ON THE FOURIER TRANSFORMS

Hwai-chiuan Wang

In this article we give a new proof of the theorem that a
positive even convex function on the real line, which vanishes at
infinity, is the Fourier transform of an integrable function.
Related results in several variables are also proved. As an
application of our results we solve the factorization problem of
Sobolev algebras.

0. Introduction

Recall several classical Fourier multiplier results: Katznelson [2,
p. 22] proved that a positive even convex sequence vanishing at infinity is
the sequence of Fourier coefficients of an integrable function. Titchmarch
[5, p. 170] proved that a positive even convex function on the real line
vanishing at infinity is the Fourier cosine transform of an integrable
function. However, [5] asserted that there exists a positive even convex
function on the real line vanishing at infinity which is not the Fourier
sine transform of an integrable function. In Section 1 we prove that a
positive even convex function on the real line vanishing at infinity is the
Fourier transform of an integrable function (see Theorem 1.1). This is not
a new result but we use a simple and delicate method through the Gauss-
Weierstrass kernels, which is closely related to partial differential

equations. We hope in the future to apply our present results to some
problems in partial differential equations. The same theory in several
variables is not clear yet. We also obtain partial results in the case of
several variables. In Section 2 we apply the theorems of Section 1 to
solve the factorization problem of Sobolev algebras on the real line.

1. Main results

Throughout this article we denote by R^n the n-dimensional euclidean
space.

THEOREM 1.1. *Let* $f(x)$ *be an even positive convex function
vanishing at infinity, then there is a positive function* F *in* $L^1(R)$
such that f *is the Fourier transform of* F *.*

Proof. We divide the proof into two steps.

(1) From the hypothesis the derivative f' of f is nonpositive,
nondecreasing and

$$f(x) = \int_{-\infty}^{x} f'(t)dt \ .$$

For nonzero real u ,

$$\int_R f(x)e^{2\pi iux}dx = 2 \int_0^\infty f(x) \ \cos \ 2\pi uxdx$$

$$= -1/\pi|u| \int_0^\infty f'(x) \ \sin \ 2\pi|u|xdx$$

$$= 1/\pi|u| \sum_{j=0}^\infty \int_{j/2|u|}^{(j+1)/2|u|} \left(-f'(x) \ \sin \ 2\pi|u|x\right)dx \ .$$

The series in the last equality is an alternating series with positive
first term, nonincreasing in absolute values of terms, and vanishing at
infinity. Therefore

$$\int_R f(x)e^{2\pi iux}dx \geq 0 \ .$$

(2) If we fix U, S , $0 < U \leq |u| \leq S$, then

$$\left|\int_{T'\geq|x|\geq T>0} f(x)e^{2\pi iux}dx\right| = \left|\int_{T'\geq|x|\geq T>0} f(x)\cos 2\pi uxdx\right|$$

$$= \left|f(c)\int_{T'\geq|x|\geq T>0}\cos 2\pi uxdx\right|$$

$$\text{where} \quad T' \geq |c| \geq T ,$$

$$\leq f(T)/\pi U \to 0 \quad \text{as} \quad T \to \infty .$$

Using the above uniform convergence arguments, we obtain

$$\int_R \left(\int_R f(x)e^{2\pi iux}dx\right)e^{-4\pi^2 a|u|^2}du$$

$$= \lim_{\substack{U\to 0 \\ S\to\infty}} \int_{0<U\leq|u|\leq S} \left(\int_R f(x)e^{2\pi iux}dx\right)e^{-4\pi^2 a|u|^2}du$$

$$= \lim_{\substack{U\to 0 \\ S\to\infty}} \lim_{T\to\infty} \int_{0<U\leq|u|\leq S} \left(\int_{|x|\leq T} f(x)e^{2\pi iux}dx\right)e^{-4\pi^2 a|u|^2}du$$

$$= \lim_{\substack{U\to 0 \\ S\to\infty}} \int_R \left(\int_{0<|U|\leq|u|\leq S} e^{-4\pi^2 a|u|^2}e^{2\pi iux}du\right)f(x)dx$$

$$= \int_R \left(\int_R e^{-4\pi^2 a|u|^2}e^{2\pi iux}du\right)f(x)dx$$

$$= \int_R f(x)(4\pi a)^{-1/2}e^{-|x|^2/4a}dx .$$

Since the Gauss-Weierstrass kernel $\{(4\pi a)^{-1/2}e^{-|x|^2/4a}\}_{a>0}$ is an approximation identity, we have

$$\lim_{a\to 0} \int_R f(x)(4\pi a)^{-1/2}e^{-x^2/4a}dx = f(0) .$$

Thus

$$\lim_{a\to 0} \int_R \left(\int_R f(x)e^{2\pi iux}dx\right)e^{-4\pi^2 a|u|^2}du = f(0) .$$

However, from part (1),

$$\int_R f(x)e^{2\pi iux}dx \geq 0 \quad \text{for all real} \quad u .$$

By the Fatou lemma,

$$\int_R \left[\int_R f(x)e^{2\pi i u x}dx\right] du \le f(0) < \infty .$$

Let

$$F(u) = \int_R f(x)e^{2\pi i u x}dx .$$

Then F is positive integrable on the real line and

$$\hat{F} = f .$$

This completes the proof.

REMARK 1.2. Let $g(x)$ be a function on the real line vanishing at infinity. We take a positive even convex function $f(x)$ on the real line vanishing at infinity with $|g(x)| \le f(x)$. Then by Theorem 1.1, there is an integrable function F such that $\hat{F} = f$. This property reveals that the Fourier transform of an integrable function on the real line can vanish at infinity arbitrarily slowly.

REMARK 1.3. By Theorem 1.1, a positive even convex function on the real line vanishing at infinity is a Fourier multiplier of p-integrable functions, $1 \le p \le \infty$.

REMARK 1.4. Theorem 1.1 may fail if in the hypothesis we replace even by odd. Let

$$g(x) = \begin{cases} 1/\log x & \text{for } x > 5 , \\ \text{tangent line of } 1/\log x \text{ at } x = 5 \text{ for } 0 \le x \le 5 , \\ -g(-x) & \text{for } x < 0 , \end{cases}$$

then g is not the Fourier transform of an integrable function (see Stein and Weiss [4, p. 31]).

We consider below some related results in several variables.

PROPOSITION 1.5. *Let* $f(x)$ *be a locally integrable function on* R^n *, which vanishes at infinity. If there is a positive even integer* $N > n$ *such that*

$$\partial_{x_i}^N f(x) \in L^1(R^n) \quad for \quad i = 1, 2, \ldots, n ,$$

and the inverse Fourier transform F of f is locally integrable, then
$F \in L^1(R^n)$ *with* $\hat{F} = f$.

Proof. By the hypothesis

$$\left(\partial_{x_1}^N + \ldots + \partial_{x_n}^N\right) f(x) \quad L^1(R^n) .$$

Let $F = \breve{f}$; then

$$(2\pi i)^N \left(x_1^N + \ldots + x_n^N\right) F(x)$$

is bounded for every $x \in R^n$. But

$$\left| x_1^N + \ldots + x_n^N \right| \geq c \left[\left(x_1^2\right)^{N/2} + \left(x_n^2\right)^{N/2} \right]$$
$$\geq c \left(x_1^2 + \ldots + x_n^2\right)^{N/2} .$$

So

$$\left| F(x) \right| \leq c/|x|^N \quad \text{for large} \quad x .$$

Thus $F \in L^1(R^n)$ and $\hat{F} = f$.

EXAMPLE 1.6. Let $f(x)$ be a function on R^n with

$$f(x) = 1/|x|^a \quad \text{for} \quad x \neq 0 , \quad 0 < a < n ,$$

then it is known that the inverse Fourier transform F of f is of the
form $c/|x|^{n-a}$, which is locally integrable. Also

$$\left| \partial_{x_i}^{2n} f(x) \right| \leq c/|x|^{a+2n} , \quad i = 1, 2, \ldots, n .$$

So, by Proposition 1.5, $F \in L^1(R^n)$ with $\hat{F} = f$.

The following locally integrable characterization is useful, in view
of Theorem 1.1.

PROPOSITION 1.7. *Let* $f(x)$ *be a radial function on* R^n *, vanishing
at infinity,* $n > 2$ *. Suppose* $|\text{grad } f(x)| \in L^1(R^n)$ *; then the inverse
Fourier transform of* f *is a locally integrable function.*

Proof. Recall that

$$\left| \operatorname{grad} f(x) \right| = -\partial f(x)/\partial |x| = -f'(r) \ , \quad r = |x| \ .$$

Using the polar representation, we have

$$\omega_{n-1} \int_0^\infty r^{n-1} |f'(r)| \, dr = \int_{R^n} |\operatorname{grad} f(x)| \, dx < \infty$$

where ω_{n-1} is the area of the unit sphere in R^n. But

$$f(r) = - \int_r^\infty f'(t) \, dt$$

or

$$|f(r)| \leq \int_r^\infty |f'(t)| \, dt$$

$$\leq 1/r^{n-1} \int_0^\infty t^{n-1} |f'(t)| \, dt$$

$$\leq c/r^{n-1} \ .$$

Thus f is locally integrable and

$$\int_{1 \leq |x|} |f(x)|^2 \, dx \leq c\omega_{n-1} \int_1^\infty 1/r^{n-1} \, dr < \infty \quad \text{if} \quad n > 2 \ .$$

Let

$$f = \chi_{\{x : |x| \leq 1\}} f + \chi_{\{x : |x| > 1\}} f$$
$$= f_1 + f_2 \ .$$

Then $f_1 \in L^1(R^n)$, $f_2 \in L^2(R^n)$ with $\hat{f} = \hat{f}_1 + \hat{f}_2$. Thus $\hat{f}$ is locally integrable.

2. Application

We begin with several definitions.

DEFINITION 2.1. A Banach algebra B is said to admit weak factorization if, given $h \in B$, there are $f_1, \ldots, f_n, g_1, \ldots, g_n \in B$

such that $h = \displaystyle\sum_{i=1}^{n} f_i g_i$.

DEFINITION 2.2. Let $\bigl(B(R), \| \ \|_B\bigr)$ be a Banach algebra of complex-valued measurable functions on R . $B(R)$ is called a homogeneous Banach algebra if it satisfies the following properties:

H1. if $f \in B(R)$, $x \in R$ then $L_x f \in B(R)$, $\|L_x f\|_B = \|f\|_B$

        where

$$L_x f(y) = f(y-x) \quad \text{for} \quad y \in R \ ;$$

H2. $x \to L_x f$ is continuous from R into $\bigl(B(R), \| \ \|_R\bigr)$.

The homogeneous Banach algebra $\bigl(B(R), \| \ \|_R\bigr)$ is a Segal algebra if

$B(R)$ is dense in $L^1(R)$. A Segal algebra $B(R)$ is character invariant if $e^{ixt} f(t) \in B(R)$ whenever $x \in R$, $f \in B(R)$. The definitions and theory of Segal algebras can be established on general locally compact abelian groups instead of the real group R (see Reiter [3] and Wang [6]).

DEFINITION 2.3. Let $H_s(R)$ be the space of all integrable functions on the real line such that

$$\int_R \bigl(1+|x|^2\bigr)^s |\hat{u}(x)|^2 dx < \infty \ .$$

Let $\Lambda_s = \bigl(I-(2\pi)^{-2}\Delta\bigr)^{s/2}$ where Δ is the Laplacian, or

$$\bigl(\widehat{\Lambda_s u}\bigr)(x) = \bigl(1+|x|^2\bigr)^{s/2}\hat{u}(x) \quad \text{for} \quad x \in R$$

define

$$\|u\|_{H_s} = \|u\|_{L^1} + \|\Lambda_s u\|_{L^2} \ .$$

Using the techniques developed in Wang [7], we have

PROPOSITION 2.4. *For any real s , $\bigl(H_s(R), \| \ \|_{H_s}\bigr)$ is a character invariant Segal algebra on R .*

We then call $H_s(R)$ a Sobolev algebra. These are interesting and

useful in both analysis and partial differential equations.

Furthermore we have the following factorization results.

THEOREM 2.5. *The Sobolev algebra* $H_s(R)$ *admits a weak factorization if and only if* $s < -1/2$.

Proof. (1) For $s < -1/2$ and $u \in L^1(R)$, then

$$\| (1+|x|^2)^{s/2} \hat{u}(x) \|_{L^2}^2 = \int_R (1+|x|^2)^s |\hat{u}(x)|^2 dx$$
$$< \infty \text{ since } \hat{u} \in C_0(R) .$$

Thus $H_s(R) = L^1(R)$ for $s < -1/2$. By a well-known factorization theorem of Cohen, the algebras $H_s(R)$, $s < -1/2$, admit weak factorizations.

(2) Feichtinger, Graham and Lakien [1] asserted that a character invariant Segal algebra $B(G)$ on a locally compact abelian group G admits weak factorization if and only if $B(G) = L^1(G)$. Hence in order to prove that the algebras $H_s(R)$, $s \geq -1/2$, do not admit weak

factorizations, it suffices to prove that $H_s(R) \subsetneqq L^1(R)$. However it is easy to see that the algebras $H_s(R)$ possess the going down property, $H_s(R) \subset H_t(R)$ if $s \geq t$. Thus it suffices to prove

$$H_{-1/2}(R) \subsetneqq L^1(R) .$$

Let $f(x)$ be an even function on R defined by

$$f(x) = \begin{cases} (\log x)^{-1/2} & \text{for } x \geq 2 , \\ \left(-1/4(\log 2)^{-3/2}\right)(x-2) + (\log 2)^{-1/2} & \text{for } 0 \leq x < 2 . \end{cases}$$

Actually the function $\left(-1/4(\log 2)^{-3/2}\right)(x-2) + (\log 2)^{-1/2}$ is the tangent line of the function $(\log x)^{-1/2}$ at $x = 2$. Then f is an even positive convex function on R vanishing at infinity. By Theorem 1.1, there is an integrable function u on R such that $\hat{u} = f$. But

$$\int_R \left(1+|x|^2\right)^{-1/2}|\hat{u}(x)|^2 dx \geq 2 \int_2^\infty \frac{1}{\left(1+x^2\right)^{\frac{1}{2}}\log x}\ dx$$

$$\geq \int_2 \frac{1}{x\log x}\ dx = \infty\ .$$

Thus $H_{-1/2}(R) \subsetneq L^1(R)$, which completes the proof.

References

[1] Hans G. Feichtinger, Colin C. Graham and Eric H. Lakien, "Nonfactor-
     ization in commutative, weakly self-adjoint Banach algebras",
     Pacific J. Math. 80 (1979), 117-125.

[2] Y. Katznelson, *An introduction to harmonic analysis* (John Wiley &
     Sons, New York, London, 1968).

[3] Hans Reiter, *Classical harmonic analysis and locally compact groups*
     (Clarendon Press, Oxford, 1968).

[4] Elias M. Stein and Guido Weiss, *Introduction to Fourier analysis on
     Euclidean spaces* (Princeton Mathematical Series, 32. Princeton
     University Press, Princeton, New Jersey, 1971).

[5] E.C. Titchmarch, *Introduction to the theory of Fourier integrals*
     (Clarendon Press, Oxford, 1959).

[6] Hwai-Chiuan Wang, *Homogeneous Banach algebras* (Lecture Notes in Pure
     and Applied Mathematics, 29. Marcel Dekker, New York, Basel,
     1977).

[7] Hwai-chiuan Wang, "A note on Segal algebras on Euclidean spaces",
     Proc. Amer. Math. Soc. 72 (1978), 513-518.

Department of Mathematics,
National Tsing Hua University,
Hsinchu,
Taiwan 300,
Republic of China.