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A NOTE ON THE VAN DER WAERDEN PERMANENT CONJECTURE

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1. Introduction and statement of results. The permanent of an n-square complex matrix $P = (p_{ij})$ is defined by

$$\operatorname{per} P = \sum_{\sigma \in S_n} p_{1\sigma(1)} p_{2\sigma(2)} \cdots p_{n\sigma(n)}$$

where the summation extends over S_n, the symmetric group of degree n. This matrix function has considerable significance in certain combinatorial problems [**6**; **7**]. The properties and many related problems about the permanent are presented in [**3**] along with an extensive bibliography.

In 1926, B. L. van der Waerden [**7**] conjectured that for all elements P of the set Ω_n the inequality

$$\operatorname{per} P \geqslant \frac{n!}{n^n}$$

holds with equality if and only if $P = J_n$. Here Ω_n denotes the set of all n-square doubly stochastic matrices and J_n is the element of Ω_n whose entries are all $1/n$. Many authors have studied this conjecture extensively and to my knowledge it has been shown true only for $n = 2$, $n = 3$ [**4**], $n = 4$ [**1**] and $n = 5$ [**2**].

For a general n, we know that if the permanent function achieves its absolute minimum value on Ω_n in $\operatorname{Int} \Omega_n$ (the interior of Ω_n), then the conjecture is true [**4**, Theorem 2 and Theorem 3]. M. Marcus and M. Newman [**4**, Theorem 4] have also shown that if the absolute minimizing matrix does not belong to $\operatorname{Int} \Omega_n$, then all its zeros cannot occur in a fixed row (or column). In addition, P. J. Eberlein and G. S. Mudholkar [**1**, Theorem 7] have shown that this absolute minimizing matrix, after suitable permutations of its rows and columns, cannot be of the form

$$\begin{bmatrix} O & Y \\ Z_1 & Z_2 \end{bmatrix}$$

where O is an $r \times k$ matrix $(1 \leqq r, k \leqq n - 1)$ whose entries are 0, and Y (respectively Z_1, Z_2) is an $r \times (n - k)$ (respectively $(n - r) \times k$, $(n - r) \times (n - k)$) matrix whose entries are positive.

The purpose of this note is to present, in the following two theorems, other results concerning the zero pattern of the absolute minimizing matrix.

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THEOREM 1. *If the absolute minimizing matrix for* per P $(P \in \Omega_n)$ *is not in* Int Ω_n, *then all its zeros cannot occur in two fixed rows (or columns).*

THEOREM 2. *If an element P in Ω_n is such that there exist $n \times n$ permutation matrices Q_1 and Q_2 with the property that the matrix Q_1PQ_2 or its transpose takes the fo m*

$$\begin{bmatrix} X & Y \\ Z_1 & Z_2 \end{bmatrix}$$

where $X = (x_{ij})$ is an $r \times 2$ matrix $(r \geqq 1)$ with $x_{11} > 0$ and the others $x_{ij} = 0$, Z_1 (respectively Z_2) is an $(n - r) \times 2$ (respectively $(n - r) \times (n - 2)$) matrix whose entries are positive and Y is any $r \times (n - 2)$ matrix, then P cannot be an absolute minimizing matrix.

The relation between Theorem 1 and Theorem 4 in Marcus and Newman's paper [**4**] is self-explanatory. In Theorem 2, the fact that Y is any $r \times (n - 2)$ matrix allows us to decide that many zero patterns are inadmissible in the construction of a minimizing (for the permanent function) element of Ω_n.

2. Proofs. Before proving these theorems we introduce some notation· An $n \times n$ matrix $P = (p_{ij})$ is expressed in terms of its column vectors as $(p^1, p^2, \ldots, p^n)$ and in terms of its row vectors as $(p_1, p_2, \ldots, p_n)$. If $1 \leqq j < k \leqq n$, $P^{\langle j,k \rangle}$ denotes the matrix

$$(p^1, \ldots, p^{j-1}, p^k - p^j, p^{j+1}, \ldots, p^k, \ldots, p^n)$$

(if $j = 1$ we adopt the obvious convention), whereas, if $1 \leqq i, j \leqq n$, $C_{ij}(P)$ denotes the permanent of the $(n - 1) \times (n - 1)$ matrix obtained by deleting row i and column j from P.

Since the permanent is a multilinear function of the column of an n-square matrix P, we have:

$$(1) \quad C_{ij}(P) - C_{ik}(P) = C_{ik}(P^{\langle j,k \rangle})$$

for $1 \leqq i \leqq n$ and $1 \leqq j < k \leqq n$. This simple relation and Theorem 1 in Marcus and Newman [**4**] will be our main tools.

We now prove Theorem 1. Assume on the contrary that $P = (p_{ij}) \in \Omega_n$ is an absolute minimizing matrix with all its zeros contained in two given rows. We can suppose that these are the first two rows (N.B. per QP = per P for any $n \times n$ permutation matrix Q). Let k (respectively r) denote the number of zero entries of P in its first (respectively, second) row. By hypothesis we have $1 \leqq k < n$ and $1 \leqq r < n$. We may also suppose without loss of generality that $k \geqq r$, that $p_{11} = p_{12} = \ldots = p_{1k} = 0$, $p_{1j} > 0$ for $k < j \leqq n$, and that s and t are two nonnegative integers satisfying $s + t = r$ and $p_{21} = p_{22} = \ldots = p_{2s} = 0$, $p_{2k+1} = p_{2k+2} = \ldots = p_{2k+t} = 0$, $p_{2j} > 0$ for $s < j \leqq k$ and $k + t < j \leqq n$. Furthermore, using the corollary to Theorem 3 in [**2**] we may suppose that the remaining $n - 3$ rows of P are all equal and so we let

$$p_3 = p_4 = \ldots = p_n = (\alpha_1, \alpha_2, \ldots, \alpha_n).$$

The case where $t = 0$ and $s = k$ is already settled (this is Eberlein and Mudholkar's result quoted earlier).

If $t = 0$ and $s \leqq k - 1$ then the stochastic constraints imply that $\alpha_s > \alpha_{s+1}$. On the other hand, using Theorem 1 of [**4**] and relation (1) above we can write

$$(2) \quad 0 \leqq C_{2s}(P) - C_{2s+1}(P) = C_{2s+1}(P^{\langle s,s+1\rangle})$$

where $P^{\langle s,s+1\rangle}$ is a matrix whose sth column is $(0, p_{2s+1}, \alpha_{s+1} - \alpha_s, \ldots, \alpha_{s+1} - \alpha_s)^T$ (as usual T denotes the transpose of a vector) and whose other entries are nonnegative. Thus it follows from (2) that $\alpha_{s+1} - \alpha_s \geqq 0$. This is a contradiction.

Finally if $t \geqq 1$, we have

$$(3) \quad 0 \leqq C_{1k+1}(P^{\langle k,k+1\rangle})$$

and

$$(4) \quad C_{2k+1}(P^{\langle k,k+1\rangle}) \leqq 0.$$

Since, in this case, $(p_{1k+1}, -p_{2k}, \alpha_{k+1} - \alpha_k, \ldots, \alpha_{k+1} - \alpha_k)^T$ is the kth column of $P^{\langle k,k+1\rangle}$ and the submatrix obtained from P by deleting its first two rows and its kth and $(k + 1)$th columns has only positive entries we infer from (3) that $\alpha_{k+1} - \alpha_k > 0$ and from (4) that $\alpha_{k+1} - \alpha_k < 0$. Again we obtain a contradiction and the proof of Theorem 1 is complete.

An obvious adaptation of this argument gives Theorem 2 and a very short proof of Theorem 4 in [**4**].

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