J. Aust. Math. Soc. **85** (2008), 51–58
doi:10.1017/S1446788708000736

AN L^p 'COUSIN OF COBOUNDARY' THEOREM FOR RANDOM FIELDS

RICHARD C. BRADLEY

(Received 14 September 2006; accepted 20 December 2006)

Communicated by V. T. Stefanov

Abstract

It is known that for a given $p \in [1, \infty)$ and a given strictly stationary sequence of random variables, the p-norms of the partial sums are bounded if and only if the sequence consists of successive differences from another strictly stationary sequence with finite p-norm. Here this is generalized to random fields, and the assumption of stationarity is relaxed. The index $p = \infty$ is included.

2000 *Mathematics subject classification*: primary 60G60, 60G10.

Keywords and phrases: cousin of coboundary, p-norm, random field.

1. Introduction

In this paper, all random variables are defined on a given probability space $(\Omega, \mathcal{F}, P)$, and are real-valued.

The following theorem is due to Robinson [8] and Leonov [6] for $p = 2$, and due to Aaronson and Weiss [1, p. 365] for general $p \in [1, \infty)$.

THEOREM 1.1. *Suppose* $1 \le p < \infty$. *Suppose* $X := (X_k, k \in \mathbb{Z})$ *is a strictly stationary sequence of random variables. Then the following two statements are equivalent: (i)* $\sup_{n \ge 1} \|X_1 + \cdots + X_n\|_p < \infty$. *(ii) There exists a strictly stationary sequence* $Y := (Y_k, k \in \mathbb{Z})$ *of random variables with* $\|Y_0\|_p < \infty$ *such that for all* $k \in \mathbb{Z}$,

$$X_k = Y_k - Y_{k+1}. \tag{1.1}$$

In Aaronson and Weiss [1, p. 365], as in Leonov [6] for the case $p = 2$, the sequence Y is a function of the sequence X, say of the form $Y_k = f(X_k, X_{k+1}, X_{k+2}, \ldots)$ (at least almost surely) where f is a real Borel function on $\mathbb{R} \times \mathbb{R} \times \ldots$.

In this note, Theorem 1.1 will be generalized to random fields $X := (X_k, k \in \mathbb{Z}^d)$ for an arbitrary positive integer d. The results will be given in Theorem 2.1 and Corollary 2.2 in Section 2. In those results the index $p = \infty$ will be included, and in Theorem 2.1 the assumption of strict stationarity will be relaxed. Equation (1.1) is sometimes put in the form $X_k = Y_k - Y_{k-1}$ (with a trivial change in the sequence Y). However, (1.1) in its present form seems to be very convenient for generalizations to random fields, as in Theorem 2.1 and Corollary 2.2 below.

Aaronson and Weiss [1, p. 365] refer to Theorem 1.1 as an '$\mathcal{L}^p$ coboundary theorem'. In the ergodic theory literature (see, for example, [1, 7, 9]), the term 'coboundary' is used in slightly different (but compatible) ways in connection with random sequences X satisfying (1.1) (with Y strictly stationary). Without being more specific, we shall simply refer to (1.1) (with Y strictly stationary) as a 'coboundary condition' for the sequence X.

In Theorem 2.1 and Corollary 2.2, Equation (2.3) gives a particular generalization of (1.1) to random fields indexed by $\mathbb{Z}^d$ for an arbitrary $d \geq 1$. However, in the case $d \geq 2$, Equation (2.3) (when, say, the random field $(Y_k, k \in \mathbb{Z}^d)$ is strictly stationary) is *not* a 'coboundary condition', but is instead a 'close relative' or 'cousin' of one. For a definition of 'coboundary' for random fields indexed by $\mathbb{Z}^d$ for $d \geq 2$, the reader is referred to the paper of Moore and Schmidt [7] (with the group G there being $\mathbb{Z}^d$); see especially Theorem 5.2 in that paper. The definition of 'coboundary' there is (for index sets $\mathbb{Z}^d$ for $d \geq 2$) quite different from, and not in any sense 'equivalent' to, Equation (2.3). We shall not give a 'name' to (2.3) (with, say, $(Y_k, k \in \mathbb{Z}^d)$ strictly stationary), but will instead informally refer to (2.3) as a 'cousin' of a coboundary condition. (An anonymous referee of an earlier version of this paper suggested the possible term 'strong coboundary' in connection with (2.3).)

Theorem 1.1 is part of a broader ongoing study of 'coboundaries' in ergodic theory. Schmidt [9, Lemma 11.7] showed that for a given strictly stationary sequence $X := (X_k, k \in \mathbb{Z})$, with no assumption of finite moments of any order, the family of distributions of the partial sums $(X_1 + \cdots + X_n, n \geq 1)$ is tight if and only if there exists a strictly stationary sequence $Y := (Y_k, k \in \mathbb{Z})$ such that (1.1) holds. One might refer to this as a 'no moments' coboundary theorem. In Schmidt [9, Theorem 11.8], Moore and Schmidt [7], Bradley [4, 5], and Aaronson and Weiss [1] that result was generalized to sequences of random variables taking their values in more general spaces than just the real numbers. Variations on it for nonstationary random sequences were given by the author [3–5]. Moore and Schmidt [7, Theorem 5.2] proved a quite general coboundary theorem which includes (as a special case) a generalization of Schmidt's [9] 'no moments' coboundary theorem to (strictly stationary) random fields indexed by $\mathbb{Z}^d$. A possible quite different generalization, to a ('cousin of coboundary') theorem involving (2.3), seems to be an open question. An anonymous referee of an earlier version of this paper pointed out that, at least for $d = 2$, such a generalization involving (2.3) can be obtained under an extra 'ergodicity' assumption, by combining the methods in [1] and [3]. That will not be pursued further here.

In Section 2, some notations will be given, and then Theorem 2.1 and Corollary 2.2 will be stated. Section 3 will be devoted to the proof of Theorem 2.1. (The derivation of Corollary 2.2 from Theorem 2.1 is elementary and therefore omitted.) In the argument in Section 3, a key role will be played by the Komlós 'subsequence' strong law of large numbers, somewhat similar to its role in arguments in [3–5] and (implicitly) in the argument in Aaronson and Weiss [1] for Theorem 1.1 (for the case $p = 1$).

2. Notations and the results

Suppose that d is a positive integer (henceforth fixed). Let $\mathbb{N}$ denote the set of all positive integers, and define the set $\overline{\mathbb{N}} := \mathbb{N} \cup \{0\}$.

For each element $j := (j_1, j_2, \ldots, j_d) \in \overline{\mathbb{N}}^d$, define the nonnegative integer $\|j\| := j_1 + j_2 + \cdots + j_d$ (the '1-norm' of the element j).

Define $C(d) := \{0, 1\}^d$, the set of 'corner points' of the d-dimensional unit cube.

The origin $(0, \ldots, 0) \in \mathbb{Z}^d$ will be denoted $\mathbf{0}$.

The usual partial ordering on $\mathbb{Z}^d$ will be used: for given elements $j := (j_1, \ldots, j_d)$ and $\ell := (\ell_1, \ldots, \ell_d) \in \mathbb{Z}^d$, the notation $j \le \ell$ means that $j_u \le \ell_u$ for every index $u \in \{1, \ldots, d\}$.

When a notation such as $\overline{\mathbb{N}}^d$ is used in a subscript or superscript, it will be denoted $\overline{\mathbb{N}} \uparrow d$ for typographical convenience. Thus $\mathbb{R}^{\overline{\mathbb{N}}\uparrow d}$ denotes the set of all mappings from $\overline{\mathbb{N}}^d$ to $\mathbb{R}$. For a given function $f : \mathbb{R}^{\overline{\mathbb{N}}\uparrow d} \longrightarrow \mathbb{R}$ and a given element $x := (x_i, i \in \overline{\mathbb{N}}^d) \in \mathbb{R}^{\overline{\mathbb{N}}\uparrow d}$, the real number $f(x)$ will also be written $f(x_i, i \in \overline{\mathbb{N}}^d)$.

Now suppose $X := (X_k, k \in \mathbb{Z}^d)$ is a random field. For any two elements $j, \ell \in \mathbb{Z}^d$ such that $j \le \ell$, define the random variable ('rectangular sum')

$$S(j, \ell) = S(X : j, \ell) := \sum_{\{i \in \mathbb{Z}\uparrow d \,|\, j \le i \le \ell\}} X_i .$$

This is the sum of the X_i for $\prod_{u=1}^{d} (\ell_u - j_u + 1)$ indices $i \in \mathbb{Z}^d$.

THEOREM 2.1. *Suppose d is a positive integer, and $p \in [1, \infty]$. Suppose $X := (X_k, k \in \mathbb{Z}^d)$ is a random field such that for every $k \in \mathbb{Z}^d$,*

$$\theta_k := \sup_{j \in \overline{\mathbb{N}}\uparrow d} \|S(k, k + j)\|_p < \infty. \tag{2.1}$$

Then there exists a Borel function $f : \mathbb{R}^{\overline{N}\uparrow d} \to \mathbb{R}$ such that, defining for each $k \in \mathbb{Z}^d$ the random variable $Y_k := f(X_{k+j}, j \in \overline{\mathbb{N}}^d)$:

$$\forall\, k \in \mathbb{Z}^d, \quad \|Y_k\|_p \le \theta_k; \quad \text{and} \tag{2.2}$$

$$\forall\, k \in \mathbb{Z}^d, \quad X_k = \sum_{j \in C(d)} (-1)^{\|j\|} Y_{k+j} \quad \text{almost surely.} \tag{2.3}$$

Here of course $[1, \infty] := [1, \infty) \cup \{\infty\}$. In Theorem 2.1, the random field X is not assumed to be strictly stationary, and the set of nonnegative numbers θ_k, $k \in \mathbb{Z}^d$, is not assumed to be bounded. By an elementary (if slightly tedious) argument, (2.3) implies that for any pair $k, \ell \in \mathbb{Z}^d$ such that $k \le \ell$, $S(k, \ell) = \sum_i (-1)^{\gamma(i)} Y_i$ where the sum is taken over the 2^d elements $i := (i_1, \ldots, i_d) \in \mathbb{Z}^d$ such that $i_u = k_u$

https://doi.org/10.1017/S1446788708000736 Published online by Cambridge University Press

or $\ell_u + 1$ for all $u \in \{1, \ldots, d\}$, and for each such i, $\gamma(i)$ denotes the number of indices $u \in \{1, \ldots, d\}$ such that $i_u = \ell_u + 1$. For strictly stationary random fields $X := (X_k, k \in \mathbb{Z}^d)$, Equation (2.3) is (as described in Section 1) a 'cousin' of a coboundary condition. The following statement generalizes Theorem 1.1 and is an elementary corollary of Theorem 2.1.

COROLLARY 2.2. *Suppose $p \in [1, \infty]$. Suppose d is a positive integer and $X := (X_k, k \in \mathbb{Z}^d)$ is a strictly stationary random field. Then the following two statements are equivalent: (i) $\sup_{j \in \overline{\mathbb{N}}^{\uparrow d}} \|S(\mathbf{0}, j)\|_p < \infty$. (ii) There exists a strictly stationary random field $Y := (Y_k, k \in \mathbb{Z}^d)$ with $\|Y_{\mathbf{0}}\|_p < \infty$ such that (2.3) holds.*

3. Proof of Theorem 2.1

We shall take for granted all notations in Section 2. We start with two preliminary lemmas.

LEMMA 3.1. *Suppose d is a positive integer. Then for every $i \in \overline{\mathbb{N}}^d - \{\mathbf{0}\}$, $\sum_{\{\ell \in C(d) | \ell \le i\}} (-1)^{\|\ell\|} = 0$.*

That is just a well-known basic fact of arithmetic. Let us quickly review it here. Suppose $i := (i_1, \ldots, i_d) \in \overline{\mathbb{N}}^d - \{\mathbf{0}\}$. Define the set $\Gamma := \{u \in \{1, \ldots, d\} \mid i_u \ge 1\}$. Then $\operatorname{card} \Gamma \ge 1$. An element $\ell := (\ell_1, \ldots, \ell_d) \in C(d)$ satisfies $\ell \le i$ precisely if $\ell_u \in \{0, 1\}$ for all $u \in \Gamma$ and $\ell_u = 0$ for all other $u \in \{1, \ldots, d\}$. For a given integer $v \in \{0, 1, \ldots, \operatorname{card} \Gamma\}$, there are precisely $\binom{\operatorname{card} \Gamma}{v}$ such elements ℓ such that $\|\ell\| = v$. Hence, the left-hand side of the equality asserted in Lemma 3.1 equals $\sum_{v=0}^{\operatorname{card} \Gamma} \binom{\operatorname{card} \Gamma}{v} \cdot (-1)^v$. That equals 0 by the Binomial theorem (since $\operatorname{card} \Gamma \ge 1$). Thus Lemma 3.1 holds.

Next, recall the Komlós 'subsequence' strong law of large numbers. (See, for example, Berkes [2] for a generalization of it.) As a simple corollary, via a routine 'Cantor diagonal' argument (left to the reader), we can state the following well-known embellishment of it.

LEMMA 3.2. *Suppose that Λ is a countably infinite set, and, for each $\lambda \in \Lambda$, $(\zeta_1^{(\lambda)}, \zeta_2^{(\lambda)}, \zeta_3^{(\lambda)}, \ldots)$ is a sequence of random variables such that $\sup_{n \in \mathbb{N}} E|\zeta_n^{(\lambda)}| < \infty$. Then there exists a strictly increasing sequence $(t(1), t(2), t(3), \ldots)$ of positive integers, and a family $(\eta^{(\lambda)}, \lambda \in \Lambda)$ of random variables with $E|\eta^{(\lambda)}| < \infty$ for all $\lambda \in \Lambda$, such that for every $\lambda \in \Lambda$ and every strictly increasing subsequence $(a(1), a(2), a(3), \ldots)$ of the integers $(t(j), j \in \mathbb{N})$, $n^{-1} \sum_{j=1}^n \zeta_{a(j)}^{(\lambda)} \to \eta^{(\lambda)}$ almost surely as $n \to \infty$.*

PROOF OF THEOREM 2.1. As in the statement of Theorem 2.1, suppose that d is a positive integer, $p \in [1, \infty]$, and $X := (X_k, k \in \mathbb{Z}^d)$ is a random field such that for every $k \in \mathbb{Z}^d$, (2.1) holds.

To prove Theorem 2.1, we (i) construct the random variables Y_k, $k \in \mathbb{Z}$, and verify (2.2), then (ii) construct the Borel function f on $\mathbb{R}^{\overline{\mathbb{N}}^{\uparrow d}}$ and verify that $Y_k = f(X_{k+j}, j \in \overline{\mathbb{N}}^d)$ for all $k \in \mathbb{Z}^d$, and finally (iii) verify (2.3).

https://doi.org/10.1017/S1446788708000736 Published online by Cambridge University Press

Construction of the Y_k's and proof of (2.2). For each positive integer n, define the set

$$F(n) := \{0, 1, \dots, n-1\}^d. \tag{3.1}$$

For each $k \in \mathbb{Z}^d$ and each positive integer n, define the random variable

$$W(k, n) := n^{-d} \sum_{j \in F(n)} S(k, k+j). \tag{3.2}$$

For each $k \in \mathbb{Z}^d$ and each positive integer n, by (2.1), (3.2), Lyapunov's and Minkowski's inequalities, and the fact that card $F(n) = n^d$ (see (3.1)),

$$\|W(k, n)\|_1 \le \|W(k, n)\|_p \le \theta_k. \tag{3.3}$$

Applying (3.3) and Lemma 3.2, let $(t(1), t(2), t(3), \dots,)$ be a strictly increasing sequence of positive integers with $t(1) \ge 2$, and let $(Y_k, k \in \mathbb{Z}^d)$ be a family of random variables such that, for every $k \in \mathbb{Z}^d$,

$$n^{-1} \sum_{j=1}^{n} W(k, t(j)) \longrightarrow Y_k \quad \text{almost surely as } n \to \infty. \tag{3.4}$$

Now $\|n^{-1} \sum_{j=1}^{n} W(k, t(j))\|_p \le \theta_k$ for all $k \in \mathbb{Z}^d$ and all $n \ge 1$ by (3.3) and Minkowski's inequality. Hence, $\|Y_k\|_p \le \theta_k$ for all $k \in \mathbb{Z}^d$ by (3.4) (and Fatou's lemma if $1 \le p < \infty$). Thus (2.2) holds.

Construction of the function f. First, for each $j \in \overline{\mathbb{N}}^d$, define the (Borel) function $s_j : \mathbb{R}^{\overline{\mathbb{N}}\uparrow d} \to \mathbb{R}$ as follows: for $x := (x_i, i \in \overline{\mathbb{N}}^d) \in \mathbb{R}^{\overline{\mathbb{N}}\uparrow d}$, define

$$s_j(x) := \sum_{\{i \in \overline{\mathbb{N}}\uparrow d \,|\, \mathbf{0} \le i \le j\}} x_i.$$

Next, for each positive integer n, define the (Borel) function $w_n : \mathbb{R}^{\overline{\mathbb{N}}\uparrow d} \to \mathbb{R}$ by (see (3.1))

$$w_n(x) := n^{-d} \sum_{j \in F(n)} s_j(x).$$

Finally, define the (Borel) function $f : \mathbb{R}^{\overline{\mathbb{N}}\uparrow d} \to \mathbb{R}$ by

$$f(x) := \begin{cases} \lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} w_{t(j)}(x) & \text{if that limit exists in } \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, for every $\omega \in \Omega$, $k \in \mathbb{Z}^d$, $j \in \overline{\mathbb{N}}^d$, and $n \in \mathbb{N}$, by (3.2) and (3.4),

$$S(k, k+j)(\omega) = s_j(X_{k+i}(\omega), i \in \overline{\mathbb{N}}^d),$$
$$W(k, n)(\omega) = w_n(X_{k+i}(\omega), i \in \overline{\mathbb{N}}^d),$$

https://doi.org/10.1017/S1446788708000736 Published online by Cambridge University Press

and (excluding the ω's in a set of probability zero)

$$Y_k(\omega) = f(X_{k+i}(\omega),\ i \in \overline{\mathbb{N}}^d). \tag{3.5}$$

For each $k \in \mathbb{Z}^d$, redefining Y_k on a set of probability zero, one can have the equality in (3.5) hold for every $\omega \in \Omega$. That does not affect (3.4) or (2.2) (proved above). Now to complete the proof of Theorem 2.1, all that remains is to prove (2.3).

Proof of (2.3). Let $k \in \mathbb{Z}^d$ be arbitrary but fixed.

For each $j \in \mathbb{N}^d$ (all indices positive), by Lemma 3.1,

$$\begin{aligned}
&\sum_{\ell \in C(d)} (-1)^{\|\ell\|} S(k+\ell, k+j) \\
&\quad = \sum_{\ell \in C(d)} (-1)^{\|\ell\|} \sum_{\{i \in \overline{\mathbb{N}}\uparrow d \mid \ell \le i \le j\}} X_{k+i} \\
&\quad = \sum_{\{i \in \overline{\mathbb{N}}\uparrow d \mid i \le j\}} X_{k+i} \left[\sum_{\{\ell \in C(d) \mid \ell \le i\}} (-1)^{\|\ell\|} \right] \\
&\quad = X_{k+\mathbf{0}} \cdot 1 + \sum_{\{i \in \overline{\mathbb{N}}\uparrow d - \{\mathbf{0}\} \mid i \le j\}} X_{k+i} \cdot 0 = X_k.
\end{aligned} \tag{3.6}$$

Now a few more index sets are needed. For each integer $n \ge 2$ and each $\ell \in C(d)$, referring to (3.1), define the set

$$A(n, \ell) := \left\{ h \in \overline{\mathbb{N}}^d \mid h - \ell \in F(n) \right\}. \tag{3.7}$$

For each integer $n \ge 2$, define the set

$$G(n) := \{1, \ldots, n-1\}^d. \tag{3.8}$$

Trivially, for each $n \ge 2$ and each $\ell \in C(d)$, by (3.1), (3.7), and (3.8), card $A(n, \ell)$ is equal to n^d,

$$A(n, \ell) \supset G(n) \quad \text{and} \quad \operatorname{card}[A(n, \ell) - G(n)] = n^d - (n-1)^d. \tag{3.9}$$

Now for each integer $n \ge 2$, by (3.2), (3.7) and (3.9),

$$\begin{aligned}
&\sum_{\ell \in C(d)} (-1)^{\|\ell\|} W(k+\ell, n) = \sum_{\ell \in C(d)} (-1)^{\|\ell\|} n^{-d} \sum_{j \in F(n)} S(k+\ell, k+\ell+j) \\
&\quad = n^{-d} \sum_{\ell \in C(d)} (-1)^{\|\ell\|} \sum_{h \in A(n,\ell)} S(k+\ell, k+h) \\
&\quad = n^{-d} \sum_{\ell \in C(d)} (-1)^{\|\ell\|} \left[\sum_{h \in G(n)} S(k+\ell, k+h) + \sum_{h \in A(n,\ell) - G(n)} S(k+\ell, k+h) \right] \\
&\quad = n^{-d} \sum_{h \in G(n)} \sum_{\ell \in C(d)} (-1)^{\|\ell\|} S(k+\ell, k+h) \\
&\qquad + n^{-d} \sum_{\ell \in C(d)} (-1)^{\|\ell\|} \sum_{h \in A(n,\ell) - G(n)} S(k+\ell, k+h).
\end{aligned} \tag{3.10}$$

https://doi.org/10.1017/S1446788708000736 Published online by Cambridge University Press

Now $(1-1/n)^d \geq 1-d/n$, for each $n \geq 2$ and hence

$$1-(n-1)^d/n^d = 1-(1-1/n)^d \leq d/n.$$

Also, for each integer $n \geq 2$, by (3.6) and (3.8), the equality of random variables

$$n^{-d} \sum_{h \in G(n)} \sum_{\ell \in C(d)} (-1)^{\|\ell\|} S(k+\ell, k+h) = n^{-d} \sum_{h \in G(n)} X_k = \Big[(n-1)^d/n^d\Big] X_k$$

holds. Hence for each $n \geq 2$, by (3.10), (3.9), and (2.1) with Lyapunov's inequality,

$$\begin{aligned}
E\Big|X_k - \sum_{\ell \in C(d)} (-1)^{\|\ell\|} W(k+\ell, n)\Big| \\
&\leq E\,\Big|X_k - \Big[(n-1)^d/n^d\Big] X_k\Big| \\
&\quad + E\Big|\Big[(n-1)^d/n^d\Big] X_k - \sum_{\ell \in C(d)} (-1)^{\|\ell\|} W(k+\ell, n)\Big| \\
&= (1-(n-1)^d/n^d) E|X_k| \\
&\quad + E\Big| -n^{-d} \sum_{\ell \in C(d)} (-1)^{\|\ell\|} \sum_{h \in A(n,\ell)-G(n)} S(k+\ell, k+h)\Big| \\
&\leq (d/n) E|X_k| + n^{-d} \sum_{\ell \in C(d)} \operatorname{card}[A(n,\ell)-G(n)] \cdot \theta_{k+\ell} \\
&\leq (d/n)\theta_k + n^{-d}(n^d-(n-1)^d) \sum_{\ell \in C(d)} \theta_{k+\ell} \\
&\leq (1/n)\Big[d \cdot \theta_k + d \cdot \sum_{\ell \in C(d)} \theta_{k+\ell}\Big].
\end{aligned}$$

Hence for each $n \geq 1$ (since $2 \leq t(1) < t(2) < t(3) < \ldots$),

$$\begin{aligned}
E\Big|X_k - \sum_{\ell \in C(d)} (-1)^{\|\ell\|} n^{-1} \sum_{j=1}^{n} W(k+\ell, t(j))\Big| \\
&= E\Big|n^{-1} \sum_{j=1}^{n} \Big[X_k - \sum_{\ell \in C(d)} (-1)^{\|\ell\|} W(k+\ell, t(j))\Big]\Big| \\
&\leq n^{-1} \sum_{j=1}^{n} (1/t(j)) \cdot \Big[d \cdot \theta_k + d \cdot \sum_{\ell \in C(d)} \theta_{k+\ell}\Big].
\end{aligned}$$

This last term converges to 0 as $n \to \infty$ by Toeplitz's lemma. Hence

$$\sum_{\ell \in C(d)} (-1)^{\|\ell\|} n^{-1} \sum_{j=1}^{n} W(k+\ell, t(j)) \longrightarrow X_k \quad \text{in probability as } n \to \infty. \tag{3.11}$$

https://doi.org/10.1017/S1446788708000736 Published online by Cambridge University Press

Also, by (3.4), the left-hand side of (3.11) converges almost surely to

$$\sum_{\ell \in C(d)} (-1)^{\|\ell\|} Y_{k+\ell} \quad \text{as } n \to \infty.$$

Hence (2.3) holds by (3.11). That completes the proof of (2.3) and of Theorem 2.1. □

Acknowledgement

The author thanks an anonymous referee of an earlier version of this paper for clarifying the difference between Equation (2.3) and the notion of 'coboundary'.

References

[1] J. Aaronson and B. Weiss, 'Remarks on the tightness of cocycles', *Colloq. Math.* **84/85** (2000), 363–376.
[2] I. Berkes, 'An extension of the Komlós subsequence theorem', *Acta. Math. Hungar.* **55** (1990), 103–110.
[3] R. C. Bradley, 'On a theorem of K. Schmidt', *Statist. Probab. Lett.* **24** (1995), 9–12.
[4] ———, 'A multiplicative coboundary theorem for some sequences of random matrices', *J. Theoret. Probab.* **9** (1996), 659–678.
[5] ———, 'A coboundary theorem for sums of random variables taking their values in a Banach space', *Pacific J. Math.* **178** (1997), 201–224.
[6] V. P. Leonov, 'On the dispersion of time averages of a stationary random process', *Theory Probab. Appl.* **6** (1961), 93–101.
[7] C. C. Moore and K. Schmidt, 'Coboundaries and homomorphisms for nonsingular actions and a problem of H. Helson', *Proc. London Math. Soc.* (3) **40** (1980), 443–475.
[8] E. A. Robinson, 'Sums of stationary random variables', *Proc. Amer. Math. Soc.* **11** (1960), 77–79.
[9] K. Schmidt, *Cocycles on Ergodic Transformation Groups* (Macmillan India, Delhi, 1977).

RICHARD C. BRADLEY, Department of Mathematics, Indiana University,
Bloomington, IN 47405, USA
e-mail: bradleyr@indiana.edu

https://doi.org/10.1017/S1446788708000736 Published online by Cambridge University Press