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ON HIGHER DIMENSIONAL ARITHMETIC PROGRESSIONS IN MEYER SETS

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Abstract

In this paper we study the existence of higher dimensional arithmetic progressions in Meyer sets. We show that the case when the ratios are linearly dependent over $\mathbb{Z}$ is trivial and focus on arithmetic progressions for which the ratios are linearly independent. Given a Meyer set Λ and a fully Euclidean model set $\curlywedge(W)$ with the property that finitely many translates of $\curlywedge(W)$ cover Λ, we prove that we can find higher dimensional arithmetic progressions of arbitrary length with k linearly independent ratios in Λ if and only if k is at most the rank of the $\mathbb{Z}$-module generated by $\curlywedge(W)$. We use this result to characterize the Meyer sets that are subsets of fully Euclidean model sets.

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1. Introduction

The Nobel Prize discovery of quasicrystals [19] sparked many questions regarding the nature of solids with long-range aperiodic order. This discovery led to establishment of a new area of mathematics, the area of aperiodic order. The goal of this new field is to study objects that show long-range order, but are not necessarily periodic.

The best mathematical models for point sets that show long-range order and are typically aperiodic were discovered in the earlier pioneering work of Meyer [13] and have been popularized in the field by Moody [14, 15] and Lagarias [9, 10]. Called model sets, they are constructed via a cut-and-project scheme, a mechanism that starts with a lattice $\mathcal{L}$ in a higher dimensional space that sits at an 'irrational slope' with respect to the real space $\mathbb{R}^d$, cuts a strip around the real space $\mathbb{R}^d$ of bounded width W (called the 'window') and projects it on the real space $\mathbb{R}^d$ (see Definition 2.13 for the exact definition). Under various weak conditions, the high order present in the lattice $\mathcal{L}$

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shows in the resulting model set, typically via a clear diffraction diagram; for example, see [5, 11, 17, 18] just to name a few.

Meyer sets are relatively dense subsets of model sets. As subsets of model sets they inherit part of the high order present in the former, which is evident in their characterization via discrete geometry, harmonic analysis and algebraic properties [9, 13, 14]. While they typically have positive entropy and hence are usually not pure point diffractive (see [4]; compare [2] for a discussion), they still show a highly ordered diffraction diagram [21, 23–25] with a relatively dense supported pure point spectrum [20–22, 25].

In [8], we showed a different type of order in Meyer sets in the form of the existence of arbitrarily long arithmetic progressions. More precisely, we proved that given a Meyer set $\Lambda \subseteq \mathbb{R}^d$, for all $N \in \mathbb{N}$ there exists some $R > 0$ such that the set $\Lambda \cap B_R(x)$ contains an arithmetic progression of length N for all $x \in \mathbb{R}^d$. Moreover, we showed that van der Waerden-type theorems hold in Meyer sets. More recently, related results have been investigated in [1, 16].

Consider now a Meyer set $\Lambda \subseteq \mathbb{R}^d$. While Λ spreads relatively densely in all directions of $\mathbb{R}^d$, any arithmetic progression is one-dimensional and hence only gives partial information about the structure of Meyer sets. This suggests that one should look for higher dimensional arithmetic progressions, which is the goal of this paper. By an m-dimensional arithmetic progression we understand a set of the form

$$A = \{s + c_1 r_1 + \cdots + c_m r_m : 0 \le c_j \le N_j \text{ for all } 1 \le j \le m\}$$

for some fixed $s, r_1, \ldots, r_m \in \mathbb{R}^d$ and $N_1, \ldots, N_m \in \mathbb{N}$. The elements $r_1, \ldots, r_m$ are called the ratios and $\vec{N} = (N_1, \ldots, N_m)$ is the vector length of the progression. The arithmetic progression is proper if all the elements $s + c_1 r_1 + \cdots + c_m r_m$ are distinct.

By a standard application of the Chinese remainder theorem, we show in Proposition 3.3 that for all $n \in \mathbb{N}$ and $\vec{N} \in \mathbb{N}^m$, every Meyer set contains a proper n-dimensional arithmetic progression of length $\vec{N}$. While the arithmetic progression is proper, every pair of ratios is linearly dependent over $\mathbb{Z}$ and hence the arithmetic progression is a subset of a one-dimensional affine $\mathbb{Q}$-space.

To make the question more interesting and meaningful, we add the extra condition that the ratios are linearly independent over $\mathbb{Z}$ (or equivalently over $\mathbb{Q}$). Given a fully Euclidean model set $⋏(W)$ in a cut-and-project scheme (or simply a CPS) $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$, we show in Theorem 4.3 that $⋏(W)$ has n-dimensional arithmetic progressions of arbitrary length with linearly independent ratios if and only if $n \le d + m$.

Next, for any Meyer set Λ, it is well known that there exist some fully Euclidean model set $⋏(W)$ in some CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ and a finite set $F \subseteq \mathbb{R}^d$ such that

$$\Lambda \subseteq ⋏(W) + F. \tag{1-1}$$

We show in Theorem 5.2 that Λ has n-dimensional arithmetic progressions of arbitrary length with linearly independent ratios if and only if $n \le d + m$. This implies that, while in general there exist multiple fully Euclidean model sets $\curlywedge(W)$ and finite sets F such that (1-1) holds, m must be the same for all these model sets.

We complete the paper by answering the following question.

QUESTION 1.1. Which Meyer sets $\Lambda \subseteq \mathbb{R}^d$ are subsets of fully Euclidean model sets?

To our knowledge, no characterization for these sets (that we call fully Euclidean Meyer sets) is known so far. We show in Theorem 6.1 that a Meyer set $\Lambda \subseteq \mathbb{R}^d$ is a subset of a fully Euclidean model set if and only if it has n-dimensional arithmetic progressions of arbitrary length with linearly independent ratios, where n is the rank of the $\mathbb{Z}$-module generated by Λ. The characterization is of number theory/combinatorics origin, emphasizing once again the nice order present in Meyer sets and model sets.

The paper is structured in the following way. In Section 2.1, we give basic definitions and prove a higher dimensional version of van der Waerden's theorem [26]. We prove, in Section 3, that Meyer sets $\Lambda \subset \mathbb{R}^d$ contain arithmetic progression of arbitrary length and dimension, albeit with linearly dependent ratios. In Section 4, we establish both the existence of arithmetic progressions with linearly independent ratios and a higher-dimensional van der Waerden-type result for fully Euclidean model sets. In Section 5, we extend these results to arbitrary Meyer sets in $\mathbb{R}^d$. We complete the paper by characterizing the fully Euclidean Meyer sets.

2. Preliminaries

In this section, we review the basic definitions and results needed in the paper.

2.1. Finitely generated free $\mathbb{Z}$-modules. We start by recalling a few basic results about finitely generated free $\mathbb{Z}$-modules. First recall [7, Theorem VIII.4.12] that if M is a free $\mathbb{Z}$-module, then all the bases of M have the same cardinality. The common cardinality of these bases is called the *rank* of M and is denoted by $\mathrm{rank}(M)$. Also, a finitely generated $\mathbb{Z}$-module is free if and only if it is torsion free [6, Theorem 12.5].

Next let us recall the following two results that we use a few times in the paper.

LEMMA 2.1 [7, Theorem VIII.6.1]. *Let M be a free $\mathbb{Z}$-module of rank k. If N is a submodule of M, then N is free and*

$$\mathrm{rank}(N) \le \mathrm{rank}(M).$$

In particular, Lemma 2.1 implies the following result.

COROLLARY 2.2. *Let M be a free $\mathbb{Z}$-module of rank k. If $v_1, \ldots, v_m \in M$ are linearly independent over $\mathbb{Z}$, then $m \le k$.*

PROOF. $v_1, \ldots, v_m$ are a basis for the submodule of M generated by $\{v_1, \ldots, v_m\}$. The claim follows from Lemma 2.1. □

Next let us recall the following result about $\mathbb{Z}$-submodules of the same rank as the full module.

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LEMMA 2.3. *Let M be a finitely generated free $\mathbb{Z}$-module and let N be a submodule of M. If*

$$\operatorname{rank}(M) = \operatorname{rank}(N),$$

then there exists some positive integer n such that

$$\{nv : v \in M\} =: n \cdot M \subseteq N.$$

PROOF. This follows from [7, Theorem VIII.6.1]. □

We complete the section by proving the following simple result. This result is surely known, but we could not find a good reference for it.

LEMMA 2.4. *Let M be a finitely generated free $\mathbb{Z}$-module with $k = \operatorname{rank}(M)$ and let S be a generating set for M. Then there exist k linearly independent elements $v_1, \ldots, v_k \in S$.*

PROOF. By a standard application of Zorn's lemma, there exists some $S' = \{v_1, \ldots, v_m\} \subseteq S$ that is a maximal linearly independent subset. Let N be the submodule of M generated by S'. Then $\operatorname{rank}(N) = m$. To complete the proof, we show that $m = k$. By [6, Theorem 12.4], there exist a basis $y_1, \ldots, y_k$ of M and $n_1, \ldots, n_m \in \mathbb{Z}$ such that $n_1y_1, \ldots, n_my_m$ is a basis for N.

Now assume by contradiction that $m < k$. Since S spans M, there exist elements $x_1, \ldots, x_l \in S$ and $k_1, \ldots, k_l \in \mathbb{Z}$ such that

$$y_k = k_1x_1 + \cdots + k_lx_l.$$

Next note that for each $1 \leq j \leq l$, we either have $x_j \in S'$ or $S' \cup x_j$ is linearly dependent. In both cases, there exists some nonzero $f_j \in \mathbb{Z}$ such that $f_jx_j \in N$. Let $f = f_1 \cdots f_l$. Then f is a nonzero integer and

$$fy_k \in N.$$

Since $n_1y_1, \ldots, n_my_m$ is a basis for N, fy_k can be written as a linear combination of $n_1y_1, \ldots, n_my_m$. As $f \neq 0$, this gives that $y_1, \ldots, y_m, y_k$ are linearly dependent over $\mathbb{Z}$, which is a contradiction. Thus, $m = k$. □

Note that one can alternatively prove the above lemma by embedding M into a $\mathbb{Q}$-vector space and looking at the subspace generated by S'.

2.2. Higher dimensional arithmetic progressions. In this section, we look at higher dimensional arithmetic progressions. Let us start with the following definition.

DEFINITION 2.5. A *higher dimensional arithmetic progression* is a set of the form

$$A := \{s + c_1r_1 + c_2r_2 + \cdots + c_nr_n : 0 \leq c_1 \leq N_1, 0 \leq c_2 \leq N_2, \ldots, 0 \leq c_n \leq N_n\}$$

for some fixed vectors $s, r_1, \ldots, r_n \in \mathbb{R}^d$ and some arbitrarily fixed natural numbers $N_1, \ldots, N_n \in \mathbb{N}$. If the elements in A are distinct, then the progression is called *proper*. We call n the *dimension* of the projection and $\vec{N} = (N_1, \ldots, N_n)$ the *vector length* of the

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progression. The *rank* of the projection is the rank of the $\mathbb{Z}$-module generated by the ratios $\{r_1, \ldots, r_n\}$. We say that the arithmetic progression is a *li-arithmetic progression* if the ratios $r_1, \ldots, r_n$ are linearly independent over $\mathbb{Z}$.

In the remainder of the paper, we simply refer to a higher dimensional arithmetic progression simply as an 'arithmetic progression'.

REMARK 2.6. The rank of a generalized arithmetic progression is simply the largest cardinality of any $\mathbb{Z}$-linearly independent subset of $\{r_1, \ldots, r_n\}$. It is obvious that the rank of any arithmetic progression is at most its dimension, with equality if and only if the arithmetic progression is a li-arithmetic progression.

Next let us note here that since our goal is to study the existence of arithmetic progressions of arbitrary length, $\vec{N} = (N_1, \ldots, N_n)$, it is sufficient to restrict to the case $N_1 = N_2 = \ldots = N_n =: N$. In this case, we say that the length of the progression is $N \in \mathbb{N}$.

Next let us review some standard notation. As usual, for $N \in \mathbb{N}$ with $N \geq 1$, we denote by $[N]$ the set

$$[N] := \{0, 1, 2, \ldots, N\} = \mathbb{N} \cap [0, N].$$

Also, $[N]^d$ denotes the Cartesian product of d copies of $[N]$, that is,

$$[N]^d = \{(k_1, \ldots, k_d) : k_j \in [N] \text{ for all } 1 \leq j \leq d\}.$$

We also need the following definition.

DEFINITION 2.7. A *d-dimensional grid of depth n* is a set of the form

$$[k_1, \ldots, k_d; l_1, \ldots, l_d; n] := \{(l_1 + m_1k_1, l_2 + m_2k_2, \ldots, l_d + m_dk_d) : m_1, \ldots, m_d \in [n]\}$$

for some fixed positive integers $k_1, \ldots, k_d$ and fixed $l_1, \ldots, l_d$.

Note that a d-dimensional grid of depth n is simply an arithmetic progression inside $\mathbb{Z}^d$ of dimension d with the ratios

$$r_j = k_je_j$$

for some $k_j \in \mathbb{N}$, where $e_j = (0, 0, \ldots, 1, 0, \ldots, 0)$ is the canonical basis.

We now prove the following higher dimensional version of van der Waerden's theorem. For the analogous statement of the one-dimensional version of this theorem, we refer the reader to [8, 26]. Note that there are already higher dimensional generalizations of van der Waerdens's theorem, such as the Gallai–Witt theorem; see [12] for a brief discussion and references therein.

THEOREM 2.8 (van der Waerden in $\mathbb{Z}^d$). *Given any natural numbers k, r, d, there exists a number $W(r, k, d)$ such that, no matter how we colour $\mathbb{Z}^d$ with r colours, for each $N \geq W(r, k, d)$, we can find a monochromatic d-dimensional grid $[k_1, \ldots, k_d; l_1, \ldots, l_d; k] \subseteq [N]^d$ of depth k.*

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PROOF. We prove the claim via induction on d.

$P(1)$: This is the standard van der Waerden theorem [26].

$P(d) \Rightarrow P(d+1)$: Let r and k be given. Let A be the set of all d-dimensional grids of depth k that are subsets of $[W(r,k,d)]^d$. Now set

$$W(r,k,d+1) = W(|A| \cdot r, k, 1).$$

We show that this choice works. Note first that the van der Waerden theorem is equivalent to the fact that given a set X with $|A| \cdot r$ elements, for any function $v : \mathbb{N} \to X$ and any $N \geq W(|A| \cdot r, k, 1)$, there exists an element $x \in X$ such that

$$v^{-1}(x) \cap [N]$$

contains an arithmetic progression of length k.

Now consider any colouring of $\mathbb{Z}^{d+1}$ with r colours $c_1, \ldots, c_r$. Let $N \geq W(r,k,d+1)$. Next, for each $1 \leq j \leq N$, consider the colouring of $\mathbb{Z}^d \times \{j\} \subseteq \mathbb{Z}^{d+1}$. By $P(d)$, the set $[W(r,k,d)]^d \times \{j\}$ contains a monochromatic grid M_j of depth k. Let $c(j)$ be the colour of this grid. We can now define a function

$$v : [N] \to A \times \{c_1, \ldots, c_r\}$$
$$v(j) = (M_j, c(j)).$$

Then there exists some $(M, c_l) \in A \times \{c_1, \ldots, c_r\}$ such that

$$v^{-1}(M, c_l) \cap [N]$$

contains an arithmetic progression of length k. Let l_{d+1}, k_{d+1} be so that $l_{d+1} + mk_{d+1} \in v^{-1}(M, c_l) \cap [N]$ for all $m \in [k]$. Next, since M is a monochromatic grid of depth k, there exists some $k_1, \ldots, k_d; l_1, \ldots, l_d$ such that

$$M = [k_1, \ldots, k_d; l_1, \ldots, l_d; k] \subseteq [W(r,k,d)]^d.$$

Then, by construction, the grid

$$[k_1, \ldots, k_d, k_{d+1}; l_1, \ldots, l_d, l_{d+1}; k] \subseteq [W(r,k,d)]^d \times [N] \subseteq [N]^{d+1}$$

is monochromatic of colour c_l. This proves the claim. □

REMARK 2.9. If we denote by $W(r,k,d)$ the smallest value that satisfies Theorem 2.8, then it is obvious that $W(r,k,1) = W(r,k)$. Moreover, the proof above yields the very poor upper bound

$$W(r,k,d+1) \leq W(l,k,1),$$

where $l = |A| \cdot r$. Note that

$$|A| = \Bigg(\sum_{j=0}^{W(n,k,d)} \Big\lfloor \frac{W(n,k,d) - j}{d} \Big\rfloor \Bigg)^d,$$

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which can be seen by observing that, for each $1 \leq i \leq d$ and for every particular choice $1 \leq j \leq W(n, k, d)$,

$$1 \leq l_j \leq \frac{W(n, k, d) - j}{d}.$$

2.3. Meyer sets and model sets. In this subsection, we review the notion of model sets and Meyer sets in $\mathbb{R}^d$. For a more detailed review of these definitions and properties, we refer the reader to the monograph [3] and to [14, 15].

We start by reviewing some of the basic definitions for point sets.

DEFINITION 2.10. Let $\Lambda \subseteq \mathbb{R}^d$ be a point set. We say that Λ is:

- *relatively dense* if there exists some $R > 0$ such that for all $x \in \mathbb{R}^d$, the set $\Lambda \cap B_R(x)$ contains at least one point;
- *uniformly discrete* if there exists some $r > 0$ such that for all $x \in \mathbb{R}^d$, the set $\Lambda \cap B_r(x)$ contains at most a point;
- *Delone* if Λ is relatively dense and uniformly discrete;
- *locally finite* if for all $R > 0$ and $x \in \mathbb{R}^d$, the set $\Lambda \cap B_R(x)$ is finite.

Relatively denseness and uniform discreteness are usually defined in arbitrary locally compact Abelian groups (LCAGs) G, using compact sets and open sets, respectively. It is easy to see that in the case of $G = \mathbb{R}^d$, the usual definitions are equivalent to Definition 2.10.

Next, in the spirit of [14], we introduce the following definition.

DEFINITION 2.11. We say that two Delone sets Λ_1, Λ_2 are *equivalent by finite translations* if there exist finite sets F_1, F_2 such that

$$\Lambda_1 \subseteq \Lambda_2 + F_2,$$
$$\Lambda_2 \subseteq \Lambda_1 + F_1.$$

REMARK 2.12.

(a) It is easy to see that being equivalent by finite translations is an equivalence relation on the set of Delone subsets of $\mathbb{R}^d$.
(b) By replacing F_1, F_2 by $F = F_1 \cup F_2$, one can assume without loss of generality that $F_1 = F_2$.

Next we review the notion of cut-and-project schemes and model sets.

DEFINITION 2.13. By a *cut-and-project scheme*, or simply CPS, we understand a triple $(\mathbb{R}^d, H, \mathcal{L})$ consisting of $\mathbb{R}^d$, a LCAG H, together with a lattice (that is, a discrete co-compact subgroup) $\mathcal{L} \subset \mathbb{R}^d \times H$ with the following two properties:

- the restriction $\pi^{\mathbb{R}^d}|_{\mathcal{L}}$ of the canonical projection $\pi^{\mathbb{R}^d} : \mathbb{R}^d \times H \to \mathbb{R}^d$ to $\mathcal{L}$ is a one-to-one function;
- the image $\pi^H(\mathcal{L})$ of $\mathcal{L}$ under the canonical projection $\pi^H : \mathbb{R}^d \times H \to H$ is dense in H.

In the special case where $H = \mathbb{R}^m$, then we refer to $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ as a *fully Euclidean* CPS.

Next we define, in the usual way, $L := \pi^{\mathbb{R}^d}(\mathcal{L})$ and $\star : L \to H$, known as the $\star$-mapping, by

$$\star = \pi^H \circ (\pi^{\mathbb{R}^d}|_{\mathcal{L}})^{-1}.$$

This allows us rewrite

$$\mathcal{L} = \{(x, x^\star) : x \in L\}.$$

Note that the range of the $\star$-mapping is

$$\star(L) =: L^\star = \pi^H(\mathcal{L}).$$

We can summarize the CPS in the diagram below.

$$\begin{tikzcd}
\mathbb{R}^d \arrow[d, phantom, "\cup"] & \mathbb{R}^d \times H \arrow[l, "\pi^{\mathbb{R}^d}"'] \arrow[r, "\pi^H"] \arrow[d, phantom, "\cup"] & H \arrow[d, phantom, "\parallel"] \\
\pi^{\mathbb{R}^d}(\mathcal{L}) \arrow[d, phantom, "\parallel"] & \mathcal{L} \arrow[l, "1-1"] \arrow[r, "\text{dense}"] & H \arrow[d, phantom, "\cup"] \\
L \arrow[rr, "\star"'] & & L^\star
\end{tikzcd}$$

We can now review the definition of model sets.

DEFINITION 2.14. Given a CPS $(\mathbb{R}^d, H, \mathcal{L})$ and some subset $W \subseteq H$, we denote by $\curlywedge(W)$ its preimage under the $\star$-mapping, that is,

$$\curlywedge(W) := \{x \in L : x^\star \in W\} = \{x \in \mathbb{R}^d : \text{ there exists } y \in W \text{ such that } (x, y) \in \mathcal{L}\}.$$

When W is precompact and has nonempty interior, the set $\curlywedge(W)$ is called a *model set*. If furthermore $H = \mathbb{R}^m$ for some m, then $\curlywedge(W)$ is called a *fully Euclidean model set*.

We want to emphasize here that the condition that W has nonempty interior is essential later in the paper in the proof of Theorem 4.3.

Next let us recall the following result.

PROPOSITION 2.15 [14, Proposition 2.6] and [3, Proposition 7.5]. *Let $(\mathbb{R}^d, H, \mathcal{L})$ be a CPS and $W \subseteq H$.*

(a) *If $W \subseteq H$ is precompact, then $\curlywedge(W)$ is uniformly discrete.*
(b) *If $W \subseteq H$ has nonempty interior, then $\curlywedge(W)$ is relatively dense.*

In particular, every model set is a Delone set.

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Next we review the concept of Meyer sets. We start by recalling the following theorem.

THEOREM 2.16 [9, 13, 14]. *Let $\Lambda \subseteq \mathbb{R}^d$ be relatively dense. Then the following are equivalent.*

(i) *Λ is a subset of a model set.*
(ii) *There exist a fully Euclidean model set $\curlywedge(W)$ and a finite set F such that*

$$\Lambda \subseteq \curlywedge(W) + F.$$

(iii) *$\Lambda - \Lambda$ is uniformly discrete.*
(iv) *Λ is locally finite and there exists a finite set F_1 such that*

$$\Lambda - \Lambda \subseteq \Lambda + F_1.$$

PROOF. The equivalence (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) can be found in [14], while (ii)$\Leftrightarrow$(iv) is [13, Theorem IV]. □

The above theorem gives the concept of a Meyer set. More precisely, we have the following definition.

DEFINITION 2.17. A relatively dense set $\Lambda \subset \mathbb{R}^n$ is called a *Meyer set* if it satisfies one (and hence all) of the equivalent conditions of Theorem 2.16.

A relatively dense set Λ is called a *fully Euclidean Meyer set* if there exists some fully Euclidean model set $\curlywedge(W)$ such that $\Lambda \subseteq \curlywedge(W)$.

One of the goals of this paper is to characterize fully Euclidean Meyer sets.

Next let us recall the following result from [14].

LEMMA 2.18. *Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set. Then the group $\langle\Lambda\rangle$ generated by Λ is finitely generated. In particular, $\langle\Lambda\rangle$ is a free $\mathbb{Z}$-module of finite rank.*

PROOF. By [14, Theorem 9.1], $\langle\Lambda\rangle$ is finitely generated. It is therefore a finitely generated $\mathbb{Z}$-module. Moreover, since $\mathbb{R}$ is torsion free as a $\mathbb{Z}$-module, so is $\langle\Lambda\rangle$. Therefore, $\langle\Lambda\rangle$ is a free $\mathbb{Z}$-module by [6, Theorem 12.5]. □

This allows us introduce the following definition.

DEFINITION 2.19. Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set. We define the *rank* of Λ to be

$$\mathrm{rank}(\Lambda) := \mathrm{rank}_{\mathbb{Z}}(\langle\Lambda\rangle).$$

We complete the section by showing that each Meyer set is equivalent by finitely many translates with a fully Euclidean model set.

LEMMA 2.20. *Let Λ be a Meyer set, F finite and $\curlywedge(W)$ a fully Euclidean model set such that*

$$\Lambda \subseteq \curlywedge(W) + F.$$

Then Λ and $\curlywedge(W)$ are equivalent by finite translations.

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PROOF. Since $\curlywedge(W)$ is a Meyer set and F is finite, $\curlywedge(W) + F$ is also a Meyer set [13, 14]. The claim follows now from [23, Lemma 5.5.1]. □

3. Higher dimensional arithmetic progressions in Meyer sets

In this section we show that Meyer sets contain arithmetic progressions of arbitrary dimensions and length. The proofs show that the existence of arithmetic progressions of arbitrary length in Meyer sets is interesting only in the case of li-arithmetic progressions. We study these in the subsequent sections.

We start by recalling the following well-known theorem.

LEMMA 3.1 (Chinese remainder theorem [6, Corollary 7.18]). *If $k_1, \ldots, k_n$ are pairwise coprime and $a_1, \ldots, a_n$ are integers, then there exists $x \in \mathbb{Z}^+$ such that*

$$\begin{cases} x & \equiv a_1 \bmod k_1, \\ x & \equiv a_2 \bmod k_2, \\ & \vdots \\ x & \equiv a_n \bmod k_n. \end{cases}$$

Moreover, any two solutions are congruent modulo $k_1, k_2, \ldots, k_n$.

As an immediate consequence, we get the following result.

COROLLARY 3.2. *For each $n, N \in \mathbb{N}$, there exist $m_1, \ldots, m_n \in \mathbb{N}$ such that $\sum_{i=1}^{n} c_i m_i$ are distinct for all $0 \leq c_i \leq N$.*

PROOF. Let $p_1, \ldots, p_n$ be distinct primes such that for all $1 \leq i \leq n$, we have $p_i > N$. By the Chinese remainder theorem, there exist $m_1, \ldots, m_n$ such that for each $1 \leq i \leq n$,

$$\begin{cases} m_i & \equiv 1 \bmod p_i \\ m_i & \equiv 0 \bmod p_j \quad \text{for all } j \neq i. \end{cases}$$

Now, if $\sum_{i=1}^{n} c_i m_i = \sum_{i=1}^{n} c_i' m_i$, then, for all $1 \leq k \leq n$,

$$\sum_{i=1}^{n} c_i m_i \equiv \sum_{i=1}^{n} c_i' m_i \bmod p_k$$

and hence $c_k \equiv c_k' \bmod p_k$. Since

$$0 \leq c_k, c_k' \leq N < p_k,$$

we get $c_k = c_k'$ for all $1 \leq k \leq n$. □

We can now prove the following result.

PROPOSITION 3.3. *Let $n, N \in \mathbb{N}$ and let $\Lambda \in \mathbb{R}^d$ be a Meyer set. Then there exists some $R > 0$ such that $\Lambda \cap B_R(x)$ contains a nontrivial and proper n-dimensional arithmetic progression of length N for all $x \in \mathbb{R}^d$.*

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PROOF. Let $n, N \in \mathbb{N}$ be given. We show that, for some $R > 0$, there exist

$$s, r_1, r_2, \ldots, r_n \in \Lambda$$

such that

$$s + \sum_{i=1}^{n} c_i r_i \in \Lambda \cap B_R(x) \quad \text{for all } 0 \le c_i \le N$$

and that the elements $s + \sum_{i=1}^{n} c_i r_i$ are distinct. Let $m_1, \ldots, m_n$ be as in Corollary 3.2 and set $N' = Nm_1 + \cdots + Nm_n$. By [8, Lemma 4.3], there exist an $R > 0$ and some nonzero $s, r \in \Lambda$ such that

$$s, s + r, \ldots, s + N'r \in \Lambda \cap B_R(x).$$

Now define $r_j := m_j r$. It follows that, for $0 \le \sum_{i=1}^{n} c_i m_i \le N'$,

$$s + \sum_{i=1}^{n} c_i r_i = s + \sum_{i=1}^{n} c_i m_i r \in \Lambda \cap B_R(x).$$

Moreover, by Corollary 3.2, the progression is proper. □

Note here in passing that, by construction, the arithmetic progression in Proposition 3.3 has rank one.

It becomes natural to ask if one can construct arithmetic progressions of higher rank. It is easy to see that one can focus on li-arithmetic progressions. Indeed, exactly as in Proposition 3.3, one can prove the following result.

LEMMA 3.4. *Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set and $k \in \mathbb{N}$. Then Λ contains li-arithmetic progressions of rank k and arbitrary length if and only if for each $d \ge k$, Λ contains arithmetic progressions of rank k, dimension d and arbitrary length.*

Since the proof is similar to that of Proposition 3.3, we skip it.

4. Higher dimensional arithmetic progressions with linearly independent ratios

In this section, we discuss the existence of li-arithmetic progressions in fully Euclidean model sets. We show that the maximal rank of any such progression is the rank of the lattice in the CPS and that for this rank, we can find li-arithmetic progressions of arbitrary length.

We start by proving the following result (compare [14, Proposition 2.6]).

LEMMA 4.1. *Let $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ be a fully Euclidean CPS and let $W \subseteq \mathbb{R}^m$ be any set with nonempty interior. Then $\curlywedge(W)$ generates $L = \pi_{\mathbb{R}^d}(\mathcal{L})$. In particular,*

$$\operatorname{rank}(\curlywedge(W)) = d + m.$$

In particular, there exist vectors $r_1, \ldots, r_{d+m} \in \curlywedge(W)$ that are linearly independent over $\mathbb{Z}$.

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PROOF. It suffices to show that $\curlywedge(W) - \curlywedge(W) = \curlywedge(W - W)$ generates L. Note that 0 is an interior point in $W - W$ and hence we can find some $r > 0$ such that $B_r(0) \subseteq W - W$.

Now let $x \in L$ be arbitrary. Pick some n such that $d(x, 0) < nr$. Then $(x/n) \in B_r(0)$. Note here that $(x/n) \in B_r(0) \cap B_{x/n}(x/n)$. Since this set is open, by the density of $\pi_{\mathbb{R}^m}(\mathcal{L})$ in $\mathbb{R}^m$, there exist some $(z, z^\star) \in \mathcal{L}$ such that $z^\star \in B_r(0) \cap B_{x/n}(x/n)$. Then $z \in \curlywedge(W - W)$ and $d(x^\star, nz^\star) < r$ and thus $x^\star - nz^\star \in B_r(0) \subseteq W - W$. Therefore,

$$z \in \curlywedge(W) - \curlywedge(W),$$

$$x - nz \in \curlywedge(W) - \curlywedge(W).$$

This gives $x \in \langle \curlywedge(W) \rangle$. The last claim now follows from Lemma 2.4. □

Next we prove the following generalization of [8, Proposition 4.2].

PROPOSITION 4.2. *Let $(\mathbb{R}^d, H, \mathcal{L})$ be any CPS and let $W \subseteq H$ be any set with nonempty interior. Then, for all $M \in \mathbb{N}$, we can find open sets $U_M, V_M \subset H$ such that $0 \in V_M$ and*

$$\curlywedge(U_M) + \underbrace{\curlywedge(V_M) + \curlywedge(V_M) + \cdots + \curlywedge(V_M)}_{M \text{ times}} \subseteq \curlywedge(W).$$

PROOF. As W has nonempty interior, there exist nonempty open sets $U_M \subseteq H$ and $0 \in V_M \subseteq H$ such that

$$U_M + \underbrace{V_M + V_M + \cdots + V_M}_{M \text{ times}} \subseteq W.$$

Then

$$\curlywedge(U_M) + \underbrace{\curlywedge(V_M) + \curlywedge(V_M) + \cdots + \curlywedge(V_M)}_{M \text{ times}} \subseteq \curlywedge(U_M + \underbrace{V_M + V_M + \cdots + V_M}_{M \text{ times}}) \subseteq \curlywedge(W).$$

This completes the proof. □

By combining Proposition 4.2 with Corollary 2.2, we get the following result.

THEOREM 4.3. *Let $\curlywedge(W)$ be a model set in a fully Euclidean CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$. Then:*

(a) *any arithmetic progression of length N in $\curlywedge(W)$ has rank at most $d + m$;*
(b) *for each N, there exists some $R > 0$ such that the set $\curlywedge(W) \cap B_R(y)$ contains a li-arithmetic progression of length N and rank $d + m$ for all $y \in \mathbb{R}^d$.*

PROOF.

(a) For any arithmetic progression of rank k in $\curlywedge(W)$, the set $\{r_1, \ldots, r_k\}$, with r_i the ratios of the progression, is $\mathbb{Z}$-linearly independent in L. Therefore, by Corollary 2.2, we have $k \leq m + d$. This proves (a).
(b) Let N be given. We show that there exists some $R > 0$ such that, for all $y \in \mathbb{R}^d$, there exist some $s \in \curlywedge(W)$ and $\mathbb{Z}$-linearly independent $r_1, \ldots, r_{m+d}$ with $s + \sum_{j=1}^{m+d} c_j r_j \in \curlywedge(W) \cap B_R(y)$ for all $0 \leq c_j \leq N$.

Set $M := N \cdot (m+d)$. By Proposition 4.2, there exist open sets $U_M, V_M \subseteq \mathbb{R}^m$ such that $0 \in V_M$ and

$$\curlywedge(U_M) + \underbrace{\curlywedge(V_M) + \curlywedge(V_M) + \cdots + \curlywedge(V_M)}_{M \text{ times}} \subseteq \curlywedge(W).$$

As U_M has nonempty interior, by Proposition 2.15, there exists $R' > 0$ such that $\curlywedge(U_M) + B_{R'}(0) = \mathbb{R}^d$. Next, by Lemma 2.4, there exist $\mathbb{Z}$-linearly independent vectors $r_1, \ldots, r_{m+d} \in \curlywedge(V_M)$. Define

$$R := N \cdot (\|r_1\| + \cdots + \|r_{m+d}\|) + R'.$$

Let $y \in \mathbb{R}^d$ be arbitrary. By the definition of R', there exists some $s \in \curlywedge(U_M) \cap B_{R'}(y)$. Then, for all $0 \le c_i \le N$, as $0 \in \curlywedge(V_M)$,

$$\begin{aligned}
s + \sum_{i=1}^{m+d} c_i r_i &= s + \underbrace{r_1 + \cdots + r_1}_{c_1 \text{ times}} + \cdots + \underbrace{r_{m+d} + \cdots + r_{m+d}}_{c_{m+d} \text{ times}} \\
&= s + \underbrace{r_1 + \cdots + r_1}_{c_1 \text{ times}} + \cdots + \underbrace{r_{m+d} + \cdots + r_{m+d}}_{c_{m+d} \text{ times}} + \underbrace{0 + \cdots + 0}_{M - \sum_{i=1}^{m+d} c_i \text{ times}} \\
&\in \curlywedge(U_M) + \underbrace{\curlywedge(V_M) + \curlywedge(V_M) + \cdots + \curlywedge(V_M)}_{c_1 \text{ times}} + \underbrace{\curlywedge(V_M) + \curlywedge(V_M) + \cdots + \curlywedge(V_M)}_{c_2 \text{ times}} \\
&\quad + \cdots + \underbrace{\curlywedge(V_M) + \curlywedge(V_M) + \cdots + \curlywedge(V_M)}_{c_{m+d} \text{ times}} + \underbrace{\curlywedge(V_M) + \curlywedge(V_M) + \cdots + \curlywedge(V_M)}_{M - \sum_{i=1}^{m+d} c_i \text{ times}} \\
&= \curlywedge(U_M) + \underbrace{\curlywedge(V_M) + \curlywedge(V_M) + \cdots + \curlywedge(V_M)}_{M \text{ times}} \subseteq \curlywedge(W).
\end{aligned}$$

Moreover, for all $0 \le c_i \le N$,

$$\begin{aligned}
d\Big(s + \sum_{i=1}^{m+d} c_i r_i, y\Big) &= \Big\| y - \Big(s + \sum_{i=1}^{m+d} c_i r_i\Big)\Big\| \le \|y - s\| + \Big\| \sum_{i=1}^{m+d} c_i r_i \Big\| \\
&\le R' + \sum_{i=1}^{m+d} \|c_i r_i\| = R' + \sum_{i=1}^{m+d} c_i \|r_i\| \le R' + \sum_{i=1}^{m+d} N\|r_i\| = R.
\end{aligned}$$

This implies that, for each $0 \le c_i \le N$,

$$s + \sum_{i=1}^{m+d} c_i r_i \in \curlywedge(W) \cap B_R(y),$$

establishing the claim. □

Theorem 4.3 suggests the following definition.

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DEFINITION 4.4. Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set.

The *ap-rank* of Λ, denoted by ap-rank(Λ), is the largest $k \in \mathbb{N}$ with the property that, for all $N \in \mathbb{N}$, there exists a li-arithmetic progression of length N and rank k in Λ.

Note here in passing that by Lemma 3.4, the ap-rank of Λ is the largest positive integer k such that for all $N \in \mathbb{N}$, the set Λ contains an arithmetic progression of length N and rank k.

The next result tells us that the definition of ap-rank makes sense.

LEMMA 4.5. *Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set. Then*

$$1 \leq \text{ap-rank}(\Lambda) \leq \text{rank}(\Lambda).$$

PROOF. The lower bound follows immediately from [8]. Note here that any nontrivial one-dimensional arithmetic progression in $\mathbb{R}^d$ has rank one.

Next consider any arithmetic progression of length $N \geq 2$ and rank k in Λ and let $r_1, \ldots, r_k$ be the linearly independent ratios. Then $r_1, \ldots, r_k$ are linearly independent vectors in $\langle \Lambda \rangle$. The claim follows from Corollary 2.2. □

REMARK 4.6.

(a) Theorem 4.3 says that for a fully Euclidean model set $\curlywedge(W)$ in the CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$,

$$\text{rank}(\curlywedge(W)) = \text{ap-rank}(\curlywedge(W)) = d + m.$$

(b) Since the ap-rank does not change under translates, but the rank changes for translates outside the $\mathbb{Z}$-module generated by the set, it is easy to construct examples of translates of fully Euclidean model sets such that

$$\text{rank}(t + \curlywedge(W)) = \text{ap-rank}(t + \curlywedge(W)) + 1.$$

In Example 5.5, for each $n \in \mathbb{N}$, we construct a Meyer set Λ such that

$$\text{rank}(\Lambda) = \text{ap-rank}(\Lambda) + n.$$

(c) If $\Lambda \subseteq \mathbb{R}^d$ is a Meyer set, we show in Corollary 5.4 that

$$\text{ap-rank}(\Lambda) \geq d.$$

We complete the section by providing a colouring version of Theorem 4.3.

THEOREM 4.7. *Let $\curlywedge(W)$ be a model set in a fully Euclidean CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$. Then, for each r, k, there exists some R such that, no matter how we colour $\curlywedge(W)$ with r colours, the set $\curlywedge(W) \cap B_R(y)$ contains a monochromatic li-arithmetic progression of length k and rank $d + m$ for all $y \in \mathbb{R}^d$.*

PROOF. Let N be such that van der Waerden's theorem (Theorem 2.8) holds for r, k applied to $[N]^{d+m}$. By Theorem 4.3, there exists some $R > 0$ such that, for all $y \in \mathbb{R}$, the set

$$\curlywedge(W) \cap B_R(y)$$

contains a nontrivial li-arithmetic progression length N. We show that this R works.

Arbitrarily colour $\curlywedge(W)$ with r colours and let $y \in \mathbb{R}^d$ be given. By Theorem 4.3, there exist $s, r_1, \ldots, r_{m+d}$ such that $r_1, \ldots, r_{m+d}$ are $\mathbb{Z}$-linearly independent and, for all $0 \le c_i \le N$,

$$s + \sum_{i=1}^{m+d} c_i r_i \in \curlywedge(W) \cap B_R(y).$$

Colour the set $[N]^{m+d}$ by colouring $(c_1, \ldots, c_{m+d})$ with the colour of $s + \sum_{i=1}^{m+d} c_i r_i$. By the choice of N, there exists a monochromatic grid

$$[k_1, \ldots, k_{d+m}; l_1, \ldots, l_{d+m}; k] \in [N]^{d+m}$$

of length k and dimension $d + m$. Then, for all $1 \le m_j \le k$, the set

$$s + \sum_{j=1}^{m+d} (l_j + m_j k_j) r_j \in \curlywedge(W) \cap B_R(y)$$

is monochromatic. Now set $s' := s + \sum_{j=1}^{m+d} l_j r_j$ and $r'_j := k_j r_j$. Then, for all $(m_1, \ldots, m_{m+d}) \in [k]^{m+d}$,

$$s' + \sum_{j=1}^{m+d} m_j r'_j \in \curlywedge(W) \cap B_R(y)$$

is a monochromatic li-arithmetic progression of length k and rank $m + d$. □

5. ap-rank of Meyer sets

In this section, we calculate the ap-rank of a Meyer set Λ. We know that Λ is equivalent by finite translates to a fully Euclidean model set $\curlywedge(W)$ and we use this to show that

$$\text{ap-rank}(\Lambda) = \text{ap-rank}(\curlywedge(W)) = \text{rank}(\curlywedge(W)).$$

We start by proving the following result.

LEMMA 5.1. *Let $\Lambda, \Gamma \subseteq \mathbb{R}^d$ be Meyer sets that are equivalent by finite translates. Then*

$$\text{ap-rank}(\Lambda) = \text{ap-rank}(\Gamma).$$

PROOF. By symmetry, it suffices to show that

$$\text{ap-rank}(\Lambda) \le \text{ap-rank}(\Gamma).$$

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Let $F = \{f_1, \ldots, f_r\}$ be such that

$$\Lambda \subseteq \Gamma + F$$

and let $k = \text{ap-rank}(\Lambda)$. We show that Γ contains arithmetic progressions of length N and rank k.

First, colour F with r colours such that no two points of F have the same colour. Next let $N \in \mathbb{N}$ be arbitrary and let $N' = W(|F|, N, d)$ be given by Theorem 2.8. Since $k = \text{ap-rank}(\Lambda)$, by definition, there exists a li-arithmetic progression

$$\{s + c_1 r_2 + \cdots + c_k r_k : 0 \leq c_j \leq N'\}$$

of length N' and rank k in Λ. Now, for each $(c_1, \ldots, c_k) \in [0, N']^k$, there exists some $f_j \in F$ so that

$$x := s + c_1 r_2 + \cdots + c_k r_k \in \bigcup_{j=1}^{r} \Gamma + f_j.$$

Pick the smallest j such that $x = y + f_j$ for some $y \in \Gamma$. Colour $(c_1, \ldots, c_k)$ with the colour of this $f_j \in F$. Then, by Theorem 2.8, there exists a monochromatic grid $[c_1' d_1, \ldots, c_k' d_k; k] \subseteq [N']^k$ of depth N. Define

$$s' = s - f_j,$$
$$r_j' = d_j r_j.$$

Then, for all $0 \leq c_j' \leq N$,

$$x = s' + c_1' r_1' + \cdots + c_k' r_k' \in \Gamma$$

is a li-arithmetic progression of length N and rank k. Since $N \in \mathbb{N}$ is arbitrary, $k \leq \text{ap-rank}(\Gamma)$. □

Now, by combining all results so far, we get the following theorem, which is the first main result in the paper.

THEOREM 5.2. *Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set, let $\curlywedge(W)$ be any fully Euclidean model set in $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ and $F \subseteq \mathbb{R}^d$ be finite such that*

$$\Lambda \subseteq \curlywedge(W) + F.$$

Then

$$\text{ap-rank}(\Lambda) = d + m = \text{ap-rank}(\curlywedge(W)) = \text{rank}(\curlywedge(W)).$$

Moreover, for each N, there exists some $R > 0$ such that the set $\Lambda \cap B_R(y)$ contains a li-arithmetic progression of length N and rank $d + m$ for all $y \in \mathbb{R}^d$.

PROOF. By Lemma 2.20, Λ and $\curlywedge(W)$ are equivalent by finite translates. Therefore, by Lemma 5.1,

$$\text{ap-rank}(\Lambda) = \text{ap-rank}(\curlywedge(W)).$$

Moreover, by Theorem 4.3,

$$\text{rank}(\curlywedge(W)) = \text{ap-rank}(\curlywedge(W)) = d + m.$$

Next, since Λ and $\curlywedge(W)$ are equivalent by finite translates, there exists some finite set $F = \{t_1, \ldots, t_r\}$ such that

$$\curlywedge(W) \subseteq \Lambda + F.$$

Colour F with r colours so that each point of F has a different colour. Let R' be the constant given by Theorem 4.7 for $\curlywedge(W)$ with r colours and length N. Define

$$R := \max\{R' + \|t_j\| : 1 \leq j \leq r\}.$$

Colour $\curlywedge(W)$ the following way: for each $x \in \curlywedge(W)$, there exists some minimal j such that $x \in t_j + \Lambda$. Colour each x by the colour of t_j for this minimal j. This gives an r-colouring of $\curlywedge(W)$. Note here that any choice of t_j works, but one needs to make a choice in case some $x \in \curlywedge(W)$ belongs to $t_j + \Lambda$ for more than one j.

Now let $y \in \mathbb{R}^d$ be arbitrary. By Theorem 4.7, there exists a monochromatic li-arithmetic progression $s + \sum_{i=1}^{m+d} c_i r_i \in \curlywedge(W) \cap B_{R'}(y)$ of rank $m + d$ for all $0 \leq c_i \leq N$.

Since the arithmetic progression is monochromatic, there exists some j such that, for all $0 \leq c_i \leq N$,

$$s + \sum_{i=1}^{m+d} c_i r_i \in t_j + \Lambda.$$

Thus, for all $0 \leq c_i \leq N$,

$$s - t_j + \sum_{i=1}^{m+d} c_i r_i \in \Lambda,$$

which is a li-arithmetic progression of length N and rank $m + d$. Moreover, for each $0 \leq c_i \leq N$,

$$d\Big(s - t_j + \sum_{i=1}^{m+d} c_i r_i, y\Big) = \Big\|s + \sum_{i=1}^{m+d} c_i r_i - t_j - y\Big\| \leq \Big\|s + \sum_{i=1}^{m+d} c_i r_i - y\Big\| + \|t_j\| \leq R.$$

This proves the last claim. □

We start by listing some consequences of this result.

As a first immediate consequence, we get that if Λ is a Meyer set, $\curlywedge(W)$ a fully Euclidean model set in $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ and F a finite set such that

$$\Lambda \subseteq \curlywedge(W) + F,$$

then m is invariant for Λ.

COROLLARY 5.3. *Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set. If $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ and $(\mathbb{R}^d, \mathbb{R}^n, \mathcal{L})$ are two different CPSs, $W \subseteq \mathbb{R}^m$, $W' \subseteq \mathbb{R}^n$ are precompact windows with nonempty interior and F, F' are finite sets such that*

$$\Lambda \subseteq \curlywedge(W) + F,$$

$$\Lambda \subseteq \curlywedge(W') + F',$$

then $m = n$.

PROOF. By Theorem 5.2,

$$\text{ap-rank}(\Lambda) = d + m,$$
$$\text{ap-rank}(\Lambda) = d + n.$$

Therefore, $d + m = d + n$ and hence $m = n$. □

As an immediate consequence, we get the following improvement on the lower bound from Lemma 4.5.

COROLLARY 5.4. *Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set. Then*

$$\text{ap-rank}(\Lambda) \geq d.$$

Next we show that in general there is no upper bound for ap-rank(Λ) in terms of rank(Λ).

EXAMPLE 5.5. Let $\curlywedge(W)$ be any fully Euclidean model set and let $r_1, \ldots, r_k$ be such that

$$\langle \curlywedge(W) \rangle = \bigoplus_{j=1}^{k} \mathbb{Z} r_k.$$

Let $s_1, \ldots, s_n$ be such that $r_1, \ldots, r_k, s_1, \ldots, s_n$ are linearly independent over $\mathbb{Z}$ and let $F = \{s_1, \ldots, s_n\}$. Then

$$\Lambda = \curlywedge(W) + F$$

is a Meyer set and

$$\text{ap-rank}(\Lambda) = k,$$
$$\text{rank}(\Lambda) = k + n.$$

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REMARK 5.6. If $\curlywedge(W)$ is a fully Euclidean model set with $k = \text{ap-rank}(\Lambda)$, then, by Theorem 4.3, *every* li-arithmetic progression in $\curlywedge(W)$ has rank at most k. The same is not true in Meyer sets.

Indeed, let $k \leq d$ and N be any positive integers. Let A be any li-arithmetic progression of rank d and length N, and let $\curlywedge(W)$ be any fully Euclidean model set of rank k such that $0 \in \curlywedge(W)$. Then

$$\Lambda := \curlywedge(W) + A$$

is a Meyer set, with $\text{ap-rank}(\Lambda) = k$, which contains the li-arithmetic progression A of rank d and length N.

Note that $\text{ap-rank}(\Lambda) = k$ means that for all $d > k$, if Λ contains li-arithmetic progressions of rank d, then they are bounded in length. We see later in Corollary 6.3 that for fully Euclidean Meyer sets, the rank of every li-arithmetic progression is also bounded by the ap-rank.

We complete the section by extending, as usual, Theorem 5.2 to colourings of Λ.

THEOREM 5.7. *Let $\Lambda \subset \mathbb{R}^d$ be a Meyer set and let $k = \text{ap-rank}(\Lambda)$. Then, for each r, N, there exists some R such that, no matter how we colour Λ with r colours, the set $\Lambda \cap B_R(y)$ contains a monochromatic li-arithmetic progression of length N and rank k for all $y \in \mathbb{R}^d$.*

PROOF. Pick N' such that van der Waerden's theorem holds for r, k applied to $[N']^{d+m}$. By Theorem 5.2, there exists $R > 0$ such that for all $y \in \mathbb{R}^d$, the set $\Lambda \cap B_R(y)$ contains an arithmetic progression of length N and dimension k. The rest of the proof is identical to that of Theorem 4.7. □

6. A characterization of fully Euclidean Meyer sets

We complete the paper by characterizing fully Euclidean Meyer sets. To our knowledge, this is the first result in this direction.

THEOREM 6.1. *Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set. Then Λ is a fully Euclidean Meyer set if and only if*

$$\text{ap-rank}(\Lambda) = \text{rank}(\Lambda).$$

PROOF. $\Longrightarrow$:

Let $\curlywedge(W)$ be a fully Euclidean model set such that

$$\Lambda \subseteq \curlywedge(W).$$

Then $\text{rank}(\Lambda) \leq \text{rank}(\curlywedge(W))$. Therefore, by Lemma 4.5 and Theorem 5.2,

$$\text{ap-rank}(\Lambda) \leq \text{rank}(\Lambda) \leq \text{rank}(\curlywedge(W)) = \text{ap-rank}(\Lambda).$$

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This gives

$$\text{ap-rank}(\Lambda) = \text{rank}(\Lambda).$$

$\Longleftarrow$:

Let $k := \text{ap-rank}(\Lambda) = \text{rank}(\Lambda)$. By Theorem 2.16, there exist a CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$, a window $W \subseteq \mathbb{R}^m$ and a finite set $F = \{f_1, \ldots, f_l\}$ such that

$$\Lambda \subseteq \curlywedge(W) + F. \tag{6-1}$$

Moreover, without loss of generality, we can assume that no proper subset F' of F satisfies (6-1). We start by showing that there exists some n so that $n\Lambda \subseteq L = \pi_{\mathbb{R}^d}(\mathcal{L})$.

First, note that by Theorem 5.2,

$$\text{rank}(\curlywedge(W)) = \text{rank}(\curlywedge(W - W)) = k.$$

By (6-1), we can partition Λ as

$$\Lambda = \dot{\bigcup_{j=1}^{l}} \Lambda_j,$$
$$\Lambda_j \subseteq \curlywedge(W) + f_j.$$

Note that for all $1 \leq j \leq l$,

$$\Lambda_j - \Lambda_j \subseteq \curlywedge(W) - \curlywedge(W)$$

and hence

$$\Gamma := \bigcup_{j=1}^{l} (\Lambda_j - \Lambda_j) \subseteq \curlywedge(W) - \curlywedge(W).$$

We claim that $\Gamma := \bigcup_{j=1}^{l} (\Lambda_j - \Lambda_j)$ is relatively dense. Indeed, set

$$J := \{j : 1 \leq j \leq l, \Lambda_j \neq \emptyset\}.$$

Then

$$\Gamma = \bigcup_{j \in J} (\Lambda_j - \Lambda_j).$$

Now, for each $j \in J$, fix some $x_j \in \Lambda_j$, that exists by the definition of J. Let

$$F' = \{x_j : j \in J\}.$$

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Then

$$\Lambda = \bigcup_{j\in J} \Lambda_j = \bigcup_{j\in J} (\Lambda_j - x_j + x_j) \subseteq \bigcup_{j\in J} (\Lambda_j - x_j + F')$$
$$= \Big(\bigcup_{j\in J} (\Lambda_j - x_j)\Big) + F' \subseteq \Big(\bigcup_{j\in J} (\Lambda_j - \Lambda_j)\Big) + F' = \Gamma + F'.$$

Since Λ is relatively dense, it follows immediately that Γ is relatively dense. In particular, Γ is a Meyer set. Now, by Theorem 5.2,

$$\text{ap-rank}(\Gamma) = \text{rank}(\curlywedge(W - W)) = k.$$

Since $\Gamma \subseteq \curlywedge(W - W)$,

$$k = \text{ap-rank}(\Gamma) \leq \text{rank}(\Gamma) \leq \text{rank}(\curlywedge(W - W)) = k$$

and hence $\text{rank}(\Gamma) = k$.

Next let $L_1 = \langle \Lambda \rangle$ and $L_2 = \langle \Gamma \rangle$ be the $\mathbb{Z}$-modules generated by Λ and Γ, respectively. Since, for each $1 \leq j \leq l$, we have $\Lambda_j \subseteq \Lambda$ and hence $\Lambda_j - \Lambda_j \subseteq \Lambda - \Lambda \subseteq L_1$, we get that L_2 is a $\mathbb{Z}$-submodule of L_1.

Now recall that by Lemma 4.1,

$$L = \langle \curlywedge(W) \rangle = \langle \curlywedge(W - W) \rangle.$$

Since $\Gamma \subseteq \curlywedge(W - W)$, we have $\Gamma \subseteq L$ and hence L_2 is a submodule of L. Moreover, by the above,

$$\text{rank}(L_1) = \text{rank}(L_2) = k$$

and by Lemma 2.3 there exists some positive integer n such that $nL_1 \subseteq L_2$. Therefore,

$$n\Lambda \subseteq nL_1 \subseteq L_2 \subseteq L,$$

as claimed. Next let $v_1, \ldots, v_{d+m}$ be the vectors such that

$$\mathcal{L} = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_{m+d}.$$

Now, by enlarging the lattice $\mathcal{L}$, we can make sure that Λ is inside the projection of the lattice on $\mathbb{R}^d$. Indeed, for each $1 \leq j \leq m + d$, let

$$w_j = \frac{1}{n} v_j$$

and set

$$\mathcal{L}' := \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_{m+d}.$$

Then it is obvious that $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L}')$ is a CPS and that

$$L' = \pi_{\mathbb{R}^d}(\mathcal{L}') = \frac{1}{m}\pi_{\mathbb{R}^d}(\mathcal{L}) = \frac{1}{n} L.$$

Moreover, by construction, $\mathcal{L} \subseteq \mathcal{L}'$. This gives

$$\Lambda \subseteq \frac{1}{n} L = L'.$$

We complete the proof by showing that $F \subseteq L'$. The result follows from this.

Let $f \in F$ be arbitrary. By the minimality of F,

$$\Lambda \nsubseteq \curlywedge(W) + (F \backslash \{f\}).$$

Therefore, there exists some $x \in \Lambda$ such that $x \notin \curlywedge(W) + (F \backslash \{f\})$. However, as $x \in \Lambda \subseteq \curlywedge(W) + F$,

$$x \in \curlywedge(W) + f.$$

Thus, there exists some $y \in \curlywedge(W)$ such that $x = y + f$. It follows that

$$f = x - y \in \Lambda - \curlywedge(W) \subseteq L' - L \subseteq L' - L' = L',$$

as claimed. Therefore, for all $1 \leq j \leq l$, there exists some $g_j \in \mathbb{R}^m$ such that $(f_j, g_j) \in \mathcal{L}'$. Define

$$W' = \bigcup_{j=1}^{l} g_j + W \subseteq \mathbb{R}^m.$$

Then W' is precompact and has nonempty interior.

We show that

$$\Lambda \subseteq \{x \in L' : \text{ there exists } y \in W' \text{such that } (x, y) \in \mathcal{L}'\} =: \curlywedge'(W'),$$

which, as $\curlywedge'(W')$ is a fully Euclidean model set in the CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L}')$, completes the proof.

Let $x \in \Lambda$ be arbitrary. Then, as $\Lambda \subseteq \curlywedge(W) + F$, there exist some $y \in \curlywedge(W)$ and $1 \leq j \leq l$ such that

$$x = y + f_j.$$

Since $y \in \curlywedge(W)$, there exists some $z \in W$ such that $(y, z) \in L \subseteq L'$. Then

$$\begin{aligned} (x, z + g_j) &= (y, z) + (f_j, g_j) \in \mathcal{L} + \mathcal{L}' \subseteq \mathcal{L}' + \mathcal{L}' = \mathcal{L}', \\ z + g_j &\in g_j + W \subseteq W' \end{aligned}$$

and hence $x \in \curlywedge'(W')$. □

REMARK 6.2. Theorem 6.1 can be equivalently stated as follows.

Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set. Then Λ is a fully Euclidean Meyer set if and only if for $k = \operatorname{rank}(\Lambda)$, for each $N \in \mathbb{N}$, there exist some $s, r_1, \ldots, r_k \in \mathbb{R}^d$ such that $r_1, \ldots, r_k$ are linearly independent over $\mathbb{Z}$ and, for all $0 \leq C_j \leq N$,

$$s + C_1 r_1 + \cdots + C_k r_k \in \Lambda.$$

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Note here that if A is a li-arithmetic progression of length $N \geq 1$ in a fully Euclidean Meyer set $\curlywedge(W)$, with ratios $r_1, r_2, \ldots, r_k$, then $r_1, r_2, \ldots, r_k \in \langle \Lambda \rangle$, which gives $k \leq \mathrm{rank}(\curlywedge(W))$. Therefore, we get the following.

COROLLARY 6.3. *Let $\Lambda \subseteq \mathbb{R}^d$ be a fully Euclidean Meyer set with* $\mathrm{rank}(\Lambda) = k$. *Then:*

(a) *every li-arithmetic progression in Λ of length $N \geq 1$ has rank at most k;*
(b) *for each $N \in \mathbb{N}$, there exists a li-arithmetic progression in Λ of rank k and length N.*

We complete the paper by giving an explicit example of a Meyer set Λ that is not fully Euclidean and we explicitly construct a CPS that produces it as a not fully Euclidean Meyer set. In fact, Λ is a regular model in this CPS.

EXAMPLE 6.4. Let *Fib* denote the well-known Fibonacci model set with its corresponding CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$, where $\mathcal{L} = \mathbb{Z}(1, 1) \oplus \mathbb{Z}(\tau, \tau')$. We refer the reader to [3] for a full description. Note that *Fib* is a fully Euclidean regular model set within a two-dimensional CPS.

Now take $\Lambda = \pi + Fib$; it follows that Λ is still relatively dense and thus a Meyer set. As the ap-rank is invariant under translates,

$$\text{ap-rank}(\Lambda) = \text{ap-rank}(Fib) = 2.$$

Now, since τ is an algebraic integer and π is transcendental, $1, \tau, \pi$ are linearly independent over $\mathbb{Z}$. It is easy to see that

$$1, \tau, \pi \subseteq \langle \pi + Fib \rangle \subseteq \mathbb{Z} + \mathbb{Z}\tau + \mathbb{Z}\pi.$$

This immediately implies that $\langle \pi + Fib \rangle = \mathbb{Z} + \mathbb{Z}\tau + \mathbb{Z}\pi$ and hence

$$\mathrm{rank}(\Lambda) = 3.$$

Therefore, by Theorem 6.1, Λ is a Meyer set that is not fully Euclidean.

In fact, Λ is a regular model set. Indeed, consider

$$\mathcal{L} := \{m + n\tau + k\pi : m, n, k \in \mathbb{Z}\} \subseteq \mathbb{R} \times (\mathbb{R} \times (\mathbb{Z}\pi)).$$

Then it is easy to see that $(\mathbb{R}, \mathbb{R} \times (\mathbb{Z}\pi), \mathcal{L})$ is a CPS and

$$\Lambda = \curlywedge([-1, \tau - 1) \times \{\pi\}).$$

REMARK 6.5. Suppose that we are given a CPS $(G, H, \mathcal{L})$, a window W and some $a \in G$. As usual, let us denote $L = \pi_G(\mathcal{L})$.

If $a \in L$, then $a + \curlywedge(W) = \curlywedge(a^\star + W)$ is a model set in the same CPS.

Otherwise, it is shown implicitly in [23, Proposition 5.6.19] that one can make $a + \curlywedge(W)$ into a model set in the following way.

Let H_0 be the cyclic subgroup of G/L generated by $a + L$. Define

$$\mathcal{L}' := \{(x + na, na + L, x^\star) : (x, x^\star) \in \mathcal{L}, n \in \mathbb{Z}\}.$$

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Then $(G, H \times H_0, \mathcal{L}')$ is a CPS and

$$a + \curlywedge(W) = \curlywedge'(W \times \{a + L\}).$$

Let us note here in passing that H_0 is a cyclic group of order at least two, so it is isomorphic either to some $\mathbb{Z}/n\mathbb{Z}$ or to $\mathbb{Z}$.

REMARK 6.6. Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set. Then, by [23, Corollary 5.9.20] and the structure theorem for compactly generated LCAGs, there exist a cut-and-project scheme$(\mathbb{R}^d, \mathbb{R}^m \times \mathbb{Z}^n \times \mathbb{K}, \mathcal{L})$, with $\mathbb{K}$ a compact Abelian group, and some compact $W \subseteq \mathbb{R}^m \times \mathbb{Z}^n \times \mathbb{K}$ such that$\Lambda \subseteq \curlywedge(W)$.

Now let $\pi : \mathbb{R}^m \times \mathbb{Z}^n \times \mathbb{K} \to \mathbb{R}^m \times \mathbb{Z}^n$ be the canonical projection and let

$$\mathcal{L}' := \{(x, \pi(x^\star)) : (x, x^\star) \in \mathcal{L}\}.$$

Then $(\mathbb{R}^d, \mathbb{R}^m \times \mathbb{Z}^n, \mathcal{L}')$ is a CPS and

$$\Lambda \subseteq \curlywedge(W) \subseteq \curlywedge'(\pi(W)).$$

This shows that every Meyer set $\Lambda \subseteq \mathbb{R}^d$ is a subset of a model set in a CPS of the form $(\mathbb{R}^d, \mathbb{R}^m \times \mathbb{Z}^n, \mathcal{L})$ for some$n \geq 0$. If the Meyer set is not fully Euclidean, then every such CPS must have $n > 0$. We suspect that the smallest possible value n can take among all the CPSs of this form is exactly

$$n = \operatorname{rank}(\Lambda) - \text{ap-rank}(\Lambda).$$

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