\mathrm{d}x;$$

that is, $\tilde{u}$ is a solution of the equation $Lu = g - \sum_{j=1}^{n} D_j f_j$. Therefore $u = \tilde{u}$ (by the uniqueness of Theorem 2.10).

(II) Now we will prove the second part of the theorem.

If $\omega \in A_2$, then there exists $\varepsilon > 0$ such that $\omega \in A_{2-\varepsilon}$ (see Theorem 15.13 (open-end property) in [**9**] or Proposition 4.5, Chapter IX, in [**11**]). We choose $\varepsilon' = n\varepsilon$ and $p > 2n - \varepsilon' = 2n - \varepsilon n$. We have that $p/n > 2 - \varepsilon$ and $\omega \in A_{2-\varepsilon} \subset A_{p/n}$.

By Theorem 2.3.15 of [**5**], if u is a weak solution of the equation $Lu = -\sum D_j f_j$, and $f_j/\omega \in L^p(\Omega, \omega)$, $\omega \in A_{p/n}$, then u is locally Hölder continuous in Ω, i.e. there exist constants $C > 0$ and λ $(0 < \lambda < 1)$, such that if $x_o \in \Omega$, $0 < \rho < R < \frac{1}{16}\operatorname{dist}(x_o, \partial\Omega)$, we have

$$\operatorname*{osc}_{B(x_o,\rho)} u \leqslant C\biggl[\biggl(\frac{1}{\omega(B_R)}\int_{B_R} u^2 \omega\,\mathrm{d}x\biggr)^{1/2} + \biggl\|\frac{f_j}{\omega}\biggr\|_{L^p(B_R,\omega)}\biggr]\rho^{\lambda}, \tag{3.7}$$

where C and λ are independent of u, ρ and x_o, and $\operatorname{osc}_{B(x_o,\rho)} u$ is the oscillation over $B(x_o, \rho)$ of u.

Applying this result for the solution of the equations

$$L_m u_m = -\sum_{j=1}^{n} D_j f_{jm},$$

$u_m \in H_0^{1,2}(\Omega, \omega_m)$, $\omega_m \in A_2$, with $f_{jm} = f_j(\omega/\omega_m)^{(1-p)/p}$, and using (3.2), we obtain

$$\begin{aligned}
\operatorname*{osc}_{B(x_o,\rho)} u_m &\leqslant C\biggl[\biggl(\frac{1}{\omega_m(B_R)}\int_{B_R} u_m^2 \omega_m\,\mathrm{d}x\biggr)^{1/2} + \biggl\|\frac{f_{jm}}{\omega_m}\biggr\|_{L^p(\Omega,\omega_m)}\biggr]\rho^{\lambda} \\
&\leqslant C\biggl[\frac{\boldsymbol{C}_1}{\tilde{\omega}_1(B_R)} + \biggl\|\frac{f_j}{\omega}\biggr\|_{L^p(\Omega,\omega)}\biggr]\rho^{\lambda}.
\end{aligned}$$

Therefore, the sequence $\{u_m\}$ is locally equicontinuous. Moreover, by Lemma 2.3.14 of [**5**], we have

$$\operatorname*{ess\,sup}_{B(x_o,\rho)} |u_m| \leqslant C\biggl\|\frac{f_{jm}}{\omega_m}\biggr\|_{L^p(\Omega,\omega_m)}\rho^{\lambda}.$$

Hence, using $\|f_{jm}/\omega_m\|_{L^p(\Omega,\omega_m)} = \|f_j/\omega\|_{L^p(\Omega,\omega)}$, we obtain

$$\operatorname*{ess\,sup}_{B(x_o,\rho)} |u_m| \leqslant C\biggl\|\frac{f_j}{\omega}\biggr\|_{L^p(\Omega,\omega)}\rho^{\lambda},$$

that is, $\{u_m\}$ is a locally uniformly bounded sequence.

We can apply the Arzelà–Ascoli Theorem and conclude that $\{u_m\}$ converges to u uniformly in compact subsets of Ω. □

4. Generalization of results in the case where $\omega \neq v$

In this section, we generalize some results in the cases $\omega \neq v$.

Definition 4.1. We shall say that the pair (v, ω) satisfies the condition A_p, $1 \leqslant p < \infty$, if there is a constant C such that

$$\left(\frac{1}{|Q|}\int_Q v(x)\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega^{-1/(p-1)}(x)\,\mathrm{d}x\right)^{p-1} \leqslant C, \quad \text{when } 1 < p < \infty,$$

and

$$\frac{1}{|Q|}\int_Q v(x)\,\mathrm{d}x \leqslant C \operatorname*{ess\,inf}_Q \omega, \quad \text{when } p = 1,$$

for all cubes Q in $\mathbb{R}^n$. The smallest constant C will be called the A_p-constant for the pair (ω, v).

Remark 4.2. Since the coefficients of the operator L satisfy (1.2), then $\omega(x) \leqslant v(x)$. In this case, if the couple $(v, \omega) \in A_p$, we have $v \in A_p$ and $\omega \in A_p$ (Definition 2.2).

Definition 4.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let ω, v be weights.

(a) The space $H^{k,2}(\Omega, \omega, v)$, $k \in \mathbb{N}$, is defined as the closure of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\|u\|_{H^{k,2}(\Omega,\omega,v)} = \left(\int_\Omega u^2 v\,\mathrm{d}x + \int_\Omega \langle A\nabla u, \nabla u\rangle\,\mathrm{d}x + \sum_{2\leqslant|\alpha|\leqslant k}\int_\Omega |D^\alpha u|^2\omega\,\mathrm{d}x\right)^{1/2},$$

where $A = (a_{ij}(x))$ is the coefficient matrix of the operator L defined in (1.1) and the symbol ∇ indicates the gradient.

(b) The space $H_0^{k,2}(\Omega, \omega, v)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0^{k,2}(\Omega,\omega,v)} = \left(\int_\Omega \langle A\nabla u, \nabla u\rangle\,\mathrm{d}x + \sum_{|\alpha|=2}^{k}\int_\Omega |D^\alpha u|^2\omega\,\mathrm{d}x\right)^{1/2}.$$

The spaces $H^{k,2}(\Omega, \omega, v)$ and $H_0^{k,2}(\Omega, \omega, v)$ are Hilbert spaces.

Definition 4.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let v, ω be weights. We define the space

$$W^{k,2}(\Omega, \omega, v) = \left\{u \in L^2(\Omega, v) : \int_\Omega \langle A\nabla u, \nabla u\rangle\,\mathrm{d}x < \infty \text{ and } D^\alpha u \in L^2(\Omega, \omega),\ 2 \leqslant |\alpha| \leqslant k\right\},$$

with the norm

$$\|u\|_{W^{k,2}(\Omega,\omega,v)} = \left(\int_\Omega u^2 v\,\mathrm{d}x + \int_\Omega \langle A\nabla u, \nabla u\rangle\,\mathrm{d}x + \sum_{2\leqslant|\alpha|\leqslant k}\int_\Omega |D^\alpha u|^2\omega\,\mathrm{d}x\right)^{1/2},$$

where $A = (a_{ij})_{i,j=1,\dots,n}$ is the coefficient matrix of the operator L.

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Definition 4.5. We shall say that the pair of weights (v,ω) satisfies the condition S_p $(p>1)$ if there is a constant C (called the S_p-constant) such that

$$\int_Q |M(\mu\chi_Q)(x)|^p v(x)\,\mathrm{d}x \leqslant C\mu(Q) < \infty, \quad \text{for every cube } Q,$$

where M is the Hardy–Littlewood maximal function, $\mu=\omega^{-1/(p-1)}$ and $\mu(Q)=\int_Q \mu(x)\,\mathrm{d}x$.

Remark 4.6. If $(v,\omega)\in S_p$, then $(v,\omega)\in A_p$.

Theorem 4.7. *If $(v,\omega)\in S_2$ and $\omega\leqslant v$, then $H^{k,2}(\Omega,\omega,v)=W^{k,2}(\Omega,\omega,v)$.*

Proof. The proof is the same as that of Theorem 2.8, using the Muckenhoupt Generalized Theorem (see [**7**, Chapter IV, Theorem 4.9]) and the WSI. □

Definition 4.8. We say that an element $u\in H^{1,2}(\Omega,\omega,v)$ is a weak solution of the equation $Lu=g-\operatorname{div}\boldsymbol{f}$ if

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x)D_iu(x)D_j\varphi(x)\,\mathrm{d}x = \int_\Omega g(x)\varphi(x)\,\mathrm{d}x + \sum_{j=1}^n \int_\Omega f_j(x)D_j\varphi(x)\,\mathrm{d}x$$

for every $\varphi\in H_0^{1,2}(\Omega,\omega,v)$, where $\boldsymbol{f}=(f_1,\dots,f_n)$.

Theorem 4.9. *Suppose that the WSI holds and that $(v,\omega)\in A_2$. Let L be the operator (1.1) with (1.2) and (1.3). If $\psi\in H^{1,2}(\Omega,\omega,v)$, $g/v\in L^{(2\sigma)'}(\Omega,v)$ and $f_j/\omega\in L^2(\Omega,\omega)$, then the Dirichlet problem*

$$\left.\begin{aligned} Lu = g-\sum_{j=1}^n D_jf_j, \quad \text{in } \Omega,\\ u-\psi\in H_0^{1,2}(\Omega,\omega,v), \end{aligned}\right\} \tag{P}$$

has a unique solution $u\in H^{1,2}(\Omega,\omega,v)$ and

$$\|u\|_{H^{1,2}(\Omega,\omega,v)} \leqslant C_{\Omega,\omega,v}\left(\left\|\frac{g}{v}\right\|_{L^{(2\sigma)'}(\Omega,v)} + \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)} + \|\psi\|_{H^{1,2}(\Omega,\omega,v)}\right),$$

where $C_{\Omega,\omega,v}$ is the Sobolev constant (see (1.5)).

Proof. The proof is very similar to that of Theorem 2.10, replacing Theorem 1.3 of [**5**] by the WSI, and by Remark 4.2 we have $\omega\in A_2$ and $v\in A_2$. □

Definition 4.10. For $k\in\mathbb{Z}$, we consider the lattice $\Gamma_k=2^{-k}\mathbb{Z}^n$ formed by those points of $\mathbb{R}^n$ whose coordinate are integral multiples of 2^{-k}. Let D_k be a collection of cubes determined by Γ_k, that is, those cubes with side length 2^{-k} and vertices in Γ_k. The cubes belonging to $D=\bigcup_{-\infty}^{\infty} D_k$ are called *dyadic cubes*.

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Note that if $Q_1, Q_2 \in D$ and $|Q_1| \leqslant |Q_2|$, then either $Q_1 \subset Q_2$ or else Q_1 and Q_2 do not overlap (by which we mean that their interiors are disjoint, $\operatorname{int} Q_1 \cap \operatorname{int} Q_2 = \emptyset$), where $|Q|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$ (see Chapter II of [**7**]).

We will need the following theorem.

Theorem 4.11 (the Jones Factorization Theorem). *For $1 < p < \infty$, $\omega \in A_p$ if and only if there exist $\omega_0, \omega_1 \in A_1$ such that $\omega = \omega_0 \omega_1^{1-p}$.*

Proof. See [**7**, Chapter IV, Corollary 5.3]. □

Remark 4.12. Now we will prove a generalization for Lemma 3.1 in the case where $(v, \omega) \in A_p$. With the condition (1.2), we know by Remark 4.2 that $v \in A_p$ and $\omega \in A_p$. We will prove the approximation in the following cases.

Case 1. $(v, \omega) \in A_1$.

Case 2. $(v, \omega) \in A_p$, $p > 1$. In this case we assume that $v \in A_p$ and $\omega \in A_p$ (see Remark 4.2). Then by the Jones Factorization Theorem there exist $v_0, v_1, \omega_0, \omega_1 \in A_1$ such that $v = v_0 v_1^{1-p}$ and $\omega = \omega_0 \omega_1^{1-p}$. We will assume the following hypotheses:

(a) $\omega_0 \leqslant v_0$ and $(v_0, \omega_0) \in A_1$; and

(b) $v_1(x) \leqslant \omega_1(x) \leqslant C_1 v_1(x)$.

We may now prove the following lemma.

Lemma 4.13. *Let $\alpha, \beta > 1$ be given and let there be a pair of weights $(v, \omega) \in A_p$, $1 \leqslant p < \infty$, satisfying the hypotheses as above and measurable real-valued functions a_{ij} satisfying*

$$\omega(x)|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant |\xi|^2 v(x), \tag{4.1}$$

$$a_{ij}(x) = a_{ji}(x), \tag{4.2}$$

for every $\xi \in \mathbb{R}^n$, a.e. $x \in \Omega$. Then there exist weights $\omega_{\alpha\beta} \geqslant 0$, $v_{\alpha\beta} \geqslant 0$ and measurable functions $a_{ij}^{\alpha\beta}$ such that:

(i) *$C_{11}(1/\beta) \leqslant \omega_{\alpha\beta} \leqslant C_{12}\alpha$, $C_{21}(1/\beta) \leqslant v_{\alpha\beta} \leqslant C_{22}\alpha$ in Ω, with C_{11} and C_{12} depending only on ω and Ω, and C_{21} and C_{22} depending only on v and Ω;*

(ii) *there exist weights $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{v}_1, \tilde{v}_2 \in A_p$, such that $\tilde{\omega}_1 \leqslant \omega_{\alpha\beta} \leqslant \tilde{\omega}_2$ and $\tilde{v}_1 \leqslant v_{\alpha\beta} \leqslant \tilde{v}_2$;*

(iii) *$\omega_{\alpha\beta}$ and $v_{\alpha\beta}$ are A_p weights;*

(iv) *there exists a closed set $G_{\alpha\beta}$ such that $\omega_{\alpha\beta}(x) = \omega(x)$ and $v_{\alpha\beta}(x) = v(x)$ in $G_{\alpha\beta}$. Moreover, $G_{\alpha\beta} \subset G_{\alpha'\beta'}$, if $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$, and the complement of $\cup_{\alpha,\beta>1} G_{\alpha,\beta}$ has zero measure;*

(v) $\omega_{\alpha\beta} \to \omega$ *and* $v_{\alpha\beta} \to v$ *a.e. in* $\mathbb{R}^n$, *as* α, β *tend to infinity; and*

(vi)

$$\tilde{\lambda}|\xi|^2\omega_{\alpha\beta}(x) \leqslant \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x)\xi_i\xi_j \leqslant |\xi|^2 v_{\alpha\beta}(x)$$

$$\sum_{i,j=1}^{n} |a_{ij}^{\alpha\beta}(x)| \leqslant C v_{\alpha\beta}(x),$$

for all $\xi \in \mathbb{R}^n$, *a.e.* $x \in \Omega$, *with* C *independent of* α, β, *where* $\tilde{\lambda}$ *is a positive constant.*

Proof. Case 1. In this case we denote $\omega_{\alpha\beta} = \omega_\alpha$, $v_{\alpha\beta} = v_\alpha$, $a_{ij}^{\alpha\beta} = a_{ij}^{\alpha}$ and $G_{\alpha\beta} = G_\alpha$ (in (i)–(vi)). Since we are interested in approximating within Ω, we can suppose, without loss of generality that $\omega, v \in L^1(\mathbb{R}^n)$.

If $(v, \omega) \in A_1$, that is,

$$\frac{1}{|Q|}\int_Q v(x)\,\mathrm{d}x \leqslant C \operatorname*{ess\,inf}_Q \omega$$

by Remark 4.2, we have $v \in A_1$ and $\omega \in A_1$. For each $\alpha > 1$, we define

$$U_\alpha^+ = \{x \in \mathbb{R}^n : M(\omega)(x) > \alpha\} \quad \text{and} \quad V_\alpha^+ = \{x \in \mathbb{R}^n : M(v)(x) > \alpha\}$$

(U_α^+ and V_α^+ are open sets because $M(\omega)$ and $M(v)$ are lower semicontinuous functions), where M is the usual Hardy–Littlewood maximal operator, i.e.

$$M(\omega)(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |\omega(y)|\,\mathrm{d}y.$$

Using the fact that $0 < \omega(x) \leqslant v(x)$, we have $M(\omega)(x) \leqslant M(v)(x)$. Consequently, $U_\alpha^+ \subset V_\alpha^+$.

By using Calderon–Zygmund decomposition (see [**7**, Chapter II, Theorem 1.12]) there are two families of non-overlapping cubes $\{Q_l^\alpha\}$ and $\{I_k^\alpha\}$, where Q_l^α and I_k^α are maximal dyadic cubes, such that

$$U_\alpha^+ = \bigcup_{l=1}^{\infty} Q_l^\alpha \quad \text{and} \quad V_\alpha^+ = \bigcup_{k=1}^{\infty} I_k^\alpha, \tag{CZ1}$$

$$\alpha < \frac{1}{|Q_l^\alpha|}\int_{Q_l^\alpha} \omega(x)\,\mathrm{d}x \leqslant 2^n\alpha \quad \text{and} \quad \alpha < \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} v(x)\,\mathrm{d}x \leqslant 2^n\alpha, \tag{CZ2}$$

$$\omega(x) \leqslant \alpha \quad \text{in } F_\alpha^+ = (U_\alpha^+)^{\mathrm{c}} \quad \text{and} \quad v(x) \leqslant \alpha \quad \text{in } G_\alpha^+ = (V_\alpha^+)^{\mathrm{c}}, \tag{CZ3}$$

where E^{c} denotes the complement of a set E;

$$|U_\alpha^+| \leqslant \frac{c}{\alpha}\int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x \quad \text{and} \quad |V_\alpha^+| \leqslant \frac{c}{\alpha}\int_{\mathbb{R}^n} v(x)\,\mathrm{d}x. \tag{CZ4}$$

Since $U_\alpha^+ \subset V_\alpha^+$ for each Q_l^α, there exists only one I_k^α ($k = k(l)$) such that $Q_l^\alpha \subset I_k^\alpha$ (recall that Q_l^α and I_k^α are maximal dyadic cubes). We define the weights ω_α and v_α by

$$\omega_\alpha(x) = \sum_{l=1}^{\infty}\left(\frac{1}{|Q_l^\alpha|}\int_{Q_l^\alpha}\omega(y)\,\mathrm{d}y\right)\chi_{Q_l^\alpha}(x) + \omega(x)\chi_{F_\alpha^+}(x),$$
$$v_\alpha(x) = \sum_{k=1}^{\infty}\left(\frac{1}{|I_k^\alpha|}\int_{I_k^\alpha}v(y)\,\mathrm{d}y\right)\chi_{I_k^\alpha}(x) + v(x)\chi_{G_\alpha^+}(x),$$

where χ_E denotes the characteristic function of a set E.

We will show that $\omega_\alpha(x) \leqslant 2^n v_\alpha(x)$.

(1) If $x \in G_\alpha^+ \subset F_\alpha^+$, then $v_\alpha(x) = v(x)$ and $\omega_\alpha(x) = \omega(x)$. Consequently, $\omega_\alpha(x) \leqslant v_\alpha(x)$.

(2) If $x \notin G_\alpha^+$, then $x \in I_k^\alpha$ (for a unique cube). Hence,

$$v_\alpha(x) = \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha}v(y)\,\mathrm{d}y.$$

For the weight ω_α, there are two possible cases as follows.

(a) $x \in Q_l^\alpha \subset I_k^\alpha$. In this case, by property (CZ2) of the Calderon–Zygmund decomposition, we have

$$\omega_\alpha(x) = \frac{1}{|Q_l^\alpha|}\int_{Q_l^\alpha}\omega(y)\,\mathrm{d}y \leqslant 2^n\alpha \leqslant 2^n\frac{1}{|I_k^\alpha|}\int_{I_k^\alpha}v(y)\,\mathrm{d}y = 2^n v_\alpha(x).$$

(b) $x \in I_k^\alpha - \mathsf{Q}_k$, where $\mathsf{Q}_k = \cup Q_l^\alpha$, with $Q_l^\alpha \subset I_k^\alpha$, that is, $x \in F_\alpha^+$. Hence, using property (CZ3) of the Calderon–Zygmund decomposition, we obtain

$$\omega_\alpha(x) = \omega(x) \leqslant \alpha < \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha}v(y)\,\mathrm{d}y = v_\alpha(x).$$

Therefore $\omega_\alpha(x) \leqslant 2^n v_\alpha(x)$ in Ω.

Since $\omega \in A_1$ and $v \in A_1$ we have that (see Lemma 2.1 of [**6**])

(I1) $\omega_\alpha \in A_1$ and $v_\alpha \in A_1$, with $C(\omega_\alpha, 1)$ depending only on $C(\omega, 1)$ and $C(v_\alpha, 1)$ depending only on $C(v, 1)$;

(II1) $\omega_\alpha \to \omega$ and $v_\alpha \to v$ a.e. in $\mathbb{R}^n$ when $\alpha \to \infty$;

(III1) $\min\{1, \omega\} \in A_1$, $\min\{1, v\} \in A_1$ and $\min\{1, \omega(x)\} \leqslant \omega_\alpha(x) \leqslant C\omega(x)$, $\min\{1, v(x)\} \leqslant v_\alpha(x) \leqslant \tilde{C}v(x)$ (C depends only on $C(\omega, 1)$ and $\tilde{C}$ depends only on $C(v, 1)$); and

(IV1)

$$C_1 \leqslant \omega_\alpha(x) \leqslant 2^n\alpha, \quad \text{with } C_1 = \frac{1}{|Q_0|}\int_{Q_0} \min\{1,\omega\}(y)\,\mathrm{d}y,$$
$$C_2 \leqslant v_\alpha(x) \leqslant 2^n\alpha, \quad \text{with } C_2 = \frac{1}{|Q_0|}\int_{Q_0} \min\{1,v\}(y)\,\mathrm{d}y,$$

where Q_0 is a fixed cube such that $\bar{\Omega} \subset Q_0$.

We need to show that ω_α and v_α satisfy properties (i)–(v). We have the following.

▷ (i) Follows from (IV1),

$$2^n\alpha \geqslant \omega_\alpha(x) \geqslant C_1 \geqslant \frac{C_1}{\alpha} \quad \text{and} \quad 2^n\alpha \geqslant v_\alpha(x) \geqslant C_2 \geqslant \frac{C_2}{\alpha}.$$

▷ (ii) We define the weights $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{v}_1$ and $\tilde{v}_2$ by

$$\tilde{\omega}_1(x) = \min\{1,\omega(x)\} \quad \text{and} \quad \tilde{\omega}_2(x) = C\omega(x),$$
$$\tilde{v}_1(x) = \min\{1,v(x)\} \quad \text{and} \quad \tilde{v}_2(x) = \tilde{C}v(x)$$

(C and $\tilde{C}$ are as in (III1)). We have that $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{v}_1$ and $\tilde{v}_2 \in A_1$ (by (III1)) and $\tilde{\omega}_1(x) \leqslant \omega_\alpha(x) \leqslant \tilde{\omega}_2(x)$, $\tilde{v}_1(x) \leqslant v_\alpha(x) \leqslant \tilde{v}_2(x)$.

▷ (iii) Follows from (I1).

▷ (iv) Define $G_\alpha = G_\alpha^+$. By definition $v_\alpha(x) = v(x)$ in G_α, and $\omega_\alpha(x) = \omega(x)$ in G_α (because $G_\alpha = G_\alpha^+ \subset F_\alpha^+$).

Moreover, if $\alpha \leqslant \alpha'$ we have $U_{\alpha'}^+ \subset U_\alpha^+$. Then

$$G_\alpha = G_\alpha^+ = (U_\alpha^+)^{\mathrm{c}} \subset (U_{\alpha'}^+)^{\mathrm{c}} = G_{\alpha'}^+ = G_{\alpha'}.$$

Furthermore, by (CZ4) we obtain $|(\cup_{\alpha>1}G_\alpha)^{\mathrm{c}}| = 0$.

▷ (v) Follows straightforwardly from (II1).

Let us now prove (vi).
We define the coefficients a_{ij}^α by

$$a_{ij}^\alpha(x) = \sum_{k=1}^\infty \biggl(\frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} a_{ij}(y)\,\mathrm{d}y\biggr)\chi_{I_k^\alpha}(x) + a_{ij}(x)\chi_{G_\alpha^+}(x).$$

We have the following.

(1) If $x \in G_\alpha^+ \subset F_\alpha^+$, then $a_{ij}^\alpha(x) = a_{ij}(x)$ and $\omega_\alpha(x) = \omega(x)$ and $v_\alpha(x) = v(x)$. Therefore, the condition

$$|\xi|^2\omega_\alpha(x) \leqslant \sum_{i,j=1}^n a_{ij}^\alpha(x)\xi_i\xi_j \leqslant |\xi|^2 v_\alpha(x)$$

is valid.

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(2) If $x \notin G_\alpha^+$, then $x \in I_k^\alpha$ (for a unique cube). In this case,

$$a_{ij}^\alpha(x) = \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} a_{ij}(y)\,\mathrm{d}y.$$

Consequently, using condition (4.1), we get

$$a_{ij}^\alpha(x)\xi_i\xi_j = \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} a_{ij}(y)\xi_i\xi_j\,\mathrm{d}y \leqslant |\xi|^2\frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} v(y)\,\mathrm{d}y = |\xi|^2 v_\alpha(x).$$

We need to show that $\sum a_{ij}^\alpha(x)\xi_i\xi_j \geqslant \tilde{\lambda}\omega_\alpha(x)|\xi|^2$. There are two possible cases as follows.

(a) If $x \in Q_l^\alpha \subset I_k^\alpha$. We have that

$$a_{ij}^\alpha(x)\xi_i\xi_j = \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} a_{ij}(y)\xi_i\xi_j\,\mathrm{d}y \geqslant |\xi|^2\frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} \omega(y)\,\mathrm{d}y. \tag{4.3}$$

Using $(v,\omega) \in A_1$ and property (CZ2) of the Calderon–Zygmund decomposition, we obtain

$$\alpha < \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} v(y)\,\mathrm{d}y \leqslant C\operatorname*{ess\,inf}_{I_k^\alpha}\omega \leqslant C\frac{1}{|I_k^\alpha|}\int_{I_k^\alpha}\omega(y)\,\mathrm{d}y. \tag{4.4}$$

Consequently,

$$\frac{1}{|I_k^\alpha|}\int_{I_k^\alpha}\omega(y)\,\mathrm{d}y \geqslant \frac{\alpha}{C} \geqslant \frac{1}{C}\frac{1}{2^n}\frac{1}{|Q_l^\alpha|}\int_{Q_l^\alpha}\omega(y)\,\mathrm{d}y = \frac{1}{2^nC}\omega_\alpha(x). \tag{4.5}$$

Hence, combining (4.3) and (4.5), we obtain

$$a_{ij}^\alpha(x)\xi_i\xi_j \geqslant \frac{1}{2^nC}|\xi|^2\omega_\alpha(x).$$

(b) If $x \in I_k^\alpha - Q_k$, ($Q_k = \cup Q_l^\alpha$, with $Q_l^\alpha \subset I_k^\alpha$), that is, $x \in F_\alpha^+$. In this case, we have $\omega_\alpha(x) = \omega(x)$.

Hence, using (4.4) and condition (CZ3) of the Calderon–Zygmund decomposition, we obtain

$$\begin{aligned}
a_{ij}^\alpha(x)\xi_i\xi_j &= \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} a_{ij}(y)\xi_i\xi_j\,\mathrm{d}y \\
&\geqslant |\xi|^2\frac{1}{|I_k^\alpha|}\int_{I_k^\alpha}\omega(y)\,\mathrm{d}y \\
&\geqslant |\xi|^2\frac{\alpha}{C} \quad \text{(by (4.4))} \\
&\geqslant |\xi|^2\frac{1}{C}\omega(x) \\
&= |\xi|^2\frac{1}{C}\omega_\alpha(x) \\
&\geqslant \frac{1}{2^nC}|\xi|^2\omega_\alpha(x).
\end{aligned}$$

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Therefore, in both cases we obtain

$$\tilde{\lambda}|\xi|^2\omega_\alpha(x) \leqslant \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x)\xi_i\xi_j \leqslant |\xi|^2 v_\alpha(x), \quad \text{where } \tilde{\lambda} = \frac{1}{2^n C}.$$

Let us now check the second condition in (vi). Again, there are two possible cases.

(a) If $x \in G_\alpha^+ \subset F_\alpha^+$, then $a_{ij}^\alpha(x) = a_{ij}(x)$ and $v_\alpha(x) = v(x)$. The condition is true.

(b) If $x \notin G_\alpha^+$, then $x \in I_k^\alpha$ (for a unique cube).

In this case,

$$a_{ij}^\alpha(x) = \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} a_{ij}(y)\,\mathrm{d}y.$$

Hence,

$$\begin{aligned}|a_{ij}^\alpha(x)| &\leqslant \frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} |a_{ij}(y)|\,\mathrm{d}y \\ &\leqslant C\frac{1}{|I_k^\alpha|}\int_{I_k^\alpha} v(y)\,\mathrm{d}y \\ &= Cv_\alpha(x).\end{aligned}$$

Therefore, in both cases we have

$$\sum_{i,j=1}^{n} |a_{ij}^\alpha(x)| \leqslant Cv_\alpha(x)$$

with constant C independent of α.

Case 2. If $(v, \omega) \in A_p$, $p > 1$. By Remark 4.2 we have $\omega \in A_p$ and $v \in A_p$, and by the Jones Factorization Theorem there are weights ω_0, ω_1, v_0 and v_1 in A_1 such that

$$v = v_0 v_1^{1-p} \quad \text{and} \quad \omega = \omega_0\omega_1^{1-p}$$

and we assume the following hypotheses (see Remark 4.12):

(a) $\omega_0 \leqslant v_0$ and $(v_0, \omega_0) \in A_1$;

(b) $v_1(x) \leqslant \omega_1(x) \leqslant C_1 v_1(x)$.

Since we are interested in approximating within Ω, we may assume, without loss of generality, that ω_0, ω_1, v_0 and $v_1 \in L^1(\mathbb{R}^n)$.

For each $\alpha > 1$, we define

$$U_{0,\alpha}^+ = \{x \in \mathbb{R}^n : M(\omega_0)(x) > \alpha\} \quad \text{and} \quad V_{0,\alpha}^+ = \{x \in \mathbb{R}^n : M(v_0)(x) > \alpha\}$$

($U_{0,\alpha}^+$ and $V_{0,\alpha}^+$ are open sets). By the Calderon–Zygmund decomposition there exist two families of maximal dyadic cubes $\{Q_{0,l}^\alpha\}$ and $\{I_{0,k}^\alpha\}$, for ω_0 and v_0, respectively, satisfying the analogous conditions of Case (1) ((CZ1)–(CZ4)). Using the hypotheses that $\omega_0 \leqslant v_0$, we have

(a$_0$) $Q^{\alpha}_{0,l} \subset I^{\alpha}_{0,k}$ (with $k = k(l)$);

(b$_0$) $G^{+}_{0,\alpha} = (V^{+}_{0,\alpha})^{\mathrm{c}} \subset (U^{+}_{0,\alpha})^{\mathrm{c}} = F^{+}_{0,\alpha}$.

For each $\beta > 1$, we define

$$\begin{aligned} U^{+}_{1,\beta} &= \{x \in \mathbb{R}^n : M(\omega_1)(x) > \beta^{1/(p-1)}\}, \\ V^{+}_{1,\beta} &= \{x \in \mathbb{R}^n : M(v_1)(x) > \beta^{1/(p-1)}\}. \end{aligned}$$

Since $v_1 \leqslant \omega_1$, we have $V^{+}_{1,\beta} \subset U^{+}_{1,\beta}$. By the Calderon–Zygmund decomposition, there exist two families of maximal dyadic cubes $\{Q^{\beta}_{1,l}\}$ and $\{I^{\beta}_{1,k}\}$, for ω_1 and v_1, respectively, satisfying conditions (CZ1)–(CZ4). We have

(a$_1$) $I^{\beta}_{1,k} \subset Q^{\beta}_{1,l}$ (with $l = l(k)$);

(b$_1$) $F^{+}_{1,\beta} = (U^{+}_{1,\beta})^{\mathrm{c}} \subset (V^{+}_{1,\beta})^{\mathrm{c}} = G^{+}_{1,\beta}$.

We define the weights

$$\begin{aligned} (\omega_0)_\alpha(x) &= \sum_{l=1}^{\infty}\biggl(\frac{1}{|Q^{\alpha}_{0,l}|}\int_{Q^{\alpha}_{0,l}} \omega_0(y)\,\mathrm{d}y\biggr)\chi_{Q^{\alpha}_{0,l}}(x) + \omega_0(x)\chi_{F^{+}_{0,\alpha}}(x), \\ (v_0)_\alpha(x) &= \sum_{k=1}^{\infty}\biggl(\frac{1}{|I^{\alpha}_{0,k}|}\int_{I^{\alpha}_{0,k}} v_0(y)\,\mathrm{d}y\biggr)\chi_{I^{\alpha}_{0,k}}(x) + v_0(x)\chi_{G^{+}_{0,\alpha}}(x), \\ (\omega_1)_\beta(x) &= \sum_{l=1}^{\infty}\biggl(\frac{1}{|Q^{\beta}_{1,l}|}\int_{Q^{\beta}_{1,l}} \omega_1(y)\,\mathrm{d}y\biggr)\chi_{Q^{\beta}_{1,l}}(x) + \omega_1(x)\chi_{F^{+}_{1,\beta}}(x), \\ (v_1)_\beta(x) &= \sum_{k=1}^{\infty}\biggl(\frac{1}{|I^{\beta}_{1,k}|}\int_{I^{\beta}_{1,k}} v_1(y)\,\mathrm{d}y\biggr)\chi_{I^{\beta}_{1,k}}(x) + v_1(x)\chi_{G^{+}_{1,\beta}}(x). \end{aligned}$$

We have, as in Case 1,

$$(\omega_0)_\alpha \leqslant 2^n (v_0)_\alpha \quad \text{and} \quad (v_1)_\beta \leqslant 2^n (\omega_1)_\beta.$$

Now we define the weights $\omega_{\alpha\beta}$ and $v_{\alpha\beta}$ by

$$\begin{aligned} \omega_{\alpha\beta}(x) &= (\omega_0)_\alpha(x)[(\omega_1)_\beta]^{1-p}(x), \\ v_{\alpha\beta}(x) &= (v_0)_\alpha(x)[(v_1)_\beta]^{1-p}(x). \end{aligned}$$

We have that $(\omega_0)_\alpha$, $(v_0)_\alpha$, $(\omega_1)_\beta$ and $(v_1)_\beta$ satisfy (using the same argument that given in Lemma 2.1 of [**6**]) the following conditions.

(I2) $(\omega_0)_\alpha$, $(v_0)_\alpha$, $(\omega_1)_\beta$ and $(v_1)_\beta \in A_1$.

(II2) $(\omega_0)_\alpha \to \omega_0$, $(v_0)_\alpha \to v_0$, $(\omega_1)_\beta \to \omega_1$ and $(v_1)_\beta \to v_1$ a.e. in $\mathbb{R}^n$ when $\alpha \to \infty$, $\beta \to \infty$.

(III2) $\min\{1,\omega_0\}$, $\min\{1,\omega_1\}$, $\min\{1,v_0\}$ and $\min\{1,v_1\}$ are weights in A_1 and

$$\min\{1,\omega_0(x)\} \leqslant (\omega_0)_\alpha(x) \leqslant C_1\omega_0(x), \quad \min\{1,\omega_1(x)\} \leqslant (\omega_1)_\beta(x) \leqslant C_2\omega_1(x),$$
$$\min\{1,v_0(x)\} \leqslant (v_0)_\alpha(x) \leqslant C_3 v_0(x), \quad \min\{1,v_1(x)\} \leqslant (v_1)_\beta(x) \leqslant C_4 v_1(x).$$

(IV2)

$$\begin{aligned}
&\tilde{C}_1 \leqslant (\omega_0)_\alpha(x) \leqslant 2^n\alpha, && \text{with } \tilde{C}_1 = \frac{1}{|Q_0|}\int_{Q_0} \min\{1,\omega_0\}\,\mathrm{d}y,\\
&\tilde{C}_2 \leqslant (v_0)_\alpha(x) \leqslant 2^n\alpha, && \text{with } \tilde{C}_2 = \frac{1}{|Q_0|}\int_{Q_0} \min\{1,v_0\}\,\mathrm{d}y,\\
&\tilde{C}_3 \leqslant (\omega_1)_\beta(x) \leqslant 2^n\beta^{1/(p-1)}, && \text{with } \tilde{C}_3 = \frac{1}{|Q_0|}\int_{Q_0} \min\{1,\omega_1\}\,\mathrm{d}y,\\
&\tilde{C}_4 \leqslant (v_1)_\beta(x) \leqslant 2^n\beta^{1/(p-1)}, && \text{with } \tilde{C}_4 = \frac{1}{|Q_0|}\int_{Q_0} \min\{1,v_1\}\,\mathrm{d}y,
\end{aligned}$$

where Q_0 is a fixed cube such that $\bar{\Omega} \subset Q_0$.

We need to show that weights $\omega_{\alpha\beta}$ and $v_{\alpha\beta}$ satisfy properties (i)–(v).

▷ (i) Follows from (IV2). In fact,

$$\begin{aligned}
\omega_{\alpha\beta}(x) &= (\omega_0)_\alpha(x)[(\omega_1)_\beta]^{1-p}(x)\\
&\geqslant \tilde{C}_1(2^n\beta^{1/(p-1)})^{1-p}\\
&= \tilde{C}_1 2^{n(1-p)}\beta^{-1},
\end{aligned}$$

$$\begin{aligned}
\omega_{\alpha\beta}(x) &= (\omega_0)_\alpha(x)[(\omega_1)_\beta]^{1-p}(x)\\
&\leqslant 2^n\alpha(\tilde{C}_3)^{1-p}\\
&= 2^n(\tilde{C}_3)^{1-p}\alpha.
\end{aligned}$$

Analogously, $\tilde{C}_2 2^{n(1-p)}\beta^{-1} \leqslant v_{\alpha\beta}(x) \leqslant 2^n(\tilde{C}_4)^{1-p}\alpha$.

▷ (ii) We define the weights $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{v}_1$ and $\tilde{v}_2$ by

$$\tilde{\omega}_1(x) = [\min\{1,\omega_0(x)\}]\omega_1^{1-p}(x) \quad \text{and} \quad \tilde{\omega}_2(x) = [\min\{1,\omega_1(x)\}]^{1-p}\omega_0(x),$$
$$\tilde{v}_1(x) = [\min\{1,v_0(x)\}]v_1^{1-p}(x) \quad \text{and} \quad \tilde{v}_2(x) = [\min\{1,v_1(x)\}]^{1-p}v_0(x).$$

By (III2) and the Jones Factorization Theorem, we have that $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{v}_1$ and $\tilde{v}_2$ are A_p-weights.

▷ (iii) Follows from (I2) and the Jones Factorization Theorem.

▷ (iv) We define the closed sets $G_{\alpha\beta}$ by

$$G_{\alpha\beta} = G^+_{0,\alpha} \bigcap F^+_{1,\beta} \quad \text{and} \quad F_{\alpha\beta} = F^+_{0,\alpha} \bigcap G^+_{1,\beta}.$$

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If $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$, we have

$$U^+_{0,\alpha'} \subset U^+_{0,\alpha}, \quad V^+_{0,\alpha'} \subset V^+_{0,\alpha}, \quad U^+_{1,\beta'} \subset U^+_{1,\beta}, \quad V^+_{1,\beta'} \subset V^+_{1,\beta}.$$

Hence,

$$F^+_{1,\beta} = (U^+_{1,\beta})^{\mathrm{c}} \subset (U^+_{1,\beta'})^{\mathrm{c}} = F^+_{1,\beta'} \quad \text{and} \quad G^+_{0,\alpha} = (V^+_{0,\alpha})^{\mathrm{c}} \subset (V^+_{0,\alpha'})^{\mathrm{c}} = G^+_{0,\alpha'}.$$

Thus

$$G_{\alpha\beta} = G^+_{0,\alpha} \cap F^+_{1,\beta} \subset G^+_{0,\alpha'} \cap F^+_{1,\beta'} = G_{\alpha'\beta'}.$$

Moreover, if $x \in G_{\alpha\beta}$, we have

$$\omega_{\alpha\beta}(x) = (\omega_0)_\alpha(x)[(\omega_1)_\beta]^{1-p}(x) = \omega_0(x)(\omega_1)^{1-p}(x) = \omega(x).$$

And by (b_0) and (b_1) we obtain $G_{\alpha\beta} \subset F_{\alpha\beta}$. In this case, if $x \in G_{\alpha\beta} \subset F_{\alpha\beta}$, we have

$$v_{\alpha\beta}(x) = (v_0)_\alpha(x)[(v_1)_\beta]^{1-p}(x) = v_0(x)(v_1)^{1-p}(x) = v(x).$$

Furthermore, by (CZ4), (b_0) and (b_1) we obtain

$$\begin{aligned}
\biggl|\biggl(\bigcup_{\alpha,\beta>1} G_{\alpha\beta}\biggr)^c\biggr| &= \biggl|\bigcap_{\alpha,\beta>1} G^{\mathrm{c}}_{\alpha\beta}\biggr| \\
&= \biggl|\bigcap_{\alpha,\beta>1} (G^+_{0,\alpha} \cap F^+_{1,\beta})^{\mathrm{c}}\biggr| \\
&= \biggl|\bigcap_{\alpha,\beta>1} [(V^+_{0,\alpha}) \cup (U^+_{1,\beta})]\biggr| \\
&\leqslant |V^+_{0,\alpha}| + |U^+_{1,\beta}| \\
&\leqslant \frac{c}{\alpha}\int_{\mathbb{R}^n} v_0(x)\,\mathrm{d}x + \frac{c}{\beta}\int_{\mathbb{R}^n} \omega_1(x)\,\mathrm{d}x, \quad \text{for all } \alpha, \beta > 1.
\end{aligned}$$

Thus $|(\cup_{\alpha,\beta>1} G_{\alpha\beta})^{\mathrm{c}}| = 0$.

▷ (v) Follows from (II2).

Let us now prove (vi). We define the coefficients $a^{\alpha\beta}_{ij}$ by

$$\begin{aligned}
a^{\alpha\beta}_{ij}(x) = [(v_1)_\beta]^{1-p}(x)\biggl\{\sum_{k=1}^{\infty}\biggl(\frac{1}{|I^\alpha_{0,k}|}\int_{I^\alpha_{0,k}} a_{ij}(y)v_1^{p-1}(y)\,\mathrm{d}y\biggr)\chi_{I^\alpha_{0,k}}(x) \\
+ a_{ij}(x)v_1^{p-1}(x)\chi_{G^+_{0,\alpha}}(x)\biggr\},
\end{aligned}$$

where $\{I^\alpha_{0,k}\}$ is the Calderon–Zygmund decomposition of v_0.

Let us check the first condition of (vi) for the coefficients $a^{\alpha\beta}_{ij}$.

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(I) If $x \in G_{\alpha\beta} = G^+_{0,\alpha} \cap F^+_{1,\beta} \subset F_{\alpha\beta}$. In this case, since $x \in G^+_{0,\alpha}$ and $x \in F^+_{1,\beta} \subset G^+_{1,\beta}$ (see (b_1)), we have

$$\begin{aligned} a^{\alpha\beta}_{i,j}(x) &= [(v_1)_\beta]^{1-p}(x) a_{ij}(x) v_1^{p-1}(x) \\ &= v_1^{1-p}(x) a_{ij}(x) v_1^{p-1}(x) \\ &= a_{ij}(x). \end{aligned}$$

And we have that $\omega_{\alpha\beta}(x) = \omega(x)$ and $v_{\alpha\beta}(x) = v(x)$. Consequently, we obtain

$$|\xi|^2 \omega_{\alpha\beta}(x) \leqslant \sum_{i,j=1}^{n} a^{\alpha\beta}_{ij}(x) \xi_i \xi_j \leqslant |\xi|^2 v_{\alpha\beta}(x). \tag{4.6}$$

(II) If $x \notin G_{\alpha\beta}$, then either $x \notin G^+_{0,\alpha}$ or else $x \notin F^+_{1,\beta}$.

▷ If $x \notin G^+_{0,\alpha}$, then $x \in I^\alpha_{0,k}$ (for only one k). In this case,

$$a^{\alpha\beta}_{ij}(x) = [(v_1)_\beta]^{1-p}(x) \frac{1}{|I^\alpha_{0,k}|} \int_{I^\alpha_{0,k}} a_{ij}(y) v_1^{p-1}(y)\, \mathrm{d}y,$$

and this implies, using condition (4.1) and $v = v_0 v_1^{1-p}$,

$$\begin{aligned} a^{\alpha\beta}_{ij}(x)\xi_i\xi_j &= [(v_1)_\beta]^{1-p}(x) \frac{1}{|I^\alpha_{0,k}|} \int_{I^\alpha_{0,k}} a_{ij}(y)\xi_i\xi_j v_1^{p-1}(y)\, \mathrm{d}y \\ &\leqslant [(v_1)_\beta]^{1-p}(x) |\xi|^2 \frac{1}{|I^\alpha_{0,k}|} \int_{I^\alpha_{0,k}} v(y) v_1^{p-1}(y)\, \mathrm{d}y \\ &= |\xi|^2 [(v_1)_\beta]^{1-p}(x) \frac{1}{|I^\alpha_{0,k}|} \int_{I^\alpha_{0,k}} v_0(y)\, \mathrm{d}y \\ &= |\xi|^2 [(v_1)_\beta]^{1-p}(x) (v_0)_\alpha(x) \\ &= |\xi|^2 v_{\alpha\beta}(x). \end{aligned}$$

On the other hand, since $\omega = \omega_0 \omega_1^{1-p}$ and $(v_1)_\beta \leqslant 2^n (\omega_1)_\beta$, we obtain

$$\begin{aligned} a^{\alpha\beta}_{ij}(x)\xi_i\xi_j &\geqslant [(v_1)_\beta]^{1-p}(x) |\xi|^2 \frac{1}{|I^\alpha_{0,k}|} \int_{I^\alpha_{0,k}} \omega(y) v_1^{p-1}(y)\, \mathrm{d}y \\ &\geqslant |\xi|^2 2^{n(1-p)} [(\omega_1)_\beta]^{1-p}(x) \frac{1}{|I^\alpha_{0,k}|} \int_{I^\alpha_{0,k}} \omega_0(y) \omega_1^{1-p}(y) v_1^{p-1}(y)\, \mathrm{d}y \\ &\geqslant 2^{n(1-p)} |\xi|^2 [(\omega_1)_\beta]^{1-p}(x) \frac{1}{|I^\alpha_{0,k}|} \int_{I^\alpha_{0,k}} \omega_0(y)\, \mathrm{d}y. \end{aligned} \tag{4.7}$$

There are two possible cases.

(1) If $x \in Q^\alpha_{0,l} \subset I^\alpha_{0,k}$. In this case, using the same argument in (4.5), we obtain

$$\frac{1}{|I^\alpha_{0,k}|} \int_{I^\alpha_{0,k}} \omega_0(y)\, \mathrm{d}y \geqslant \frac{1}{2^n C_0} \frac{1}{|Q^\alpha_{0,l}|} \int_{Q^\alpha_{0,l}} \omega_0(y)\, \mathrm{d}y = \frac{1}{2^n C_0} (\omega_0)_\alpha(x),$$

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where C_0 is the A_1-constant for the pair (v_0, ω_0). Hence, in (4.7) we obtain

$$\begin{aligned} a_{ij}^{\alpha\beta}(x)\xi_i\xi_j &\geqslant \frac{2^{n(1-p)}}{2^n C_0}|\xi|^2[(\omega_1)_\beta]^{1-p}(x)(\omega_0)_\alpha(x) \\ &= \frac{1}{2^{np}C_0}|\xi|^2\omega_{\alpha\beta}(x). \end{aligned} \tag{4.8}$$

(2) If $x \in I_{0,k}^\alpha - Q_{0,k}$, where $Q_{0,k} = \cup Q_{0,l}^\alpha$, with $Q_{0,l}^\alpha \subset I_{0,k}^\alpha$, that is, $x \in F_{0,\alpha}^+$. In this case $(\omega_0)_\alpha = \omega_0$ and using the properties of the Calderon–Zygmund decomposition, we obtain

$$\frac{1}{|I_{0,k}^\alpha|}\int_{I_{0,k}^\alpha}\omega_0(y)\,\mathrm{d}y \geqslant \frac{\alpha}{C_0} \geqslant \frac{1}{C_0}\omega_0(x) = \frac{1}{C_0}(\omega_0)_\alpha(x). \tag{4.9}$$

Therefore,

$$\frac{1}{2^{np}C_0}|\xi|^2\omega_{\alpha\beta}(x) \leqslant \sum_{i,j=1}^n a_{ij}^{\alpha\beta}(x)\xi_i\xi_j \leqslant |\xi|^2 v_{\alpha\beta}(x).$$

▹ If $x \notin F_{1,\beta}^+$ and $x \in G_{0,\alpha}^+ \subset F_{o,\alpha}^+$. In this case, we have $x \in I_{1,k}^\beta \subset Q_{1,l}^\beta$ (for only one k and $l = l(k)$, by (a_1)). Thus

$$\begin{aligned} a_{ij}^{\alpha\beta}(x)\xi_1\xi_j &= (v_1)_\beta^{1-p}(x)a_{ij}(x)\xi_i\xi_j v_1^{p-1}(x) \\ &\leqslant (v_1)_\beta^{1-p}(x)v(x)v_1^{p-1}(x)|\xi|^2 \\ &= (v_1)_\beta^{1-p}(x)v_0(x)|\xi|^2 \\ &= (v_1)_\beta^{1-p}(x)(v_0)_\alpha(x)|\xi|^2 \\ &= v_{\alpha\beta}(x)|\xi|^2. \end{aligned}$$

On the other hand, since $\omega_1(x) \leqslant C_1 v_1(x)$ (Remark 4.12) and $(v_1)_\beta(x) \leqslant 2^n(\omega_1)_\beta(x)$, we obtain

$$\begin{aligned} a_{ij}^{\alpha\beta}(x)\xi_i\xi_j &= (v_1)_\beta^{1-p}(x)a_{ij}(x)\xi_i\xi_j v_1^{p-1}(x) \\ &\geqslant (v_1)_\beta^{1-p}(x)\omega(x)v_1^{p-1}(x)|\xi|^2 \\ &\geqslant 2^{n(1-p)}(\omega_1)_\beta^{1-p}(x)\omega(x)v_1^{p-1}(x)|\xi|^2 \\ &\geqslant 2^{n(1-p)}\omega_\beta^{1-p}(x)\omega(x)C_1^{1-p}\omega_1^{p-1}(x)|\xi|^2 \\ &= 2^{n(1-p)}C_1^{1-p}(\omega_1)_\beta^{1-p}(x)\omega_0(x)|\xi|^2 \\ &= (2^n C_1)^{1-p}(\omega_1)_\beta^{1-p}(x)(\omega_0)_\alpha(x) \\ &= (2^n C_1)^{1-p}\omega_{\alpha\beta}(x)|\xi|^2. \end{aligned} \tag{4.10}$$

Therefore, by (4.6) and (4.8)–(4.10) we conclude that

$$\tilde{\lambda}|\xi|^2\omega_{\alpha\beta}(x) \leqslant \sum_{i,j=1}^n a_{ij}^{\alpha\beta}(x)\xi_i\xi_j \leqslant |\xi|^2 v_{\alpha\beta}(x),$$

where $\tilde{\lambda} = \min\{1, 1/2^{np}C_0, 1/(2^n C_1)^{p-1}\}$.

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We need to show the second condition of (vi). Again, there are two possible cases.

(1) If $x \in G_{\alpha\beta}$, then $a_{ij}^{\alpha\beta}(x) = a_{ij}(x)$ and $v_{\alpha\beta}(x) = v(x)$. Hence the condition is valid.

(2) If $x \notin G_{\alpha\beta}$, then $x \in I_{0,k}^{\alpha}$. In this case, since $v = v_0 v_1^{1-p}$, we have

$$\begin{aligned}
|a_{ij}^{\alpha\beta}(x)| &\leqslant [(v_1)_\beta]^{1-p}(x) \frac{1}{|I_{0,k}^{\alpha}|} \int_{I_{0,k}^{\alpha}} |a_{ij}(y)| v_1^{p-1}(y)\, \mathrm{d}y \\
&\leqslant [(v_1)_\beta]^{1-p}(x) \frac{1}{|I_{0,k}^{\alpha}|} \int_{I_{0,k}^{\alpha}} C v(y) v_1^{p-1}(y)\, \mathrm{d}y \\
&= C[(v_1)_\beta]^{1-p}(x) \frac{1}{|I_{0,k}^{\alpha}|} \int_{I_{0,k}^{\alpha}} v_0(y) v_1^{1-p}(y) v_1^{p-1}(y)\, \mathrm{d}y \\
&= C[(v_1)_\beta]^{1-p}(x) \frac{1}{|I_{0,k}^{\alpha}|} \int_{I_{0,k}^{\alpha}} v_0(y)\, \mathrm{d}y \\
&= C[(v_1)_\beta]^{1-p}(x)(v_0)_\alpha(x) \\
&= C v_{\alpha\beta}(x).
\end{aligned}$$

Therefore, in both cases we obtain

$$\sum_{i,j=1}^{n} |a_{ij}^{\alpha\beta}(x)| \leqslant C v_{\alpha\beta}(x)$$

with constant C independent of α and β.

□

Let us now state the main result of this section. Using Lemma 4.13 we can prove that the solution $u \in H^{1,2}(\Omega, \omega, v)$ for the Dirichlet problem

$$\begin{gathered}
Lu = g - \sum_{j=1}^{n} D_j f_j, \quad \text{in } \Omega, \\
u - \psi \in H_0^{1,2}(\Omega, \omega, v),
\end{gathered}$$

with the conditions (4.1) and (4.2), can be approximated by a sequence of solutions of non-degenerate elliptic equations.

Theorem 4.14. *Suppose that the WSI holds and let $(v, \omega) \in S_2$. If $g/v \in L^{(2\sigma)'}(\Omega, v)$, $f_j/\omega \in L^2(\Omega, \omega)$ and $\psi \in H^{1,2}(\Omega, \omega, v)$, then the solution $u \in H^{1,2}(\Omega, \omega, v)$ of the problem*

$$\left.\begin{aligned}
Lu = g - \sum_{j=1}^{n} D_j f_j, \quad &\text{in } \Omega, \\
u - \psi \in H_0^{1,2}(\Omega, \omega, v),&
\end{aligned}\right\} \tag{P}$$

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is the weak limit in $H^{1,2}(\Omega,\tilde{\omega}_1,\tilde{v}_1)$ *of a sequence of solutions* $u_m \in H^{1,2}(\Omega,\omega_m,v_m)$ *of the problems*

$$\left.\begin{aligned} L_m u_m &= g_m - \sum_{j=1}^{n} D_j f_{jm}, \\ u_m - \psi &\in H_0^{1,2}(\Omega,\omega_m,v_m), \end{aligned}\right\} \tag{P$_m$}$$

with $\omega_m = \omega_{mm}$ *and* $v_m = v_{mm}$, $L_m u_m = -\sum D_j(a_{ij}^{mm} D_i u_m)$, $f_{jm} = f_j(\omega/\omega_m)^{-1/2}$ *and* $g_m = g(v/v_m)^{-1/q}$ *(where* ω_{mm}, v_{mm}, $\tilde{\omega}_1$, $\tilde{v}_1$ *and* a_{ij}^{mm} *are as in Lemma 4.13).*

Proof. By Remark 4.6 we have that $(v,\omega) \in A_2$, and by Theorem 4.7 we have $H^{1,2}(\Omega,\omega,v) = W^{1,2}(\Omega,\omega,v)$. We denote by $A^m = (a_{ij}^{mm})_{i,j=1,\dots,n}$ the coefficient matrix of the operator L_m.

First, we note that

$$\left\|\frac{g_m}{v_m}\right\|_{L^{(2\sigma)'}(\Omega,v_m)} = \left\|\frac{g}{v}\right\|_{L^{(2\sigma)'}(\Omega,v)} \quad \text{and} \quad \left\|\frac{f_{jm}}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} = \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)}.$$

Since u_m is a solution of (P_m), by Theorem 4.9 we obtain

$$\begin{aligned} \|u_m\|_{H^{1,2}(\Omega,\omega_m,v_m)} &\leqslant C_{\Omega,\omega_m,v_m}\left(\left\|\frac{g_m}{v_m}\right\|_{L^{(2\sigma)'}(\Omega,v_m)} + \left\|\frac{f_{jm}}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} + \|\psi\|_{H^{1,2}(\Omega,\omega,v)}\right) \\ &\leqslant C\left(\left\|\frac{g}{v}\right\|_{L^{(2\sigma)'}(\Omega,v)} + \left\|\frac{f_j}{\omega}\right\|_{L^2(\Omega,\omega)} + \|\psi\|_{H^{1,2}(\Omega,\omega,v)}\right) \\ &= C_1, \end{aligned}$$

where the constant C is independent of m, since $\tilde{\omega}_1 \leqslant \omega_m \leqslant \tilde{\omega}_2$ and $\tilde{v}_1 \leqslant v_m \leqslant \tilde{v}_2$, and we have

$$\begin{aligned} C_{\Omega,\omega_m,v_m} &= \frac{[v_m(\Omega)]^{1/q}}{[\omega_m(\Omega)]^{1/2}} C_\Omega \\ &\leqslant \frac{[\tilde{v}_2(\Omega)]^{1/q}}{[\tilde{\omega}_1(\Omega)]^{1/2}} C_\Omega. \end{aligned}$$

Consequently,

$$\|u_m\|_{H^{1,2}(\Omega,\tilde{\omega}_1,\tilde{v}_1)} \leqslant \|u_m\|_{H^{1,2}(\Omega,\omega_m,v_m)} \leqslant C_1,$$

that is, u_m is a bounded sequence in $H^{1,2}(\Omega,\tilde{\omega}_1,\tilde{v}_1)$. Hence, there exists a subsequence, again denoted by $\{u_m\}$, which converges weakly to an element $\tilde{u} \in H^{1,2}(\Omega,\tilde{\omega}_1,\tilde{v}_1)$,

$$u_m \rightharpoonup \tilde{u} \quad \text{in } H^{1,2}(\Omega,\tilde{\omega}_1,\tilde{v}_1).$$

We need to show that $\tilde{u} \in H^{1,2}(\Omega,\omega,v)$ and that $\tilde{u}$ is a solution of the equation

$$L\tilde{u} = g - \sum D_j f_j.$$

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The proof that $\tilde{u} \in H^{1,2}(\Omega,\omega,v)$ is similar to the proof in Theorem 3.2 (I) (by Theorem 4.7 we have $H^{1,2}(\Omega,\omega,v) = W^{1,2}(\Omega,\omega,v)$). To prove that $L\tilde{u} = g - \sum D_j f_j$, we need to show that

$$\int_\Omega a_{ij} D_i \tilde{u} D_j \varphi \,\mathrm{d}x = \int_\Omega g\varphi \,\mathrm{d}x + \int_\Omega f_j D_j \varphi \,\mathrm{d}x, \quad \forall \varphi \in H_0^{1,2}(\Omega,\omega,v).$$

Fix $G_k = G_{kk}$ (as in Lemma 4.13). Using the fact that u_m is a solution of (P_m), and in G_k we have $\omega_m = \omega$, $v_m = v$, $g_m = g$, $f_{jm} = f_j$ and $a_{ij}^{mm} = a_{ij}$ (with $m \geqslant k$), we obtain

$$\begin{aligned}\int_{G_k} a_{ij} D_i \tilde{u} D_j \varphi \,\mathrm{d}x &= \lim_{m\to\infty} \int_{G_k} a_{ij}^{mm} D_i u_m D_j \varphi \,\mathrm{d}x \\ &= \lim_{m\to\infty} \biggl(\int_\Omega a_{ij}^{mm} D_i u_m D_j \varphi \,\mathrm{d}x - \int_{\Omega\cap G_k^{\mathrm{c}}} a_{ij}^{mm} D_i u_m D_j \varphi \,\mathrm{d}x \biggr) \\ &= \lim_{m\to\infty} \biggl(\int_\Omega g_m \varphi \,\mathrm{d}x + \int_\Omega f_{jm} D_j \varphi \,\mathrm{d}x - \int_{\Omega\cap G_k^{\mathrm{c}}} a_{ij}^{mm} D_i u_m D_j \varphi \,\mathrm{d}x \biggr).\end{aligned}$$

As demonstrated in Theorem 3.2 (I), we have

(a) $\displaystyle\lim_{m\to\infty} \int_\Omega g_m \varphi = \int_\Omega g\varphi \,\mathrm{d}x$;

(b) $\displaystyle\lim_{m\to\infty} \int_\Omega f_{jm} D_j \varphi \,\mathrm{d}x = \int_\Omega f_j D_j \varphi \,\mathrm{d}x$; and

(c) since $\|\tilde{u}\|_{H^{1,2}(\Omega,\omega_m,v_m)} \leqslant C_1$ (C_1 independent of m), and the matrix A^m is symmetric and $v_m \leqslant \tilde{v}_2$, we have

$$\begin{aligned}\biggl| \int_{\Omega\cap G_k^{\mathrm{c}}} a_{ij}^{mm} D_i u_m D_j \varphi \,\mathrm{d}x \biggr| &\leqslant \int_{\Omega\cap G_k^{\mathrm{c}}} |\langle A^m \nabla u_m, \nabla\varphi\rangle| \,\mathrm{d}x \\ &\leqslant \int_{\Omega\cap G_k^{\mathrm{c}}} \langle A^m \nabla u_m, \nabla u_m\rangle^{1/2} \langle A^m \nabla\varphi, \nabla\varphi\rangle^{1/2} \,\mathrm{d}x \\ &\leqslant \biggl(\int_{\Omega\cap G_k^{\mathrm{c}}} \langle A^m \nabla u_m, \nabla u_m\rangle \,\mathrm{d}x \biggr)^{1/2} \\ &\qquad \times \biggl(\int_{\Omega\cap G_k^{\mathrm{c}}} \langle A^m \nabla\varphi, \nabla\varphi\rangle \,\mathrm{d}x \biggr)^{1/2} \\ &\leqslant \|u_m\|_{H^{1,2}(\Omega,\omega_m,v_m)} \biggl(\int_{\Omega\cap G_k^{\mathrm{c}}} |\nabla\varphi|^2 v_m \,\mathrm{d}x \biggr)^{1/2} \\ &\leqslant C_1 \biggl(\int_{\Omega\cap G_k^{\mathrm{c}}} |\nabla\varphi|^2 v_m \,\mathrm{d}x \biggr)^{1/2} \\ &\leqslant C_1 C_\varphi [v_m(\Omega\cap G_k^{\mathrm{c}})]^{1/2}.\end{aligned}$$

Applying [**7**, Theorem 2.9, Chapter IV], there exist constants $\delta > 0$ and $C > 0$ such that, if $\bar{\Omega} \subset Q_0$ (Q_0 is a fixed cube), then

$$v_m(\Omega\cap G_k^{\mathrm{c}}) \leqslant \tilde{v}_2(\Omega\cap G_k^{\mathrm{c}}) \leqslant C\tilde{v}_2(Q_0) \biggl(\frac{|\Omega\cap G_k^{\mathrm{c}}|}{|Q_0|} \biggr)^{\delta}.$$

Hence, since $|\Omega \cap G_k^{\mathrm{c}}| \to 0$ when $k \to \infty$ (by construction of G_k), we obtain

$$v_m(\Omega \cap G_k^{\mathrm{c}}) \to 0 \quad \text{when } k \to \infty.$$

Therefore, by (a), (b) and (c), we can conclude that

$$\int_\Omega a_{ij}(x) D_i \tilde{u}(x) D_j \varphi(x)\,\mathrm{d}x = \int_\Omega g(x)\varphi(x)\,\mathrm{d}x + \int_\Omega f_j(x) D_j \varphi(x)\,\mathrm{d}x,$$

that is, $\tilde{u}$ is a solution of the equation

$$L\tilde{u} = g - \sum_{j=1}^{n} D_j f_j.$$

Therefore, we conclude that $\tilde{u} = u$ (by the uniqueness of the Theorem 4.9). □

Remark 4.15. The uniform convergence on compacts does not immediately generalize to the unequal weights cases.

In cases where $\omega = v$, it is necessary to prove the Hölder continuity (3.7) that: if u is a solution of $Lu = 0$, then

$$\operatorname*{osc}_{B(x,\rho)} u \leqslant \frac{M-1}{M+1} \operatorname*{osc}_{B(x,8\rho)} u,$$

where M is independent of ρ, u and x (see Lemma 2.3.11 in [**5**]).

But in cases where $\omega \neq v$, if u is a solution of $Lu = 0$, then

$$\operatorname*{osc}_{B(x,\rho)} u \leqslant \frac{\mathrm{e}^{C\mu(x,\rho)} - 1}{\mathrm{e}^{C\mu(x,\rho)} + 1} \operatorname*{osc}_{B(x,2\rho)} u,$$

where

$$\mu(x,\rho) = \left(\frac{v(B(x,\rho))}{\omega(B(x,\rho))}\right)$$

(see § 5 in [**2**]).

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