4

Quantum fields on time-dependent backgrounds: Particle creation

Beginning with this chapter we will introduce quantum field theory (QFT) and develop the necessary ideas and methods which form the basis of nonequilibrium (NEq) QFT. We focus on quantum field systems in external fields or in a time-varying background spacetime. The latter is included here because many basic concepts and techniques in QFT in external fields were developed historically in the area of QFT in curved spacetimes, especially in time-dependent backgrounds used in relativistic cosmology. Cosmology is also the arena where some of the basic tenets of NEqQFT were established and tested out.

In a dynamical background some basic concepts of QFT need to be reexamined. We point out the problem in straightforwardly extending the methodology of Minkowski spacetime QFT, such as the definition of particles by way of instantaneous diagonalization of the Hamiltonian. The vacuum state defined this way is nonviable since particles are being created as the system evolves. We introduce the Bogoliubov transformation between two sets of mode functions of the field, and discuss how two different particle models defined at different times are related to each other. Particle creation is a nonadiabatic process. We introduce the nth order adiabatic vacuum and number state as the proper way to construct a QFT in dynamical backgrounds. We derive expressions for spontaneous particle production as parametric amplification of vacuum fluctuations, and stimulated production as amplification of particles already present in the quantum or thermal state.

Following this we give two examples for the problem of charged particle motion in an external field. The first one is for a uniform electric field. We show how to use the adiabatic number state and the Bogoliubov transformation to obtain the famous result of Schwinger. In the second problem we study periodically driven fields based on the Floquet theory of parametric resonance. For charged particles in an external field we derive a quantum Vlasov equation for the rate of particle creation and show that particle creation is a non-Markovian (history dependent) process. We point out the intrinsic relation between number and phase of a quantum system, and under what conditions particle number may increase and others when it may decrease.

We then turn to a discussion of the second class of problems, that of quantum fields in dynamic background spacetimes. These are useful for the study of quantum processes in the early universe. We introduce the wave equation in curved spacetime, and discuss the conditions where one can construct a physically meaningful particle model, including the conformal vacuum for conformal fields in conformally flat spacetimes, which are relevant to the standard model in cosmology. We use a simple observation to show why gravitons are not produced in a radiation-dominated universe, and a simple model to illustrate how thermal particle creation arises. We then demonstrate how one can identify and remove the ultraviolet divergences in the stress–energy tensor of the quantum field by the method of adiabatic regularization. Obtaining a physically reasonable regularized stress–energy tensor is an essential step in approaching the so-called "back-reaction problem," i.e. finding a self-consistent solution of the quantum particle–EM field or quantum field–background spacetime system.

This is followed by a self-contained description of particle creation in the squeezed state language which can better elucidate the relation between number and phase representations. We first give the result of spontaneous and stimulated production, discuss the difference between bosons and fermions, and their dependence on the initial state. We then introduce the statistical mechanics of particle creation and relate entropy generation to the specification of the initial state and the choice of representations, such as the number state, the coherent and the squeezed state. Finally we present results for the fluctuations in particle number as it is relevant to defining noise in quantum fields and the vacuum susceptibility of spacetime. In the last section we give a description of squeezed quantum open systems. These discussions bring out some basic issues in the statistical mechanics of quantum fields and prepare the ground for investigating the statistical, kinetic, and stochastic features of quantum processes such as back-reaction and dissipation, entropy generation, fluctuations, correlations, noise and decoherence, which will be elaborated in later chapters.

4.1 Basic field theory

4.1.1 Classical fields

A field theory is concerned with extended physical systems, whose configurations are defined by giving some set of numbers at each spacetime point associated with an event, with coordinates denoted by a 4-vector $x^\mu = (t, \mathbf{x})$ containing the time and space components respectively. The simplest field theories have only one (real) number assigned to each event (or, attached to each spacetime point) and this number is prescribed to be the same for all observers. These are the so-called scalar field theories. For example, if we imagine spacetime as a continuous fluid, we may define a temperature (scalar) field $T(x)$ whose field configuration is given by the temperature T reading (a number) at each spatial point $\mathbf{x}$ at a given time t as measured by an observer at rest with respect to the fluid. Another familiar example of a scalar field is the magnetization density $\mu(x)$ in a ferromagnetic material, again in the continuous spacetime approximation.

For pedagogical reasons we shall be using the scalar field theory to illustrate new ideas and methods in this book. Extensions to vector (e.g. electromagnetic),

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

tensor (e.g. gravitational) and spinor (e.g. electron) fields can be made with proper treatment of their specific tensor characters. In most parts of this book, except Chapters 9 and 15, we shall work in flat spacetime, endowed with the Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, with time being the zeroth coordinate.

A scalar field theory describes the field variable $\phi(x)$, namely, the single real number to be prescribed at every event. Its dynamics is given by the action $S[\phi]$ of the theory; for example

$$S[\phi] = \int d^4x \left\{ -\frac{1}{2} (\nabla\phi)^2 - V[\phi(x)] \right\} \tag{4.1}$$

where $(\nabla\phi)^2$ in Minkowski space is equal to $\partial_\mu\phi\partial^\mu\phi = (\partial\phi)^2$ and the potential $V(\phi)$ is a real functional of the field variable ϕ. In this chapter we choose units such that the speed of light $c = 1$. A common example for massive interacting fields is the ϕ^4 potential

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \tag{4.2}$$

where m is the *mass* of the field (also known as the inverse correlation length) and λ is the coupling constant. The equations of motion are given by the variational principle $\delta S/\delta\phi = S_{,\phi} = 0$. In our case they read

$$\nabla^2 \phi - V'(\phi) = 0 \tag{4.3}$$

where $\nabla^2 = \partial_\mu \partial^\mu$ in Minkowski space and $V' = dV/d\phi$. We can define the field momentum as $\pi = \phi_{,t}$ $\left(\phi_{,t} \equiv \phi_{,0} \text{ or } \dot{\phi}\right)$. A particular solution of the equations of motion is identified by its *Cauchy data* ϕ, π on a constant time surface. (There are more general surfaces one can use, the so-called *Cauchy surfaces*, but we won't go into that here.) The dynamics inherits the symmetries of the action, which in Minkowski spacetime possesses Poincaré invariance, and, for an even potential such as in equation (4.2), $\phi \to -\phi$ symmetry.

The second-order equation (4.3) can also be written as a first-order equation for π, namely

$$\frac{\partial \pi}{\partial t} = \nabla^2 \phi - V'(\phi) \tag{4.4}$$

The definition of π and (4.4) together have the structure of canonical equations derivable from a Hamiltonian

$$H = \int d^3\mathbf{x} \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla_i \phi)^2 + V(\phi) \right] \tag{4.5}$$

Observe that the integral extends over space variables only. In other words, the nondenumerable set $\left\{\phi(t, \mathbf{x}), \mathbf{x} \epsilon R^3\right\}$ defines the canonical coordinates at time t, and the π's are their conjugate momenta. These canonical variables obey the equal-time Poisson brackets

$$\{\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')\} = \{\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')\} = 0; \ \{\pi(t, \mathbf{x}), \phi(t, \mathbf{x}')\} = \delta(\mathbf{x} - \mathbf{x}') \tag{4.6}$$

This formulation is called the canonical formalism of field theory.

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

4.1.2 Quantum fields

The theory is quantized by replacing the field variable ϕ by an operator-valued distribution Φ. In the *Heisenberg picture*, for each event x there is an operator $\Phi(x)$ acting on some Hilbert space $\mathcal{H}$ of states. The conjugate momentum π goes over to the momentum operator Π, and the Poisson brackets equation (4.6) become the *equal-time canonical commutation relations* (ETCCRs)

$$[\Phi(t,\mathbf{x}),\Phi(t,\mathbf{x}')] = [\Pi(t,\mathbf{x}),\Pi(t,\mathbf{x}')] = 0;\ \ [\Pi(t,\mathbf{x}),\Phi(t,\mathbf{x}')] = -i\hbar\delta(\mathbf{x}-\mathbf{x}') \tag{4.7}$$

The field operator moreover obeys the equation

$$\nabla^2\Phi - V'(\Phi) = 0 \tag{4.8}$$

which is equivalent to the first-order system

$$\dot{\Phi} = \frac{i}{\hbar}[H,\Phi];\ \ \dot{\Pi} = \frac{i}{\hbar}[H,\Pi] \tag{4.9}$$

leading to the rule

$$\Phi(t,\mathbf{x}) = U^\dagger(t,t')\,\Phi(t',\mathbf{x})\,U(t,t') \tag{4.10}$$

where U is the *evolution operator*

$$U(t,t') = e^{-i(t-t')H/\hbar} \tag{4.11}$$

More generally, we may introduce the generators P^μ of translations. The P^μ operators commute among themselves, as dictated by the algebra of the Poincaré group, and equation (4.11) is a particular case of the transformation rule

$$\Phi(x) = e^{-iPx/\hbar}\Phi(0)\,e^{iPx/\hbar} \tag{4.12}$$

after identifying the Hamiltonian $H = P^0$.

4.1.3 Free fields

A free field corresponds to a quadratic potential $V(\Phi)$. A generic example is a free massive scalar field with $V(\Phi) = (1/2)m^2\Phi^2$. The Heisenberg equation of motion for this field becomes the *Klein–Gordon equation* $\nabla^2\Phi(x) - m^2\Phi(x) = 0$.

Assuming that the field lives in a finite large volume V and expanding the scalar field operator in (spatial) Fourier modes, we have

$$\Phi(t,\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\varphi_{\mathbf{k}}(t)u_{\mathbf{k}}(\mathbf{x}) \tag{4.13}$$

where $\mathbf{k} = 2\pi\mathbf{n}/L$, and $\mathbf{n} = (n_1,n_2,n_3)$ in general consists of a triplet of integers. In Minkowski space the spatial mode functions are simply $u_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{x}}$. In the infinite volume continuum limit this becomes

$$\Phi(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\mathbf{x}} \varphi_{\mathbf{k}}(t) \tag{4.14}$$

The (operator-valued) amplitude function $\varphi_{\mathbf{k}}(t)$ for each mode $\mathbf{k}$ obeys a harmonic oscillator equation

$$\frac{d^2\varphi_{\mathbf{k}}}{dt^2} + \omega_{\mathbf{k}}^2 \varphi_{\mathbf{k}} = 0 \tag{4.15}$$

where $\omega_{\mathbf{k}}^2 = |\mathbf{k}|^2 + m^2$ in Minkowski space.

Given two complex independent solutions $f_{\mathbf{k}}$, $f_{\mathbf{k}}^*$ of equation (4.15), we may write

$$\varphi_{\mathbf{k}}(t) = f_{\mathbf{k}}(t)\, a_{\mathbf{k}} + f_{\mathbf{k}}^*(t)\, a_{-\mathbf{k}}^{\dagger} \tag{4.16}$$

Let us introduce the Wronskian $(f, g) = f\dot{g} - g\dot{f}$, which is conserved by equation (4.15), and impose the normalization

$$(f_k, f_k^*) = i\hbar \tag{4.17}$$

The ETCCRs are equivalent to

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = \left[a_{\mathbf{k}}^{\dagger}, a_{\mathbf{k}'}^{\dagger}\right] = 0; \quad \left[a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}\right] = \delta(\mathbf{k} - \mathbf{k}') \tag{4.18}$$

These operators may be interpreted as particle destruction and creation operators. We say that each choice of the basis functions $f_{\mathbf{k}}$ constitutes a *particle model*, where $f_{\mathbf{k}}$ is the *positive frequency* component and $f_{\mathbf{k}}^*$ is the *negative frequency* component of the $\mathbf{k}$th mode; the state which is destroyed by all the $a_{\mathbf{k}}$'s is the vacuum of the particle model. The vacua of different particle models are in general inequivalent. This situation becomes more challenging for quantum fields in a dynamical background field or spacetime, which is the central theme of this chapter.

In terms of the creation and destruction operators, the Hamiltonian is

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left\{ \mathcal{A}\hbar\omega_{\mathbf{k}} \left(\hat{N}_{\mathbf{k}} + \frac{1}{2} \right) + F_{\mathbf{k}} a_{\mathbf{k}} a_{-\mathbf{k}} + F_{\mathbf{k}}^* a_{\mathbf{k}}^{\dagger} a_{-\mathbf{k}}^{\dagger} \right\} \tag{4.19}$$

Here,

$$\hat{N}_{\mathbf{k}} = a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}; \quad \mathcal{A}\hbar\omega_{\mathbf{k}} \equiv \left(\left| \dot{f}_{\mathbf{k}} \right|^2 + \omega_{\mathbf{k}}^2 |f_{\mathbf{k}}|^2 \right); \quad F_{\mathbf{k}} \equiv \dot{f}_{\mathbf{k}}^2 + \omega_{\mathbf{k}}^2 f_{\mathbf{k}}^2 \tag{4.20}$$

We may diagonalize the Hamiltonian at any time $t = 0$ by imposing the condition $\dot{f}_{\mathbf{k}}(0) = -i\omega_{\mathbf{k}} f_{\mathbf{k}}(0)$, making $F_{\mathbf{k}}(0) = 0$. In Minkowski space, and with the natural time coordinate, the Hamiltonian stays diagonal at all times. The corresponding particle model in *Minkowski* space is given by

$$f_{\mathbf{k}}(t) = \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}} e^{-i\omega_{\mathbf{k}} t}; \; \mathcal{A} = 1 \tag{4.21}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

which possesses a well-defined meaning of particles at all times. This is the framework of (flat space) quantum field theory implicitly assumed in textbooks.

4.1.4 Particle creation

We now consider quantum fields propagating on dynamic backgrounds. When a mode decomposition is available the (c-number) amplitude function of the **k**th mode obeys, from equation (4.15), the wave equation

$$\frac{d^2 f_{\mathbf{k}}}{dt^2} + \omega_{\mathbf{k}}^2(t) f_{\mathbf{k}}(t) = 0 \tag{4.22}$$

where the natural frequency $\omega_{\mathbf{k}}$ now acquires an explicit time dependence.

In Minkowski space QFT we are accustomed to the notion that positive energy solutions to the wave equation for every normal mode correspond to particles while negative energy solutions correspond to antiparticles. One can diagonalize the Hamiltonian to select a preferred particle model, e.g. the Minkowski modes (4.21). However for a time-dependent background field this notion becomes meaningless and the criterion of instantaneous diagonalization of the Hamiltonian is inviable as a particle model. This is because the mode equation (4.22) generally possesses time-dependent solutions which have no clear a priori physical meaning in terms of particles or antiparticles. The energy of individual particle/antiparticle modes is not conserved, and a consistent separation into positive and negative energy solutions of the wave equation is not always possible. This is just a reflection of the fact that physical particle number does not correspond to an operator which commutes with the Hamiltonian. We can see this point more clearly by way of the Bogoliubov transformation.

The transformation between any two Fock space bases $a_{\mathbf{k}}$ and $\tilde{a}_{\mathbf{k}}$ is known as the Bogoliubov transformation. Let the first basis $a_{\mathbf{k}}$ be associated with modes $(f_{\mathbf{k}}, f_{\mathbf{k}}^*)$, the second basis $\tilde{a}_{\mathbf{k}}$ with modes $\left(\tilde{f}_{\mathbf{k}}, \tilde{f}_{\mathbf{k}}^*\right)$. We may expand the field operators in either base, leading to equation (4.16) in the first case, and to

$$\varphi_{\mathbf{k}}(t) = \tilde{f}_{\mathbf{k}}(t)\, \tilde{a}_{\mathbf{k}} + \tilde{f}_{\mathbf{k}}^*(t)\, \tilde{a}_{-\mathbf{k}}^{\dagger} \tag{4.23}$$

in the second. Since both sets of solutions of the mode equations are complete, we must have

$$f_{\mathbf{k}}(t) = \alpha_{\mathbf{k}} \tilde{f}_{\mathbf{k}}(t) + \beta_{\mathbf{k}} \tilde{f}_{\mathbf{k}}^*(t) \tag{4.24}$$

and its inverse

$$\tilde{f}_{\mathbf{k}}(t) = \alpha_{\mathbf{k}}^* f_{\mathbf{k}}(t) - \beta_{\mathbf{k}} f_{\mathbf{k}}^*(t) \tag{4.25}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

The Wronskian condition $(f_{\mathbf{k}}, f_{\mathbf{k}}^*) = \left(\tilde{f}_{\mathbf{k}}, \tilde{f}_{\mathbf{k}}^*\right) = i\hbar$ imposes a condition on the Bogoliubov coefficients

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1 \tag{4.26}$$

for each $\mathbf{k}$. We can thus write

$$\begin{aligned} |\alpha_{\mathbf{k}}(t)| &= \cosh r_{\mathbf{k}}(t) \\ |\beta_{\mathbf{k}}(t)| &= \sinh r_{\mathbf{k}}(t) \end{aligned} \tag{4.27}$$

where $r_{\mathbf{k}}(t)$ is called the squeeze parameter for mode $\mathbf{k}$, a terminology adopted from quantum optics. In Section 4.7 we will give a description of particle creation in the squeezed state language.

The linear relationship between the $\tilde{f}$'s and f's induces a corresponding transformation between $a, \tilde{a}$

$$\tilde{a}_{\mathbf{k}} = \alpha_{\mathbf{k}} a_{\mathbf{k}} + \beta_{\mathbf{k}}^* a_{-\mathbf{k}}^\dagger \tag{4.28}$$

with inverse

$$a_{\mathbf{k}} = \alpha_k^* \tilde{a}_{\mathbf{k}} - \beta_{\mathbf{k}} \tilde{a}_{-\mathbf{k}}^\dagger \tag{4.29}$$

Each particle model is associated with a particular vacuum state, in this case, $|0\rangle$ and $|\tilde{0}\rangle$, defined by

$$a_{\mathbf{k}}\,|0\rangle = 0 \quad \text{and} \quad \tilde{a}_{\mathbf{k}}\,|\tilde{0}\rangle = 0 \tag{4.30}$$

separately for all $\mathbf{k}$. Fock spaces can be constructed from the vacuum states by the action of the creation operators. One can easily see that generally $\tilde{a}_{\mathbf{k}}\,|0\rangle \neq 0$ because the two vacua are different by the coefficients α, β. Introducing the particle number operator $\left(\tilde{N}_{\mathbf{k}}\right)^\wedge \equiv \tilde{a}_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}}$ of the second particle model, we see that its expectation value with respect to the vacuum of the first model is nonzero, but equal to

$$\tilde{N}_{\mathbf{k}} = \langle 0 \mid \left(\tilde{N}_{\mathbf{k}}\right)^\wedge \mid 0\rangle = V \mid \beta_{\mathbf{k}} \mid^2 \tag{4.31}$$

where V is the "volume" of space. An observer of the second particle model would say that $\tilde{N}_{\mathbf{k}}$ particles have been created from the first vacuum (from now on, we shall disregard factors of V, assuming that particle counts are always referred to a unit volume). When we think of the second particle model as defined at a time t while the first particle model is defined at the initial time t_0, we may write the particle numbers at these two times as $\langle \hat{N}_{\mathbf{k}}(t)\rangle_t, \langle \hat{N}_{\mathbf{k}}(t_0)\rangle_0$ respectively, i.e. $\hat{N}$ denotes a generic number operator which takes on eigenvalues N and $\tilde{N}$ in the two Fock spaces respectively. (We may at times use the notation n and $\mathcal{N}$ for these two values also.) In an S-matrix formulation of quantum field theory in a dynamical background (field or spacetime), where one assumes an asymptotic region where the background field is constant or the spacetime is static (so the modes obtained by the diagonalization of the Hamiltonian in those regions give a preferred particle model), the states of the first ($a_{\mathbf{k}}$) and second ($\tilde{a}_{\mathbf{k}}$) particle models are conventionally called the *in* and the *out* states respectively. We will use these nomenclatures interchangeably.

It is interesting to give a closed expression for the amplitude for finding $\tilde{n}$ pairs in the $|0\rangle$ state in terms of the Bogoliubov coefficients. We have

$$\langle \tilde{n}_k, \tilde{n}_{-k} | \, 0\rangle = \frac{1}{\alpha_k^*} \left[\frac{\beta_k^*}{\alpha_k^*} \right]^{\tilde{n}_k} \tag{4.32}$$

4.1.5 Adiabatic vacua

The transformation of the Fock space operators described by the Bogoliubov transformation (4.28), despite its appearance, is only a formal expression. The creation and annihilation operators do not give particle creation unless the vacuum state is well defined. We will discuss below situations where there are preferred particle models asymptotically, such as constant background fields or stationary spacetimes at $t = \pm\infty$, or conformally-invariant fields in conformally-static spacetimes without asymptotic conditions. Then the Fock spaces are well defined and one can calculate the amplitude for particle creation in a S-matrix sense. Under general conditions the particle number at any one time during the evolution is not well-defined. A straightforward intuitive generalization from flat space field theory – the so-called method of instantaneous diagonalization of the Hamiltonian – leads to severe problems; see e.g. [Ful89]. One has to appeal to other methods. If the external field (or background spacetime) does not change too rapidly (to be quantified below by the nonadiabaticity parameter) there is a conceptually clear and technically simple method which has proven to be useful in problems involving time-dependent fields (as in the external field problem) and spacetimes (as in cosmological particle creation). It is the nth order *adiabatic vacuum or number state*, and, when applied to the removal of ultraviolet divergences in the current or energy–momentum tensor, it is called *adiabatic regularization*. A selection of influential papers on this subject is [Park66, Park69, ParFul74, FulPar74, Park76, Park77].

Both the time-dependence of the Fock space operators and the evolution of the amplitude functions are dictated by the wave equations for the normal modes of the quantum scalar field with time-dependent natural frequency ω_k as in equation (4.22). To single out a solution, we need to specify initial data for $f_{\mathbf{k}}$ and $df_{\mathbf{k}}/dt$ at some time t_0. When ω_k is constant one can use the same Fock space representation of the field theory as it remains the same as originally defined at t_0. Staticity means that the dynamics is invariant in time, and implies the existence of a Killing vector in time ∂_t, which enforces the positive and negative frequency components to remain separated. This means, in second quantized language, that the particles and antiparticles are separately well-defined and their number remains a constant. Therefore the possibility of defining a positive frequency

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

component in a field theory is the precondition for a vacuum state to exist. We learned that maintaining such a condition in the evolution is not always possible.

If the external field or background spacetime changes gradually one can extend this idea and define an adiabatic vacuum or number state. Recall from elementary wave or quantum theory that a WKB solution can give a reasonable approximation to the wave equation when the system changes gradually enough. Successively higher order WKB (or adiabatic) solutions can encompass more rapid changes in the background field as they show up in the natural frequency function. This is the lead idea behind the adiabatic method.

The sequence of successively higher order WKB solutions to this wave equation has been explored quite extensively by researchers working on wave propagation in inhomogeneous media. There, the reflection of waves due to successively higher order derivatives in the dielectric media can be treated with successively higher order WKB solutions. Translating the variation in spatial homogeneity to time dependence is a physically intuitive way to understanding the adiabatic vacuum. This route explored by Hu [Hu72, Hu74] gives the same result as that established first by Parker and Fulling. It was also shown to be equivalent to the result obtained by Zeldovich and Starobinsky [ZelSta71, FuPaHu74] in their "n-wave regularization."

Consider the wave equation (4.22) in t time for the amplitude function of the $\mathbf{k}$th mode. (We shall omit the $\mathbf{k}$ subscript, as only one mode is being considered.) The idea is to use a transformation of both time t and dependent variable f to reduce this equation to one we can solve.

Define a new time variable $t_1 = t_1(t)$, and write equation (4.22) as

$$\left(\frac{dt_1}{dt}\right)^2 \frac{d^2 f}{dt_1^2} + \left(\frac{d^2 t_1}{dt^2}\right)\frac{df}{dt_1} + \omega^2 f = 0 \tag{4.33}$$

The equation is simplified by choosing

$$\frac{dt_1}{dt} = \omega(t) \tag{4.34}$$

whereby

$$\frac{d^2 f}{dt_1^2} + \frac{1}{\omega}\left(\frac{d\omega}{dt_1}\right)\frac{df}{dt_1} + f = 0 \tag{4.35}$$

The first-order term is eliminated by writing

$$f = \omega^{-1/2} f_1 \tag{4.36}$$

obtaining

$$\frac{d^2}{dt_1^2} f_1 + w_1^2 f_1 = 0 \tag{4.37}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

where

$$w_1^2 = 1 + \epsilon_2, \; \epsilon_2 = -\frac{1}{\omega^{1/2}} \frac{d^2}{dt_1^2}(\omega^{1/2}) \tag{4.38}$$

Observe that equation (4.37) has the same structure as the original equation (4.22). If ω varies sufficiently slowly, we can neglect ϵ_2, and it becomes trivial.

Higher order WKB approximations to the wave equation are obtained by iterating this procedure. Define (note r here is an adiabatic order parameter, not the squeeze parameter introduced earlier)

$$dt_r \equiv w_{r-1} dt_{r-1} \equiv W_r dt \quad (w_0 \equiv \omega, t_0 \equiv t) \tag{4.39}$$

$$f_r \equiv w_{r-1}^{1/2} f_{r-1} = W_r^{1/2} f \tag{4.40}$$

$$W_r \equiv w_0 w_1 \cdots w_{r-1} \tag{4.41}$$

$$\Theta_r \equiv \int W_r dt \tag{4.42}$$

The $n(=2r)$th-order WKB equation is given by $(r = 1, 2, \ldots)$

$$\frac{d^2}{dt_r^2} f_r + w_r^2 f_r = 0 \tag{4.43}$$

where, for $r = 1, 2, 3 \ldots$,

$$w_r^2 = 1 + \epsilon_{2r}, \quad \epsilon_{2r} = -\frac{1}{w_{r-1}^{1/2}} \frac{d^2}{dt_r^2}(w_{r-1}^{1/2}) \tag{4.44}$$

The quantities ϵ_{2r} are called the *adiabatic frequency corrections* [FuPaHu74]. If $|\epsilon_{2r}| \ll 1$, the solution of the wave equation correct up to the $n(=2r)$th order of derivatives of the natural frequency $w^2(t)$ with respect to t_r is given by

$$f_{(n)}(t) = \frac{\hbar^{1/2}}{(2W_r)^{1/2}} \left[Ae^{-i\int W_r dt} + Be^{i\int W_r dt} \right] \tag{4.45}$$

where A, B are complex functions. The subscript (n) on f indicates that a solution to the full wave equation is sought *including up to* the nth adiabatic order. In contradistinction, we define a $n(=2r)$th-order adiabatic solution as the solution with ϵ_{2r} set equal to zero.

The *nth-order adiabatic vacuum* is defined such that there is no negative frequency component in the nth-order WKB solution. What this means is that, at the nth adiabatic order approximation the nth-order adiabatic number state is obtained by assuming that the wavefunction $f(t)$ is given only by the positive frequency nth-order WKB solution

$$f(t) \simeq f_{(n)}^{+}(t) = \frac{Ae^{-i\int^t W_{n/2} dt}}{\sqrt{2W_{n/2}}} \tag{4.46}$$

So intrinsically this is a quasi-local (in time) expansion counting time derivative orders, which can be translated to frequency ranges. In terms of what adiabatic order will encompass what range of frequencies we shall see how this method

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

becomes useful for identifying and isolating ultraviolet divergences in quantum field theory in dynamical spacetimes, as in cosmology. This method, known as adiabatic regularization, will be discussed in a later section.

4.1.6 Hamiltonian mean field dynamics and general Gaussian ansatz

Let us broaden our scope somewhat to introduce an important class of approximations in quantum field theory which shares the same dynamics as the problem under discussion so far. This is the mean field (or Gaussian) approximation. Mean field methods have a long history in such diverse areas as atomic physics (Born–Oppenheimer), nuclear physics (Hartree–Fock), condensed matter (BCS) and statistical physics (Landau–Ginzburg), quantum optics (coherent/squeezed states), and semiclassical gravity. Because no higher than second moments of the fluctuations are incorporated, the mean field approximation is related to a Gaussian variational ansatz for the wavefunction of the system.

For the mixed state density matrix ρ Habib *et al.* [HKMP96] have shown that the time-dependent mean field approximation is equivalent to the general Gaussian ansatz. It is instructive to follow the exposition of this feature.

As a matter of principle, the Hamiltonian nature of the evolution makes it clear from the outset that the mean field approximation does *not* introduce dissipation or time irreversibility at a fundamental level. Any such behavior must come from some assumption in coarse graining some information of this closed system away. We shall remark on this aspect at the end of this section and further in Chapter 9 on entropy generation.

Consider again the one-dimensional harmonic oscillator with Hamiltonian

$$H_{osc}(q,p;t) = \frac{1}{2}\left(p^2 + \omega^2(t)\, q^2\right) \tag{4.47}$$

where $\omega(t)$ is the natural frequency. The most general Gaussian ansatz for the mixed state normalized density matrix is

$$\begin{aligned}\langle x'|\rho|x\rangle = (2\pi\xi^2)^{-\frac{1}{2}} \exp\Bigg\{ & i\frac{\bar{p}}{\hbar}(x'-x) - \frac{\zeta^2+1}{8\xi^2}\left[(x'-\bar{q})^2 + (x-\bar{q})^2\right] \\ & + i\,\frac{\eta}{2\hbar\xi}\left[(x'-\bar{q})^2 - (x-\bar{q})^2\right] + \frac{\zeta^2-1}{4\xi^2}(x'-\bar{q})(x-\bar{q}) \Bigg\}\end{aligned} \tag{4.48}$$

in the coordinate representation. The five parameters $(\bar{q},\bar{p},\xi,\eta,\zeta)$ of this Gaussian may be identified with the two mean values, $\bar{q} = \langle q\rangle \equiv \mathrm{Tr}(\rho q)$, $\bar{p} = \langle p\rangle \equiv \mathrm{Tr}(\rho p)$, and the three symmetrized variances via

$$\begin{aligned}\langle (q-\bar{q})^2\rangle = \xi^2, \qquad & \langle (pq + qp - 2\bar{q}\bar{p})\rangle = 2\xi\eta \\ \langle (p-\bar{p})^2\rangle = \eta^2 + & \frac{\hbar^2\zeta^2}{4\xi^2}\end{aligned} \tag{4.49}$$

The one antisymmetrized variance is fixed by the commutation relation, $[q, p] = i\hbar$. The parameter ζ measures the degree to which the state is mixed: Tr $\rho^2 = \zeta^{-1} \leq 1$, with unity for pure states. If the state is pure, $\rho = |\psi\rangle\langle\psi|$, and only two of the three symmetrized variances in (4.49) are independent.

The Gaussian ansatz for the density matrix is preserved under time evolution. In the Schrödinger picture ρ evolves according to the Liouville equation, $\dot{\rho} = -i[H, \rho]$. Substitution of the Gaussian form (4.48) into this equation with Hamiltonian (4.47) and equating coefficients of x, x', x^2, x'^2 and xx' gives five evolution equations for the five parameters specifying the Gaussian,

$$\begin{aligned} \bar{q}_{,t} &= \bar{p}\ ; \qquad \bar{p}_{,t} = -\omega^2(t)\bar{q} \\ \xi_{,t} &= \eta\ ; \qquad \eta_{,t} = -\omega^2(t)\xi + \tfrac{\hbar^2\zeta^2}{4\xi^3} \end{aligned} \tag{4.50}$$

and $\dot{\zeta} = 0$. Since ζ is a constant and the von Neumann entropy $-$Tr $\rho \ln \rho$ of the state (4.48) is a (monotonic) function of ζ alone, this quantity is also a constant of the motion. This establishes the equivalence between mean field methods and Gaussian density matrices for all evolutions of the form of equations (4.50).

An essential property of the evolution equations (4.50) is that they are Hamilton's equations (hence, time reversible) for an effective classical Hamiltonian [RajMar82], with η playing the role of the momentum conjugate to ξ,

$$H_{\rm eff}(\bar{q}, \bar{p}; \xi, \eta) = {\rm Tr}(\rho H) = \frac{1}{2}\left(\bar{p}^2 + \eta^2\right) + V_{\rm eff} \tag{4.51}$$

and $V_{\rm eff}(\bar{q}, \xi)$ depending on the particular form of $\omega^2(\bar{q}(t), \xi(t); t)$.

The unitary time evolution operator $U(t)$ for the density matrix (4.48),

$$\rho(t) = U(t)\rho(0)U^\dagger(t)\,, \quad U(t) = \exp\left(-i\hbar^{-1}\int_0^t H dt\right) \tag{4.52}$$

is given explicitly in the coordinate basis by

$$\langle x'|U(t)|x\rangle = (2\pi i\hbar v(t))^{-\frac{1}{2}} \exp\left\{\frac{i}{2\hbar v\,(t)}\left(u(t)x^2 + \dot{v}(t)x'^2 - 2xx'\right)\right\} \tag{4.53}$$

in terms of the two linearly independent solutions to the classical evolution equation,

$$\left(\frac{d^2}{dt^2} + \omega^2(t)\right)\begin{pmatrix} u \\ v \end{pmatrix} = 0\ ; \qquad \begin{aligned} u(0) = \dot{v}(0) = 1 \\ \dot{u}(0) = v(0) = 0 \end{aligned} \tag{4.54}$$

The Gaussian dynamics may be expressed as well by means of a Fock representation of the time-dependent Heisenberg operators,

$$\begin{aligned} q(t) &= U^\dagger(t)\, q(0)\, U(t) = \bar{q}(t) + af(t) + a^\dagger f^*(t) \\ p(t) &= U^\dagger(t)\, p(0)\, U(t) = \bar{p}(t) + a\dot{f}(t) + a^\dagger \dot{f}^*(t) \end{aligned} \tag{4.55}$$

where $[a, a^\dagger] = 1$. The complex mode functions f satisfy the evolution equation (4.54) and the Wronskian condition (4.17). This shows that Gaussian time evolution is essentially classical, with $\hbar$ appearing only in the time-independent condition (4.17) enforcing the quantum uncertainty relation.

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

Time-dependent basis

One can choose a basis in which all expectation values vanish, except

$$\langle a^\dagger a\rangle = \langle aa^\dagger\rangle - 1 \equiv N \geq 0 \tag{4.56}$$

The Gaussian density matrix is diagonal in the corresponding $a^\dagger a$ time-independent number basis,

$$\langle n'|\rho|n\rangle = \frac{2\delta_{n'n}}{\zeta+1}\left(\frac{\zeta-1}{\zeta+1}\right)^n \tag{4.57}$$

with $\zeta = 2N+1 = \mathcal{A}$ (the parametric amplification factor as introduced in equation (4.20)) and $\xi^2(t) = \zeta|f(t)|^2$. Upon identifying $\zeta = \coth(\hbar\omega/2k_\mathrm{B}T)$, the diagonal form (4.57) will be recognized as a thermal density matrix at temperature T. The pure state Gaussian wavefunction ($\zeta = 1$) corresponds therefore to a coherent, squeezed zero temperature vacuum state. The smoothness of the finite temperature classical limit $\hbar\zeta \to 2k_\mathrm{B}T/\omega$ as $\hbar \to 0$, $\zeta \to \infty$ shows that quantum and thermal fluctuations are treated by the mean field approximation in a unified way.

Instantaneous diagonalization

It is always possible to diagonalize (4.47) at any given time, bringing the quadratic Hamiltonian into the standard harmonic oscillator form, $H_\mathrm{osc} = \frac{\hbar\omega}{2}\left(\tilde{a}\tilde{a}^\dagger + \tilde{a}^\dagger\tilde{a}\right)$ with $\tilde{a}$ time dependent. This time-dependent basis is defined by the relations,

$$\begin{aligned} q(t) &= \tilde{a}\tilde{f} + \tilde{a}^\dagger\tilde{f}^*, \qquad p(t) = -i\omega\tilde{a}\tilde{f} + i\omega\tilde{a}^\dagger\tilde{f}^* \\ \tilde{f}(t) &= \sqrt{\frac{\hbar}{2\omega(t)}}\exp\left(-i\int_0^t dt'\omega(t')\right) \end{aligned} \tag{4.58}$$

in place of (4.55). In the $\tilde{a}^\dagger\tilde{a}$ number basis, ρ is no longer diagonal, $\langle\tilde{a}\rangle$, $\langle\tilde{a}\tilde{a}\rangle$, etc. are nonvanishing, and $\tilde{N} \equiv \langle\tilde{a}^\dagger\tilde{a}\rangle \neq N$ in general, becoming equal only in the static case of constant ω. As cautioned by Fulling [Ful89] this is the incorrect way to establish a quantum field theory in dynamical backgrounds.

Adiabatic basis

If $\omega(t)$ varies slowly in time, an adiabatic invariant may be constructed from the Hamilton–Jacobi equation corresponding to the effective classical Hamiltonian (4.51). By a simple quadrature we find the adiabatic invariant,

$$\frac{W}{2\pi\hbar} = \frac{\langle H\rangle}{\hbar\omega} - \frac{\zeta}{2} = \tilde{N}(t) - N \tag{4.59}$$

Since N is time independent, $\tilde{N}(t)$ is an adiabatic invariant of the evolution. On the other hand, the phase angle conjugate to the action variable W varies rapidly in time. Since the diagonal matrix elements of ρ in the $\tilde{N}$ basis are independent of this phase angle, they are slowly varying, whereas the *off-diagonal* matrix elements of ρ in this basis (which depend on the phase angle) are *rapidly* varying functions of time. If we are interested only in the effects of the fluctuations on the more slowly varying mean fields it is natural to define an *effective* density matrix $\rho_{\rm eff}(t)$ by *time-averaging* the density matrix (4.48), thereby truncating ρ to its diagonal elements only, in the adiabatic $\tilde{N}$ basis [HuPav86, Kan88a, Kan88b]. Clearly, for this truncation to be justified there must be very efficient phase cancellation, *i.e. dephasing*, either by averaging the fluctuations over time or by summing over many independent fluctuating degrees of freedom at a fixed time. This is perhaps the most direct way to understand the decoherence of the mean field. We shall discuss this issue in Chapter 8.

4.2 Particle production in external fields

After the above simple introduction we can begin to explore two classes of problems involving quantum fields in dynamical backgrounds. In this section we study the production of charged scalar particles in an external field, relevant to problems of collective excitations in QED plasma (and by extension to QCD quark–gluon processes). There are good introductions to this topic in standard texts, such as [ItzZub80]. In the next section we study a neutral scalar field in a dynamical spacetime, applicable to cosmological problems, such as vacuum particle creation at the Planck time or reheating after GUT (Grand-Unified Theory) scale inflationary expansion. Both problems have been studied extensively; the former began with the works of Klein [Kle29], Sauter [Sau31, Sau32], Heisenberg and Euler [HeiEul36], Schwinger [Sch51], and others [Greiner, GrMaMo88, FrGiSh91, Ginz87, Ginz95]; the latter by Parker, Sexl and Urbantke, Zel'dovich and Starobinsky, Fulling, Hu, and many others. For later and current developments, see [DeW75, BirDav82, Bordag]. For the first part in this section our treatment follows the work of Kluger, Mottola and Eisenberg [KlMoEi98]. For the second part in the next section, we follow the approach of Zel'dovich, Starobinsky [ZelSta71] and Hu [Hu72, Hu74, FuPaHu74].

Assuming that the electric field is spatially homogeneous, in the Coulomb gauge, we can express the vector potential as

$$\mathbf{A} = A(t)\hat{\mathbf{z}}, \;\; A_0 = 0 \tag{4.60}$$

and the electric field as

$$\mathbf{E} = -\dot{A}\hat{\mathbf{z}} = E\hat{\mathbf{z}} \tag{4.61}$$

Assuming also the field lives in a finite large volume V we can expand the charged scalar field operator in Fock space in Fourier modes. Since particles are physically distinct from antiparticles, we need two independent sets of

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

destruction operators

$$\Phi(\mathbf{x},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\varphi_{\mathbf{k}}(t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\left\{e^{i\mathbf{k}\cdot\mathbf{x}}f_{\mathbf{k}}(t)a_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{x}}f^*_{-\mathbf{k}}(t)b^\dagger_{\mathbf{k}}\right\} \tag{4.62}$$

Denote the time-independent annihilation operator of a particle in mode $\mathbf{k}$ by $a_{\mathbf{k}}$ and the creation of an antiparticle in mode $-\mathbf{k}$ by $b^\dagger_{\mathbf{k}}$. They obey the commutation relations

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = [b_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'} \tag{4.63}$$

Therefore

$$\begin{aligned} N_+(\mathbf{k}) &\equiv \langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle \\ N_-(\mathbf{k}) &\equiv \langle b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\rangle \end{aligned} \tag{4.64}$$

are the mean numbers of particles and antiparticles respectively. Without loss of generality we can make use of the freedom in defining the initial phases of the mode functions to set the correlation densities $\langle a_{\mathbf{k}} a_{\mathbf{k}}\rangle = \langle b_{\mathbf{k}} b_{\mathbf{k}}\rangle = 0$. In a Hamiltonian description we can take for each mode $\mathbf{k}$

$$\varphi_{\mathbf{k}}(t) \equiv f_{\mathbf{k}}(t)a_{\mathbf{k}} + f^*_{\mathbf{k}}(t)b^\dagger_{-\mathbf{k}} \tag{4.65}$$

as the (complex) generalized coordinates of the field Φ and

$$\pi_{\mathbf{k}}(t) = \dot\varphi^\dagger_{\mathbf{k}}(t) = \dot f^*_{\mathbf{k}}(t)a^\dagger_{\mathbf{k}} + \dot f_{\mathbf{k}}(t)b_{-\mathbf{k}} \tag{4.66}$$

as the momentum canonically conjugate to it. By virtue of the commutation relation (4.63) they obey the canonical commutation relation,

$$[\varphi_{\mathbf{k}}, \pi_{\mathbf{k}'}] = i\hbar\delta_{\mathbf{k}\mathbf{k}'} \tag{4.67}$$

provided that the mode functions satisfy the Wronskian condition (4.17).

The complex amplitude function $f_{\mathbf{k}}(t)$ of the $\mathbf{k}$th mode satisfies the equations of motion (4.22), where the time-dependent frequency $\omega^2_{\mathbf{k}}(t)$ is given by

$$\omega^2_{\mathbf{k}}(t) = (\mathbf{k} - e\mathbf{A})^2 + m^2 = (k_z - eA(t))^2 + k^2_\perp + m^2 \tag{4.68}$$

where k_z is the constant canonical momentum in the $\hat{\mathbf{z}}$ direction while the physical (gauge-invariant) kinetic momentum is given by

$$p_z(t) = k_z - eA(t); \dot p_z = -e\dot A = eE \tag{4.69}$$

(In the directions transverse to the electric field the kinetic and canonical momenta are the same: $p_\perp = k_\perp$.) Any function of the kinetic momenta contains these two components, e.g. $\omega(p_z, p_\perp) = \sqrt{p_z^2 + p_\perp^2 + m^2}$.

Since the definition of particle number becomes very different from that conceived in QFT in Minkowski space, especially in arbitrarily strong and rapidly

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

time-varying fields, it is often easier to deal with the conserved physical currents like $j(t)$ in an external field problem (or the stress–energy tensor $T_{\mu\nu}(x)$ in curved spacetimes). For a spatially homogeneous electric field (i.e. $\nabla\cdot\mathbf{E}=0$), by Gauss' law, the mean charge density must vanish,

$$j^0(t) = e\int d^3\mathbf{k}\,[N_+(\mathbf{k}) - N_-(-\mathbf{k})] = 0 \tag{4.70}$$

The mean current in the $\hat{\mathbf{z}}$ direction is

$$j(t) = 2e\int d^3\mathbf{k}\;[k_z - eA(t)]|f_{\mathbf{k}}(t)|^2(1 + N_+(\mathbf{k}) + N_-(-\mathbf{k})) \tag{4.71}$$

One can further restrict to the subspace of states for which

$$N_+(\mathbf{k}) = N_-(-\mathbf{k}) \equiv N_{\mathbf{k}} \tag{4.72}$$

Clearly the vacuum $N_+(\mathbf{k}) = N_-(-\mathbf{k}) = 0$ (as well as a thermal state) belongs to this class of states.

Particle pairs will be produced in a strong background field, and in turn, affect the strength and evolution of this background field. At the first level of sophistication (simplification), one can assume the background field (electric field or spacetime) is fixed in what is called a "test field" approximation (language also used in QFT in curved spacetime). At the second level, one looks for a self-consistent solution of the mean electric field $\mathbf{E}(t)$ (or the classical background spacetime) coupled to the expectation value of the current $j(t)$ of the quantum charged scalar field (or, in the case of cosmology, the energy–momentum tensor of the quantized matter field). This is known as the dynamical back-reaction problem. For the creation of charged particles in a homogeneous electric field, the back-reaction problem involves solving for the current $j(t)$ from the charge field $\varphi_{\mathbf{k}}(t)$, and using it as source in the Maxwell equation for the vector potential $\mathbf{A}$. In a spatially homogeneous electric field, the only nontrivial Maxwell equation is simply

$$-\dot{E}(t) = \ddot{A}(t) = j(t) \tag{4.73}$$

where the current is given by (4.71). Since the charged scalar field depends on the vector potential A to begin with, $f_{\mathbf{k}}(t)$ and $A(t)$ need to be solved self-consistently from equations (4.22) with (4.68) and (4.73).

4.2.1 Particle creation in a constant electric field

As a concrete example of particle creation in strong fields, let us review the well-known case of a uniform time-independent electric field worked out by Schwinger [Sch51]. There is a very detailed treatment of this problem in [KlMoEi98]. We may take E to be along the z direction with $A\,(t) = -Et$. The wave equation for the amplitude function of the kth mode can be written in terms of a new time τ (we omit the mode subscripts, since only one mode is considered):

$$\frac{d^2 f}{d\tau^2} + \omega^2(\tau) f = 0\;,\;\omega^2(\tau) = \nu_0^2 + \nu_1^4\tau^2 \tag{4.74}$$

where

$$\tau = t + \frac{k_z}{eE}, \ \nu_0^2 = k_\perp^2 + m^2, \ \nu_1^4 = e^2E^2 \tag{4.75}$$

We are interested in the strong-field case $\nu_1^2 \geq \nu_0^2$. It is obvious that the natural frequency is never constant. However, the second-order adiabatic frequency correction (4.38)

$$\epsilon_2 = -\frac{1}{\omega^{\frac{1}{2}}}\frac{d^2}{d\tau_1^2}(\omega^{\frac{1}{2}}) = \frac{\nu_1^4}{[\nu_0^2+\nu_1^4\tau^2]^2}\left\{\frac{3}{4} - \frac{5}{4}\frac{\nu_0^2}{[\nu_0^2+\nu_1^4\tau^2]}\right\} \tag{4.76}$$

is small provided $|\tau| \gg \nu_1^{-1} \geq \nu_0/\nu_1^2$. Therefore, for this problem, the zeroth-order adiabatic vacua already can provide a consistent particle definition both in the distant past and future.

Let us then consider the nth adiabatic order positive frequency solution $f_{(n)}^+$ in equation (4.46) with $n = 0$ and $W_0 = \omega$. To be precise, we adopt the convention that the WKB exponent (adiabatic phase) takes on the values

$$\Theta(\tau) \equiv \int_0^\tau \omega(\tau')\ d\tau' \ \ (\tau \geq 0) \quad \text{and} \quad \Theta(\tau) \equiv -\Theta(-\tau) \ \ (\tau < 0) \tag{4.77}$$

Computing the integral, we get the parametric form

$$\Theta_0 = \frac{\nu_0^2}{2\nu_1^2}[u + \sinh u\ \cosh u], \qquad \tau = \frac{\nu_0}{\nu_1^2}\sinh u \tag{4.78}$$

For large τ,

$$u \sim \ln\left[\frac{2\nu_1^2\tau}{\nu_0}\right] + O\left(\tau^{-2}\right) \tag{4.79}$$

$$\Theta = \frac{\nu_1^2}{2}\tau^2 + \frac{\nu_0^2}{2\nu_1^2}\ln\left[\frac{2\nu_1^2\tau}{\nu_0}\right] + \frac{\nu_0^2}{4\nu_1^2} + O\left(\tau^{-2}\right) \tag{4.80}$$

$$\omega \sim \nu_1^2\tau + O\left(\tau^{-1}\right) \tag{4.81}$$

Using equation (4.77) we obtain the corresponding form for $\tau \to -\infty$,

$$\Theta(\tau) \sim \frac{-\nu_1^2}{2}\tau^2 - \frac{\nu_0^2}{2\nu_1^2}\ln\left[\frac{2\nu_1^2|\tau|}{\nu_0}\right] - \frac{\nu_0^2}{4\nu_1^2} + O\left(\tau^{-2}\right) \tag{4.82}$$

$$\omega \sim \nu_1^2|\tau| + O\left(\tau^{-1}\right) \tag{4.83}$$

The asymptotic behavior of the WKB-approximate positive frequency mode function f^+ is

$$f^+(\tau) \sim \frac{\hbar^{1/2}}{\nu_1}\frac{1}{\sqrt{2|\tau|}}\left[\frac{2\nu_1^2|\tau|}{\nu_0}\right]^{i\nu_0^2/2\nu_1^2} \exp\left\{\frac{i\nu_1^2}{2}\tau^2 + \frac{i\nu_0^2}{4\nu_1^2}\right\} \ (\tau \to -\infty) \tag{4.84}$$

We define the positive frequency mode associated to the *in* vacuum as the exact solution f_{in} of equation (4.74) which matches this behavior in the distant past.

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

Similarly, for $\tau \to \infty$,

$$f^{+}(\tau) \sim \frac{\hbar^{1/2}}{\nu_1}\frac{1}{\sqrt{2\tau}}\left[\frac{2\nu_1^2\tau}{\nu_0}\right]^{-i\nu_0^2/2\nu_1^2} \exp\left\{\frac{-i\nu_1^2}{2}\tau^2 - \frac{i\nu_0^2}{4\nu_1^2}\right\} \quad (\tau \to \infty) \tag{4.85}$$

and we define the positive frequency mode associated with the *out* vacuum as the exact solution f_{out} of equation (4.74) which matches this behavior in the distant future. The whole point of the analysis is that $f_{in} \neq f_{out}$.

A basis of solutions of equation (4.74) is given by the parabolic cylinder functions $D_p(z)$ and its conjugate $D_{p^*}(z^*)$, where $z = (-1+i)\nu_1\tau$ and $p = \left(i(\nu_0/\nu_1)^2 - 1\right)/2$. When $\tau \to -\infty$, $z \sim \sqrt{2}\nu_1 |\tau| e^{-i\pi/4}$, and

$$\begin{aligned} D_p(z) \sim z^p e^{-z^2/4} = \left(\sqrt{2}\nu_1|\tau|\right)^{-1/2}\left(\sqrt{2}\nu_1|\tau|\right)^{i\nu_0^2/2\nu_1^2} \\ \times e^{i\pi/8} e^{\pi(\nu_0/\nu_1)^2/8}\exp\left\{\frac{i}{2}(\nu_1\tau)^2\right\} \end{aligned} \tag{4.86}$$

$(\tau \to -\infty)$. Comparing with the corresponding expansion of f^+, equation (4.84), we find that the normalized mode function associated with the *in* vacuum is

$$f_{in} = \frac{\hbar^{1/2}}{\sqrt{\sqrt{2}\nu_1}}\left[\frac{\sqrt{2}\nu_1}{\nu_0}\right]^{i\nu_0^2/2\nu_1^2} e^{-i\pi/8}e^{-\pi(\nu_0/\nu_1)^2/8}\exp\left\{\frac{i\nu_0^2}{4\nu_1^2}\right\} D_p(z) \tag{4.87}$$

When $\tau \to \infty$, $z \sim \sqrt{2}\nu_1\tau\, e^{3i\pi/4}$, and

$$\begin{aligned} D_p(z) &\sim z^p e^{-z^2/4} - \frac{\sqrt{2\pi}}{\Gamma[-p]}e^{i\pi p}z^{-p-1}e^{z^2/4} \\ &= \left(\sqrt{2}\nu_1\tau\right)^{-1/2} e^{-3i\pi/8}\left\{\left(\sqrt{2}\nu_1\tau\right)^{i\nu_0^2/2\nu_1^2} e^{-3\pi(\nu_0/\nu_1)^2/8}\exp\left\{\frac{i}{2}(\nu_1\tau)^2\right\}\right. \\ &\quad \left. - \frac{\sqrt{2\pi}}{\Gamma[-p]}e^{-i\pi/2}\left(\sqrt{2}\nu_1\tau\right)^{-i\nu_0^2/2\nu_1^2} e^{-\pi(\nu_0/\nu_1)^2/8}\exp\left\{-\frac{i}{2}(\nu_1\tau)^2\right\}\right\} \end{aligned} \tag{4.88}$$

$(\tau \to \infty)$. Substituting this into equation (4.87) and comparing with the development equation (4.85) we find that f_{in} and f_{out} are related in a way given exactly by the Bogoliubov transformation

$$f_{in} = \alpha\, f_{out} + \beta\, f_{out}^* = \alpha\, f^+ + \beta\, f^- \qquad (\tau \to \infty) \tag{4.89}$$

where f^- is the corresponding negative frequency solution of the same adiabatic order [the term with coefficient B in (4.45)]. Note that the first identity actually holds everywhere. Hence we can identify the Bogoliubov coefficients as

$$\alpha = \left[\frac{\sqrt{2}\nu_1}{\nu_0}\right]^{i\nu_0^2/\nu_1^2}\exp\left\{\frac{i\nu_0^2}{2\nu_1^2}\right\}\left(\frac{\sqrt{2\pi}}{\Gamma[-p]}\right)e^{-\pi(\nu_0/\nu_1)^2/4} \tag{4.90}$$

and

$$\beta = e^{-i\pi/2}e^{-\pi(\nu_0/\nu_1)^2/2} \tag{4.91}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

As a check, observe that

$$|\alpha|^2 = 2\cosh\left[\frac{\pi}{2}\left(\frac{\nu_0}{\nu_1}\right)^2\right]\exp\left[\frac{-\pi}{2}\left(\frac{\nu_0}{\nu_1}\right)^2\right] \tag{4.92}$$

$$|\beta|^2 = \exp\left[-\pi\left(\frac{\nu_0}{\nu_1}\right)^2\right] \tag{4.93}$$

obeys the Wronskian condition $|\alpha|^2 - |\beta|^2 = 1$.

It is clear that if we set up the quantum state to be the *in* vacuum, when we arrive at the *out* region we find

$$N = |\beta|^2 = \exp\left\{-\pi\left(\frac{k_\perp^2 + m^2}{eE}\right)\right\} \tag{4.94}$$

particles in each mode. This is Schwinger's celebrated result [Sch51].

4.3 Spontaneous and stimulated production

So far we have focused on how to define a physically meaningful vacuum state and the number of particles produced in a changing external field or dynamical spacetime. We learn how to define adiabatic vacuum states in a dynamical setting, via the adiabatic expansion. The nth order adiabatic number state is well-defined to the nth adiabatic order. In this section we will show how to derive the energy density of these particles produced in adiabatic orders. A related problem is the identification and subtraction of ultraviolet divergences in the stress–energy tensor of quantum fields in a dynamical background. Here we will explain how to apply the adiabatic method introduced above in what is called the adiabatic regularization scheme.

We begin with a formal rendition to the parametric oscillator equation (4.22) describing the amplitude function of the kth normal mode. We want an expression of $s_{\mathbf{k}} \equiv |\beta_{\mathbf{k}}|^2$ in terms of $|f_{\mathbf{k}}|$ and $|\dot{f}_{\mathbf{k}}|$. Here following [ZelSta71, Hu74] we seek a solution in the form:

$$f_{\mathbf{k}}(t) = \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}}\left\{\alpha_{\mathbf{k}} e_{\mathbf{k}}^- + \beta_{\mathbf{k}} e_{\mathbf{k}}^+\right\}; \quad e_{\mathbf{k}}^{\pm} \equiv \exp\left\{\pm i \int \omega_{\mathbf{k}} dt\right\} \tag{4.95}$$

The two functions $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}$ are the positive and negative frequency components of a formal solution $f_{\mathbf{k}}$, but without a well-defined vacuum they do not convey the meaning of particles and antiparticles, as we forewarned with regard to the Bogoliubov coefficients. Since the single equation (4.95) does not determine the coefficients $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ uniquely, we need another condition, which is chosen so that the Wronskian condition equation (4.17) is satisfied. The auxiliary condition imposed on $\dot{f}_{\mathbf{k}}$ is

$$\dot{f}_{\mathbf{k}}(t) = -i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2}}\left(\alpha_{\mathbf{k}} e_{\mathbf{k}}^- - \beta_{\mathbf{k}} e_{\mathbf{k}}^+\right) \tag{4.96}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

Inverting these two equations we can express the complex function $\beta_{\mathbf{k}}$ in terms of $|f_{\mathbf{k}}|^2$, $|\dot{f}_{\mathbf{k}}|^2$ as follows:

$$\alpha_{\mathbf{k}} = \sqrt{\frac{\omega_{\mathbf{k}}}{2\hbar}}\left(f_{\mathbf{k}} + \frac{i}{\omega_{\mathbf{k}}}\dot{f}_{\mathbf{k}}\right)e^{+}_{\mathbf{k}}\,, \qquad \beta_{\mathbf{k}} = \sqrt{\frac{\omega_{\mathbf{k}}}{2\hbar}}\left(f_{\mathbf{k}} - \frac{i}{\omega_{\mathbf{k}}}\dot{f}_{\mathbf{k}}\right)e^{-}_{\mathbf{k}} \tag{4.97}$$

Making use of the Wronskian condition we obtain

$$s_{\mathbf{k}} \equiv \mid \beta_{\mathbf{k}} \mid^2 = \frac{1}{2\hbar\omega_{\mathbf{k}}}\left(\left|\dot{f}_{\mathbf{k}}\right|^2 + \omega_{\mathbf{k}}^2\,|f_{\mathbf{k}}|^2\right) - \frac{1}{2} \tag{4.98}$$

It is tempting to regard $s_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2$ as the amount of particle production. However, we need to be careful that the vacuum state is well-defined to make sense of particles. To which adiabatic order one needs to carry out the expansion is determined by the physical conditions (foremost how rapidly the natural frequency changes) of the system and by the accuracy demanded in its description. For slowly varying fields if one is interested in problems concerning the adiabatic particle number or mean current distribution as used in quantum kinetic theory [GrLeWe80] (low particle creation rate and minimal phase information) the adiabatic number state of [KlMoEi98] to be introduced in a later section, which is in the lowest adiabatic order, will suffice.

4.3.1 Spontaneous production

The energy–momentum tensor of a massive scalar field in flat space is

$$T^{\text{Mink}}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi - \frac{1}{2}\eta_{\mu\nu}m^2\phi^2 \tag{4.99}$$

The energy density associated with these particles is given by the expectation value of the 00 component of $T_{\mu\nu}$ with respect to the Minkowski vacuum, i.e.

$$\rho_0^{\text{Mink}} \equiv \langle 0 \mid T_{00} \mid 0\rangle = \int \frac{d^3\mathbf{k}}{2(2\pi)^3}(\mid \dot{f}_{\mathbf{k}} \mid^2 + \omega_{\mathbf{k}}^2 \mid f_{\mathbf{k}} \mid^2) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}(2s_{\mathbf{k}}+1)\frac{\hbar\omega_{\mathbf{k}}}{2} \tag{4.100}$$

In a Hamiltonian description of the dynamics of a finite system of parametric oscillators, the Hamiltonian is simply

$$H^{\text{Mink}}(t) = \frac{1}{2}\sum_k(\pi_{\mathbf{k}}^2 + \omega_{\mathbf{k}}^2 q_{\mathbf{k}}^2) = \sum_k\left(N_{\mathbf{k}} + \frac{1}{2}\right)\hbar\omega_{\mathbf{k}} \tag{4.101}$$

Comparing this with (4.100) one can identify $\mid f_{\mathbf{k}} \mid^2$ and $\mid \dot{f}_{\mathbf{k}} \mid^2$ with the canonical coordinates $q_{\mathbf{k}}^2$ and moment $\pi_{\mathbf{k}}^2$, the eigenvalue of H_0 being the energy $E_{\mathbf{k}} = (N_{\mathbf{k}} + \frac{1}{2})\hbar\omega_{\mathbf{k}}$. The analogy of particle creation with parametric amplification is formally clear: equation (4.98) defines the number operator

$$N_{\mathbf{k}}(t) = \frac{1}{2\hbar\omega_{\mathbf{k}}}(\pi_{\mathbf{k}}^2 + \omega_{\mathbf{k}}^2 q_{\mathbf{k}}^2) - \frac{1}{2} = s_{\mathbf{k}} \tag{4.102}$$

and equation (4.100) says that the energy density of vacuum particle creation comes from the amplification of vacuum fluctuations $\hbar\omega_{\mathbf{k}}/2$ by the factor $\mathcal{A}_{\mathbf{k}} = 2s_{\mathbf{k}} + 1$. Now it is easy to recognize that the Minkowski result in equation (4.21) corresponds to $\mathcal{A}_{\mathbf{k}} = 1$, no particle creation or zero amplification.

In general there are ultraviolet divergences appearing in the integral (4.100) which requires a subtraction scheme. The adiabatic method comes in handy for such a task, because as we have explained before, the lowest few orders of the WKB solutions encompass particle production from the high-frequency range downwards in the spectrum. This is just what one needs for the subtraction of ultraviolet divergences. For renormalization of the energy–momentum tensor of quantum fields in curved spacetimes the zeroth, second and fourth adiabatic order expressions give the quartic, quadratic and logarithmic divergences. We will discuss this method in the context of cosmological particle creation in Section 4.6. To facilitate adoption of the formula there for flat space field theory in a dynamical background field, just replace χ by ϕ, η by t (thus primes by overdots) and set $a = 1$.

A quantity which enters in the expressions for the adiabatic expansion of the energy–momentum tensor of quantum fields is the *nonadiabaticity parameter* defined as (for the **k**th mode with natural frequency $\omega_{\mathbf{k}}$ in t time) $\bar{\omega}_{\mathbf{k}} \equiv \dot{\omega}_{\mathbf{k}}/\omega_{\mathbf{k}}^2$. Particle production is more pronounced in modes which evolve nonadiabatically, i.e. $\bar{\omega}_{\mathbf{k}}(t) \simeq 1$ (or $\bar{\omega}_{\mathbf{k}}(\eta) \simeq 1$ in the conformal wave equation of Section 4.6). Thus particle production is a nonadiabatic process. We will learn soon that it is also a non-Markovian process (nonlocal in time, memory, or history, dependent).

4.3.2 Stimulated production

Equation (4.98) gives the vacuum energy density of particles produced from an initial vacuum, a pure state. If the initial state at t_0 is a statistical mixture of pure states, each of which contains a definite number of particles, then an additional mechanism of particle creation enters. This is known as induced or stimulated creation. In particular, if the statistical density matrix μ is diagonal in the representation whose basis consists of the eigenstates of the number operators $a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}$ at time t_0, then for bosons this process increases the average number of particles (in mode **k** in a unit volume) at a later time t over and above the initial amount present. From (4.28) we have

$$\tilde{N} \equiv \langle N_{\mathbf{k}}(t)\rangle_t = \mathrm{Tr}[\mu\tilde{a}_{\mathbf{k}}^{\dagger}(t)\tilde{a}_{\mathbf{k}}(t)] = \langle N_{\mathbf{k}}(t_0)\rangle + \mid \beta_{\mathbf{k}}(t) \mid^2 [1 + 2\langle N_{\mathbf{k}}(t_0)\rangle] \quad (4.103)$$

where angular brackets without a subscript t refers to that taken at the initial time t_0, $\langle N_{\mathbf{k}}(t_0)\rangle = \mathrm{Tr}[\mu a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}]$, if the system is in a pure state at t_0. For fermions induced (or stimulated) particle creation decreases the initial number.

The above result can be understood in the parametric oscillator description as the sum of two parts: First, the amount $s_{\mathbf{k}} = |\beta_{\mathbf{k}}(t)|^2$ from spontaneous production of particles from the amplification of vacuum fluctuations by the factor

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

$\mathcal{A}_{\mathbf{k}} = 2s_{\mathbf{k}} + 1$. Second, an amplification by the same factor $\mathcal{A}_{\mathbf{k}}$, of the particles already present $N_{\mathbf{k}}(t_0)$, i.e.

$$\langle N_{\mathbf{k}}(t)\rangle_t = \mid \beta_{\mathbf{k}}(t) \mid^2 + \mathcal{A}_{\mathbf{k}}\langle N_{\mathbf{k}}(t_0)\rangle \tag{4.104}$$

where $s_{\mathbf{k}} = |\beta_{\mathbf{k}}(t)|^2$. The second part is called stimulated production. It yields an energy density ρ_n with respect to the n-particle state at t_0 given by

$$\begin{aligned}\rho_n^{\text{Mink}} = \langle n \mid T_{00} \mid n\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}(\mid \dot{f}_{\mathbf{k}} \mid^2 + \omega_{\mathbf{k}}^2 \mid f_{\mathbf{k}} \mid^2)\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}(2s_{\mathbf{k}} + 1)\hbar\omega_{\mathbf{k}}\langle N_{\mathbf{k}}(t_0)\rangle \end{aligned} \tag{4.105}$$

Combining (4.100) and (4.105), for a density matrix diagonal in the number state, the total energy density of particles created from the vacuum and from those already present in the n-particle state is given by

$$\begin{aligned}\rho^{\text{Mink}} = \rho_0^{\text{Mink}} + \rho_n^{\text{Mink}} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}(\mid \dot{f}_{\mathbf{k}} \mid^2 + \omega_{\mathbf{k}}^2 \mid f_{\mathbf{k}} \mid^2)\left(\frac{1}{2} + \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle\right) \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\mathcal{A}_{\mathbf{k}}\hbar\omega_{\mathbf{k}}\left(\frac{1}{2} + \langle N_{\mathbf{k}}(t_0)\rangle\right) \end{aligned} \tag{4.106}$$

This can be understood as the result of parametric amplification by the factor $\mathcal{A}_{\mathbf{k}}$ of the energy density of vacuum fluctuations $\hbar\omega_{\mathbf{k}}/2$ plus that of the particles originally present in the $\mathbf{k}$th mode at t_0, i.e. $\langle N_{\mathbf{k}}(t_0)\rangle\hbar\omega_{\mathbf{k}}$.

4.4 Quantum Vlasov equation

Having familiarized ourselves with the general scheme of adiabatic vacuum and number states, we now return to the problem of charged particle production in an external electromagnetic field. We continue to follow the treatment given by Kluger, Mottola and Eisenberg [KlMoEi98] for pedagogical advantage.

4.4.1 Adiabatic number state

An adiabatic number state $\tilde{f}^+_{\mathbf{k}(0)}(t)$ was suggested by [KlMoEi98] for the description of a kinetic theory of charged particles moving in an electromagnetic field. That corresponds to the $n = 0$ adiabatic state defined in equation (4.46)

$$\tilde{f}_{\mathbf{k}}^{(0)}(t)\ \left(= f^+_{\mathbf{k}(0)} \text{ equation (4.46)}\right) \equiv \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}(t)}}\exp\left(-i\Theta_{\mathbf{k}(n=0)}\right) \tag{4.107}$$

where $\Theta_{\mathbf{k}(n=0)} \equiv \int^t \omega_{\mathbf{k}}(t')dt'$ is the $(n = 0)$th-order adiabatic phase. At this level of accuracy one measures particle numbers at all times with respect to the initial vacuum state at time t_0. This definition of a number state makes use of the fact that under adiabatic evolution, particle number is an adiabatic invariant. This restricts its validity from the start to weak or slowing varying background fields.

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

The adiabatic particle number is defined to be [KlMoEi98]

$$\begin{aligned}\tilde{N}_{\mathbf{k}}(t) &\equiv \langle \tilde{a}^\dagger_{\mathbf{k}}(t)\tilde{a}_{\mathbf{k}}(t)\rangle + \langle \tilde{b}^\dagger_{-\mathbf{k}}(t)\tilde{b}_{-\mathbf{k}}(t)\rangle = |\alpha_{\mathbf{k}}|^2\langle a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\rangle + |\beta_{\mathbf{k}}|^2\langle b_{-\mathbf{k}}b^\dagger_{-\mathbf{k}}\rangle \\ &= \left(1+|\beta_{\mathbf{k}}|^2\right)N_+(\mathbf{k}) + |\beta_{\mathbf{k}}|^2\left(1+N_-(-\mathbf{k})\right) \\ &= |\beta_{\mathbf{k}}|^2 + (1+2|\beta_{\mathbf{k}}|^2)N_{\mathbf{k}} = N_{\mathbf{k}} + (1+2N_{\mathbf{k}})\;|\beta_{\mathbf{k}}(t)|^2 \end{aligned} \tag{4.108}$$

where the last line is valid only if the number of positive and negative charges are equal (cf. (4.72)). To verify that $\tilde{N}_{\mathbf{k}}$ is an adiabatic invariant we show that it is proportional to the ratio of the energy to frequency for any mode $\mathbf{k}$, $\epsilon_{\mathbf{k}}(t)/\hbar\omega_{\mathbf{k}}(t)$, which is known as such for a harmonic oscillator with time-dependent frequency. After the discussions on spontaneous and stimulated production we can actually read off this expression from (4.106): Viewing $\int d^3\mathbf{k}/(2\pi)^3$as $1/V$, the inverse volume, the integrand there is the energy in mode $\mathbf{k}$. Dividing by $\hbar\omega$ and multiplying it by 2 for the presence of both $\pm$ charges gives the expression we are looking for:

$$\frac{\epsilon_{\mathbf{k}}(t)}{\hbar\omega_{\mathbf{k}}(t)} = 1 + 2\tilde{N}_{\mathbf{k}}(t) \tag{4.109}$$

The amount of particle production at time t in this basis is given by the expectation value of the number operator $\tilde{a}^\dagger\tilde{a}$ at time t with respect to the vacuum state $|\rangle_0$ defined at t_0 (not the vacuum state $|\rangle_t$ defined at t). As discussed above this is fine if the vacuum states are well defined at the initial t_0 *and* final times t, as in an asymptotically-static evolution. Otherwise one needs to specify the adiabatic order to make the vacuum well-defined: the adiabatic number state of [KlMoEi98] corresponds to the lowest adiabatic order.

From earlier discussions, we know that this level of approximation will not give a good measure for on-going particle creation, as particle creation is basically a nonadiabatic process. It is however useful for quantum kinetic theory descriptions, where a quasi-particle approximation is usually introduced which amounts to incorporating only the quantum radiative corrections to the particles but not fully field theoretical effects such as particle creation. In other words, quantum kinetic theory is usually treated at the same level of approximation described by the adiabatic number basis. We will describe quantum kinetic *field* theory in Chapter 11.

4.4.2 Number and correlation

We now proceed to derive an equation for the time rate of change of the number of particles created in each mode with respect to the time-dependent particle number basis. Differentiating (4.108), we obtain

$$\frac{d}{dt}\tilde{N}_{\mathbf{k}} = 2\,(1+2N_{\mathbf{k}})\,\mathrm{Re}\,(\beta^*_{\mathbf{k}}\dot{\beta}_{\mathbf{k}}) \tag{4.110}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

We need an expression for $\dot{\beta}_{\mathbf{k}}$ in terms of α, β and $\Theta_{\mathbf{k}0}(t) \equiv \int^t \omega_{\mathbf{k}}(t')dt'$ (we will omit the subscript 0 on Θ_0 in this subsection). To do so we use equations (4.97) and (4.22) to get

$$\dot{\alpha}_{\mathbf{k}} = \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\beta_{\mathbf{k}}\exp(2i\Theta_{\mathbf{k}}), \quad \dot{\beta}_{\mathbf{k}} = \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\alpha_{\mathbf{k}}\exp(-2i\Theta_{\mathbf{k}}) \tag{4.111}$$

thus

$$\frac{d}{dt}\tilde{N}_{\mathbf{k}} = \frac{\dot{\omega}_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(1 + 2N_{\mathbf{k}}\right) \mathrm{Re}\left\{\alpha_{\mathbf{k}}\beta^*_{\mathbf{k}}\exp(-2i\Theta_{\mathbf{k}})\right\} = \frac{\dot{\omega}_{\mathbf{k}}}{\omega_{\mathbf{k}}}\mathrm{Re}\left\{\mathcal{C}_{\mathbf{k}}\exp(-2i\Theta_{\mathbf{k}})\right\} \tag{4.112}$$

where we have defined the time-dependent pair correlation function

$$\mathcal{C}_{\mathbf{k}}(t) \equiv \langle\tilde{a}_{\mathbf{k}}(t)\tilde{b}_{-\mathbf{k}}(t)\rangle = \left(1 + 2N_{\mathbf{k}}\right)\alpha_{\mathbf{k}}\beta^*_{\mathbf{k}} \tag{4.113}$$

The pair correlation $\mathcal{C}_{\mathbf{k}}(t)$ is a very rapidly varying function, since the time-dependent phases on the right side of (4.113) *add* rather than cancel. The phases, however, nearly cancel in the final combination of (4.112) to render $\tilde{N}_{\mathbf{k}}$ a slowly varying function. The time derivative of the pair correlation function is given by

$$\frac{d}{dt}\mathcal{C}_{\mathbf{k}} = \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\left(1 + 2N_{\mathbf{k}}\right)\exp(2i\Theta_{\mathbf{k}})\,\left(1 + 2|\beta_{\mathbf{k}}|^2\right) = \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\left(1 + 2\tilde{N}_{\mathbf{k}}\right)\exp(2i\Theta_{\mathbf{k}}) \tag{4.114}$$

4.4.3 Current and energy density

To obtain the current (4.71) in terms of the particle number and its time derivative, we need to express $|f_{\mathbf{k}}(t)|^2$ in terms of the Bogoliubov coefficients. For this we use equations (4.98), (4.96) and obtain

$$j(t) = e\hbar\int d^3\mathbf{k}\frac{(k_z - eA(t))}{\omega_{\mathbf{k}}(t)}(1 + 2|\beta_{\mathbf{k}}(t)|^2 + 2\mathrm{Re}\{\alpha_{\mathbf{k}}\beta^*_{\mathbf{k}}e^{-2i\Theta_{\mathbf{k}}(t)}\})(1 + 2N_{\mathbf{k}}) \tag{4.115}$$

The vacuum term in this expression, $\int d^3\mathbf{k}\,(k_z - eA(t))\,/\omega_{\mathbf{k}}(t)$, vanishes by charge conjugation symmetry, when proper gauge invariant integration boundaries are chosen. Using the mean value of particles in the adiabatic number basis (4.108), its time derivative and the equations of motion (4.112), we can rewrite the current as

$$j(t) = 2e\hbar\int d^3\mathbf{k}\frac{(k - eA(t))}{\omega_{\mathbf{k}}(t)}\tilde{N}_{\mathbf{k}}(t) + \frac{2\hbar}{E}\int d^3\mathbf{k}\,\omega_{\mathbf{k}}(t)\frac{d\tilde{N}_{\mathbf{k}}}{dt}(t) = j_{\mathrm{cond}} + j_{\mathrm{pol}} \tag{4.116}$$

Classically, if the particle distribution $\tilde{N}_{\mathbf{k}}$ is coupled to a uniform electric field the energy density and its time derivative are given by

$$\varepsilon = \frac{E^2}{2} + 2\int d^3\mathbf{k}\;\hbar\omega_{\mathbf{k}}\tilde{N}_{\mathbf{k}} \tag{4.117a}$$

$$\dot{\varepsilon} = \dot{E}E + 2\int d^3\mathbf{k}\left(e\hbar E\frac{(k - eA)}{\omega_{\mathbf{k}}}\tilde{N}_{\mathbf{k}} + \omega_{\mathbf{k}}\hbar\frac{d\tilde{N}_{\mathbf{k}}}{dt}\right) = 0 \tag{4.117b}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

Using the Maxwell equation $-\dot{E} = j$ this last relation is precisely the same as the mean value of the quantum current in (4.116). Hence we may identify the adiabatic particle number $\tilde{N}_{\mathbf{k}}(t)$ with the (quasi) classical single-particle distribution. This is the starting point of a quantum kinetic theory description.

4.4.4 Quantum Vlasov equation

Let us return now to the two equations for the rates of change of the particle number and the quantum correlations. Solving equation (4.114) formally for $\mathcal{C}_{\mathbf{k}}$, assuming that $\mathcal{C}_{\mathbf{k}}$ vanishes at some $t = t_0$ which could be taken to $-\infty$, and substituting into (4.112) we obtain

$$\frac{d}{dt}\tilde{N}_{\mathbf{k}} = \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \int_{t_0}^{t} dt' \left\{ \frac{\dot{\omega}_{\mathbf{k}}}{\omega_{\mathbf{k}}}(t') \left(1 + 2\tilde{N}_{\mathbf{k}}(t')\right) \cos\left[2\Theta_{\mathbf{k}}(t) - 2\Theta_{\mathbf{k}}(t')\right] \right\} \tag{4.118}$$

Equation (4.118) may be called a "quantum Vlasov equation," in the sense that it gives the rate of particle creation in an arbitrary time-varying mean field. Note the appearance of the Bose enhancement factor $(1 + 2\tilde{N}_{\mathbf{k}})$ in (4.118) indicates that both spontaneous and induced particle creation are present. One important feature of equation (4.118) is that it is nonlocal in time, the particle creation rate depending on the entire previous history of the system. Thus particle creation in general is a *non-Markovian* process [BirDav82, Rau94, RauMue96, SRSBTP97]. Note that the nonlocal form of (4.118) results from solving one variable $\mathcal{C}$ in terms of the other $\tilde{N}$, each obeying a Hamiltonian equation of motion. This is a general feature of coupled subsystems.

Equation (4.118) becomes exact in the limit in which the electric field can be treated classically, i.e. the limit in which real and virtual photon emission is neglected, and there is no scattering. We will learn later that this *semiclassical limit* is obtained at the leading order of a large N approximation [CoJaPo74, Roo74].

Inclusion of scattering processes leads to collision terms on the right side of (4.118) which are also nonlocal in general. This nonlocality is essential to the quantum description in which phase information is retained for all times. The phase oscillations in the cosine term are a result of the quantum coherence between the created pairs, which must be present in principle in any unitary evolution. However, precisely because these phase oscillations are so rapid it is clear that the integral in (4.118) receives most of its contribution from t' close to t, which suggests that some local approximation to the integral should be possible, provided that we are not interested in resolving the short-time structure or measuring the phase coherence effects. The time-scale for these quantum phase coherence effects to wash out is the time-scale of several oscillations of the phase factor $\Theta_{\mathbf{k}}(t) - \Theta_{\mathbf{k}}(t')$, which is of order $\tau_{qu} = 2\pi/\omega_{\mathbf{k}} = 2\pi\hbar/\epsilon_{\mathbf{k}}$, where $\epsilon_{\mathbf{k}}$ is the single-particle energy.

We will return to this equation in Chapter 9 to construct the density matrix and discuss entropy generation in these quantum field processes.

4.5 Periodically driven fields

As another example of particle production from parametric amplification, we give in this section a brief discussion of the solutions of equation (4.22) in the important case when the natural frequency depends periodically on time, that is, $\omega^2(t+T) = \omega^2(t)$ for some period T. Again we drop the mode label $\mathbf{k}$, as only one mode will be considered. This is a case of parametric resonance. In the mathematical literature, the corresponding problem is the subject of the so-called Floquet theory [WhiWat40, Inc56]. In physics there are many applications (e.g. [Shi65, MilWya83, MonPaz01]). One such area in cosmology which has drawn considerable attention is particle creation by parametric resonance during the preheating epoch after the universe came out of inflation, see Chapter 15. Our treatment here is influenced by the work of Kofman, Linde and Starobinsky [KoLiSt97].

The key insight is that, if $f(t)$ is a solution, then $f(T+t)$ is a solution too. If f_1 and f_2 are linearly independent solutions, then we must have

$$f_i(t+T) = A_{i1} f_1(t) + B_{i2} f_2(t) \qquad (i = 1, 2) \tag{4.119}$$

Thus there must exist solutions $F_1(t)$, $F_2(t)$ such that

$$F_i(t+T) = e^{\mu_i T} F_i(t) \tag{4.120}$$

or equivalently

$$F_i(t) = e^{\mu_i t} \tilde{f}_i(t) \tag{4.121}$$

where the functions $\tilde{f}_i(t)$ are periodic with period T. The eigenvalues μ_i are the so-called Floquet exponents. Sometimes Floquet energies $i\hbar\mu_i$ are introduced. As we shall see presently, the Floquet exponents may be real, leading to exponential amplification of the solution (or, in quantum language, exponential squeezing of the quantum state, see later in this chapter).

The second key insight is that if μ is a Floquet exponent, then $-\mu$ and μ^* must be exponents as well. The first follows from the fact that the Wronskian of two solutions must be a constant, and the second because the equation is real. So we only have two possibilities, either the Floquet exponents are imaginary and complex conjugate to each other, or real and opposite to each other. In the second case, we say there is parametric resonance.

To be concrete, we shall restrict ourselves to the Mathieu equation, which is obtained when

$$\omega^2(t) = \omega_0^2 + \omega_1^2 [1 + \cos\gamma t] \tag{4.122}$$

where ω_0, ω_1 and γ are constants. There are two interesting regimes, namely, the so-called broad resonance when $\omega_1 \gg \omega_0, \gamma$, and the narrow resonance when the opposite obtains. Of course, we may take $\gamma = 1$ with no loss of generality.

4.5.1 Broad resonance

In the broad resonance regime, $\omega \gg \omega_0, 1$ unless $t \sim (2j+1)\pi$, where j is an integer. The second-order adiabatic frequency correction is given by

$$\epsilon_2 = \left(-\frac{1}{4\omega_1^2}\right) \frac{1 + \left[1 + \left(\frac{\omega_0}{\omega_1}\right)^2\right] \cos t + \frac{1}{4}\sin^2 t}{\left[1 + \left(\frac{\omega_0}{\omega_1}\right)^2 + \cos t\right]^3} \tag{4.123}$$

which is much smaller than 1 unless $\cos t \sim -1$. Therefore we may describe the evolution as a series of adiabatic periods, separated by nonadiabatic transitions when $t \sim t_j = (2j+1)\pi$. Between transitions we may use the $(n=0)$ adiabatic function $\tilde{f}^0 = f_0^+$ of (4.107), there being no net amplification. Near t_j, we may approximate $\omega^2(t) = \omega_0^2 + \omega_1^2 (t - t_j)^2/2$. We have already encountered the resulting equation in our study of pair creation by a constant electric field.

Let $t_{k-1} < t \leq t_k$, and consider the exact solution f which behaves as a positive frequency $(+)$ lowest WKB order $(n=0)$ solution near t. For $t \geq t_k$ this solution plays the same role as the *in*-region positive frequency wave in the calculation of particle creation. Thus, for $t \geq t_k$, it assumes the form (cf. equation (4.89))

$$f = \alpha f_0^+ + \beta f_0^- \tag{4.124}$$

with α and β given in equations (4.90) and (4.91) respectively. Neglect any further evolution of the Bogoliubov coefficients, and write

$$f_0^+(t + 2\pi) = e^{-i\Theta_0} f_0^+(t) \tag{4.125}$$

Therefore

$$f(t + 2\pi) = \alpha e^{-i\Theta_0} f(t) + \beta e^{i\Theta_0} f^*(t) \tag{4.126}$$

The general solution is $F = Af + Bf^*$, and the eigenvalue condition (4.120) becomes a set of linear equations for the coefficients

$$\begin{pmatrix} \alpha e^{-i\Theta_0} & \beta^* e^{-i\Theta_0} \\ \beta e^{i\Theta_0} & \alpha^* e^{i\Theta_0} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} A \\ B \end{pmatrix} \tag{4.127}$$

where $\lambda = \exp(2\pi\mu)$. We see that the Floquet exponents must satisfy

$$\left(\alpha^* - \lambda e^{-i\Theta_0}\right)\left(\alpha - \lambda e^{i\Theta_0}\right) - |\beta|^2 = 0 \tag{4.128}$$

The condition for μ to be real is

$$\mathrm{Re}\left[\alpha e^{-i\Theta_0}\right] > 1 \tag{4.129}$$

We see that it is not sufficient to have $\beta \neq 0$.

4.5.2 Narrow resonance

Let us now consider the case of narrow resonance. Consider the case when at the boundary of a resonant region the Floquet exponents vanish, meaning that there are purely periodic solutions; a second family of unstable regions corresponds to antiperiodic solutions at the boundary, and can be treated in a similar way. When $\omega_1 \to 0$, we obtain periodic solutions if $\omega_0 = \ell$, where ℓ is an integer. So we expect to find an infinite sequence of resonant regions in the (ω_1, ω_0) plane, the ℓth region reducing to $\omega_0 = \ell$ when $\omega_1 = 0$. Our goal is to describe these regions and the corresponding Floquet exponents when $\omega_1 \ll 1, \omega_0$.

To this end, observe that if we write a solution as a linear combination of $\pm$ frequency solutions in the WKB form, as in equations (4.95) and (4.96) (exact), then the evolution of the α and β coefficients to a sufficiently high adiabatic order (presently $r = 0$) is dictated by equations (4.111) and (4.107), where

$$\frac{\dot{\omega}}{2\omega} \sim \left(\frac{-\omega_1^2}{4\omega_0^2}\right) \sin t \tag{4.130}$$

$$\Theta_0(t) \sim \left[1 + \frac{\omega_1^2}{2\omega_0^2}\right] \omega_0 t + \frac{\omega_1^2}{2\omega_0} \sin t \tag{4.131}$$

leading to

$$\exp\{2i\Theta_0\} = \exp\left\{2i\left[1 + \frac{\omega_1^2}{2\omega_0^2}\right]\omega_0 t\right\} \sum_{n=-\infty}^{\infty} J_n\left[\frac{\omega_1^2}{\omega_0}\right] e^{int} \tag{4.132}$$

where the J_n are Bessel functions (recall that for integer n, $J_{-n} = (-1)^n J_n$).

Now consider the ℓ-th resonant region where $\omega_0 = \ell + \delta_\ell$. Keeping only the slowly varying terms in equation (4.111), we get

$$\dot{\alpha}_\ell = i\beta_\ell \kappa_\ell \exp(2i\sigma_\ell t) \tag{4.133}$$

$$\dot{\beta}_\ell = -i\alpha_\ell \kappa_\ell \exp(-2i\sigma_\ell t) \tag{4.134}$$

where

$$\kappa_\ell \equiv \frac{\omega_1^2 K_\ell}{8\omega_0^2}, \qquad \sigma_\ell \equiv \frac{\omega_1^2}{2\omega_0} + \delta_\ell \tag{4.135}$$

and

$$K_\ell = J_{2\ell-1}\left[\frac{\omega_1^2}{\omega_0}\right] - J_{2\ell+1}\left[\frac{\omega_1^2}{\omega_0}\right] \sim \frac{1}{(2\ell-1)!}\left(\frac{\omega_1^2}{2\omega_0}\right)^{2\ell-1} \tag{4.136}$$

We seek a solution of the form

$$\alpha_\ell(t) = \alpha_{\ell 0} e^{\mu_\ell t} \exp(i\sigma_\ell t) \tag{4.137}$$

$$\beta_\ell(t) = \beta_{\ell 0} e^{\mu_\ell t} \exp(-i\sigma_\ell t) \tag{4.138}$$

where μ_ℓ is the Floquet exponent of the ℓth resonance band. We get

$$(\mu_\ell + i\sigma_\ell)\,\alpha_{\ell 0} = i\beta_{\ell 0}\kappa_\ell \tag{4.139}$$

$$(\mu_\ell - i\sigma_\ell)\,\beta_{\ell 0} = -i\alpha_{\ell 0}\kappa_\ell \tag{4.140}$$

Therefore

$$\mu_\ell^2 = \kappa_\ell^2 - \sigma_\ell^2 \tag{4.141}$$

The boundaries of the resonant region are given by

$$\delta_\ell \sim -\frac{\omega_1^2}{2\ell} \pm \frac{\omega_1^2 K_\ell}{8\ell^2} \tag{4.142}$$

We see that the regions become narrower, and the Floquet indices become weaker, as we go to higher resonance bands. Particle production by parametric resonance is the principal mechanism in the pre-heating stage when the universe is warmed up after an inflationary expansion.

4.6 Particle creation in a dynamical spacetime

Another important class of problems similar to the external field model above is cosmological particle creation. There, a classical dynamical background spacetime governs the quantum field and imparts a time-dependence in the natural frequencies of its normal modes. Historically this is a major arena where nonequilibrium field theory was inculcated and constructed. It has wide ranging implications in modern cosmology since many late era phenomena have originated from quantum effects in the very early universe including inflationary cosmology. The era from the Planck to the GUT era is depicted by quantum field theory in curved spacetime (the test field description) and semiclassical gravity (including the back-reaction).

Cosmological particle creation is a physical process of basic theoretical interest in quantum field theory in curved spacetime [Park66, Park68, Park69, Park71, ZelSta71, SexUrb69, Zel70, Hu72, Hu74, FuPaHu74, Gri74, Berger74, Berger75a, Berger75b, HuPar77, HuPar78, HarHu79, DeW67, DeW75, BirDav82], and important practical interest in the quantum dynamics of the early universe. Our summary here is based on earlier work of [Hu74, ZelSta71]. We begin with the underlying physics, which is rooted in parametric amplification of classical waves [Zel70]. This effect in second quantized language manifests itself as particle creation. A modern representation of such processes is by means of the squeezed state language developed in quantum optics. It is useful for the discussion of entropy and coherence issues. We defer such a discussion to Section 4.7.

4.6.1 Wave equations in curved spacetimes

Consider a massive (m) neutral scalar field ϕ coupled arbitrarily (ξ) to a background spacetime with metric $g_{\mu\nu}$ and scalar curvature R. Its dynamics is described by the action

$$S = \int d^4x \mathcal{L}(\phi, \nabla\phi, g_{\mu\nu}) \tag{4.143}$$

where the Lagrangian density is given by

$$\mathcal{L}(\phi, \nabla\phi, g_{\mu\nu}) = -\frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}(x)\nabla_\mu\phi\nabla_\nu\phi + (m^2 + \xi R)\phi^2(x)\right] \tag{4.144}$$

where $g \equiv \det g_{\mu\nu}$ and ∇ denotes taking the covariant derivative defined on the background spacetime. Here $\xi = 1/6$ and 0 denote, respectively, conformal and minimal coupling. The indices $\mu = (0, 1, 2, 3)$ denote time and spatial components. The scalar field satisfies the wave equation

$$[-\nabla^2 + m^2 + \xi R]\phi(\mathbf{x}, t) = 0 \tag{4.145}$$

where

$$\nabla^2 \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}\sqrt{-g}\frac{\partial}{\partial x^\nu}\right) \tag{4.146}$$

is the Laplace–Beltrami operator defined on the background spacetime.

In the canonical quantization approach, one assumes a foliation of spacetime into dynamically evolving, time-ordered, spacelike hypersurfaces Σ. If the three-dimensional space Σ possesses some symmetry, such as a homogeneous space with a group of motion, a separation of variables is usually possible which permits a normal mode decomposition of the field. (The spacetimes considered in this book, e.g. Friedmann–Lemaitre–Robertson–Walker (FLRW) and De Sitter (DS) all possess these properties.) One can then impose canonical commutation relations on the creation and annihilation operators corresponding to the (time-dependent) amplitude functions of each normal mode, define the vacuum and number states, and then construct the Fock space. In flat space, Poincaré invariance guarantees the existence of a unique global Killing vector ∂_t orthogonal to all constant-time spacelike hypersurfaces, an unambiguous separation of the positive- and negative-frequency modes, and a unique and well-defined vacuum. In curved spacetime, general covariance precludes any such privileged choice of time and slicing. There is no natural mode decomposition and no unique vacuum [Ful72, Ful89]. We assume the background spacetime under consideration has at least enough symmetry to allow for a normal mode decomposition of the invariant operator at any constant-time slice.

The classical field theory is quantized by replacing the field variable ϕ by the operator-valued distribution Φ. In the Heisenberg picture, Φ and its conjugate momentum $\Pi = \delta\mathcal{L}/\delta(\partial_0\Phi)$ obey the equal time commutation relation

(4.7). Note that the scalar delta function $\delta(\mathbf{x}, \mathbf{x}')$ in curved spacetime is defined by $\int \sqrt{-g}\delta(\mathbf{x}, \mathbf{x}')h(\mathbf{x}) = h(\mathbf{x}')$, where h is any test function.

Consider the field Φ in a coordinate volume $V = L^3$ with coordinate length L. We can expand the field Φ in terms of a complete set of (spatial) orthonormal modes $u_{\mathbf{k}}(\mathbf{x})$ as in equation (4.13). We use $\mathbf{x}$ as a generic notation for the spatial coordinates. (This is also applicable for spatially nonflat spacetimes, e.g. in S^3 with radius a, $V = 2\pi^2 a^3$, one can use the hyperspherical coordinates, $\mathbf{x} = (\chi, \theta, \phi)$, and the wavenumbers are then labeled by the corresponding principal quantum numbers $\mathbf{k} = (n, l, m)$. See, e.g. [Wig68].) As before, we write the operator-valued amplitude function $\varphi_{\mathbf{k}}(t)$ in terms of the time-independent annihilation operators $a_{\mathbf{k}}$ and the (c-number) amplitude functions $f_{\mathbf{k}}(t)$ as in (4.16). The canonical commutation rules on Φ then imply the conditions on $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}'}$ as in (4.18).

For the spatially-flat Friedmann–Lemaitre–Robertson–Walker (FLRW) spacetime [Park69], the spatial mode functions are simply $u_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{x}}$ and the wave equation for the amplitude function of the $\mathbf{k}$th mode in cosmic time t becomes (because of spatial isotropy, f depends only on $k \equiv |\mathbf{k}|$)

$$\ddot{f}_k(t) + 3H\ \dot{f}_k(t) + [\omega_k^2(t) + q(t)]f_k(t) = 0 \tag{4.147}$$

where an overdot denotes taking the derivative with respect to cosmic time, $\cdot = d/dt$. Here

$$\omega_k^2(t) = \frac{k^2}{a^2} + m^2; \quad q = \xi R \tag{4.148}$$

$$R = 6\left[\dot{H}(t) + 2H^2(t)\right] \tag{4.149}$$

$H(t) \equiv \frac{\dot{a}}{a}$ being the expansion (Hubble) rate of the background space. We have grouped terms containing two time derivatives of a (second derivative or first derivative squared) and call them q. As we will define below, they are of second adiabatic order while ω_k is of zero adiabatic order.

In curved space the inequivalence of Fock representation due to the lack of a global time-like Killing vector makes the constant separation of positive and negative-frequency components in general impossible. The mixing of positive- and negative-frequency components is the source of particle creation (in the second quantization description). Particle creation may arise from topological, geometrical, or dynamical causes. In cosmological spacetimes the inequivalence of vacua appears at different times of evolution, and thus cosmological particle creation is by nature a dynamically induced effect. Note that we are dealing here with a free field: particles are not produced from interactions, but rather from the excitation (parametric amplification [Zel70]) of vacuum fluctuations (or quantum noise) by the changing background gravitational field. The basic mechanism is also different from thermal particle creation in black holes [Haw75], accelerated detectors [Unr76] or moving mirrors [FulDav76, DavFul77], which involves the presence of an event horizon or the exponential red-shifting of outgoing modes [HuRav96, RaHuAn96, RaHuKo97].

4.6.2 Conformal vacuum in conformally-static spacetimes

In the class of conformally-static spacetimes where the metric is conformally related to a static spacetime by a conformal factor a there exists a global conformal Killing vector ∂_η, where $\eta = \int dt/a(t)$ is the conformal time. For example, the spatially-flat FRW spacetime with metric

$$g_{\mu\nu}(x) = a^2(\eta)\eta_{\mu\nu} \tag{4.150}$$

is conformally related to the Minkowski metric $\eta_{\mu\nu}$:

$$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2) \tag{4.151}$$

In this case the vacuum defined by the mode decomposition with respect to ∂_η is globally well-defined, known as the conformal vacuum. For conformally-invariant fields (e.g. a massless scalar field with $\xi = 1/6$ in equation (4.145)) in conformally-static spacetimes, it is easy to see that there is no particle creation [Park69]. Thus any small deviation from these conditions, e.g. small $m, \xi - (1/6)$, can be treated perturbatively from these states.

Consider a neutral massive scalar field coupled to a spatially-flat FRW metric with constant ξ. It is convenient to define a conformal amplitude function $\chi_{\mathbf{k}}(\eta) \equiv a(\eta) f_{\mathbf{k}}(\eta)$ related to the c-number amplitude function $f_{\mathbf{k}}$ for the $\mathbf{k}$th normal mode. It satisfies the following wave equation (cf. equation (4.147))

$$\chi''_{\mathbf{k}}(\eta) + [\omega^2_{\mathbf{k}}(\eta) + Q]\chi_{\mathbf{k}}(\eta) = 0 \tag{4.152}$$

where a prime denotes differentiation: $' \equiv d/d\eta$ and

$$\omega^2_k(\eta) \equiv \omega^2_k(t)a^2 = k^2 + m^2a^2 \tag{4.153}$$

is the time-dependent natural frequency. For spatially flat FRW spacetime $Q = Q_\xi = (\xi - \frac{1}{6})Ra^2$. For anisotropic spatially homogeneous universe (Bianchi type-I) where the expansion rates $H_i(t) \equiv \frac{\dot{a}_i}{a_i}$ are different in the three directions $i = 1, 2, 3$ $(a^3 = a_1a_2a_3)$, the wave equation in conformal time has in addition to Q_ξ another term $Q_\beta \equiv -\frac{1}{2}\sum_{i>j}(H_i - H_j)^2$, which, like Q_ξ, is also of second adiabatic order.

One sees that, for massless $(m = 0)$ conformally coupled $(\xi = \frac{1}{6})$ fields in a spatially flat FLRW universe $(Q = 0)$, the conformal wave equation admits solutions

$$\chi_{\mathbf{k}}(\eta) = Ae^{i\omega_{\mathbf{k}}\eta} + Be^{-i\omega_{\mathbf{k}}\eta} \tag{4.154}$$

which are of the same form as traveling waves in flat space. Since $\omega_{\mathbf{k}}(m = 0, \xi = \frac{1}{6}) = k =$const., the positive- and negative-frequency components remain separated and there is no particle production.

In this connection, Grishchuk [Gri74] showed that there is no production of gravitons in a radiation-dominated FLRW universe. This is easily seen as follows:

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

The gravitons are quantized linear perturbations. In a FLRW universe, just as in Minkowski spacetime, there are two polarizations, each obeying an equation (the Lifshitz equation [Lif46]) which has the same form as a massless ($m = 0$) minimally coupled ($\xi = 0$) scalar field [ForPar77]. For a FLRW universe $R = 6a''/a^3$, the wave equation (4.152) reads, in conformal time,

$$\chi''_{\mathbf{k}}(\eta) + (k^2 - a''/a)\chi_{\mathbf{k}}(\eta) = 0 \tag{4.155}$$

For a radiation-dominated FLRW universe, $a \sim \sqrt{t} \sim \eta$, and thus $R = 0$. The natural frequency is a constant and there is no production of massless minimally coupled scalar particles or gravitons in the conformal vacuum.

More generally, the wave equation for each mode has a time-dependent natural frequency. The negative-frequency modes can thus be excited by the dynamics of the background through $a(\eta)$ and $R(\eta)$. In analogy with the time-dependent Schrödinger equation, one can view the $\omega_k^2 + Q$ term in (4.152) as a time-dependent potential $V(\eta)$ which can induce back-scattering of waves [Zel70, Hu74], thus mixing the positive and negative frequency components in each mode. This, as we have learned, signifies particle creation.

4.6.3 Thermal radiance

It is rather commonly known that black holes emit thermal radiation, known as the Hawking effect [Haw75]. Hawking radiation has a deep meaning and many ways to derive and understand it. One way is to view it as arising from the exponential red-shifting of outgoing modes from the black hole. This condition is responsible for thermal radiance observed in uniformly accelerated detectors, known as the Unruh effect [Unr76], and in an exponential expansion of the early universe [Park76]. We can see this from the simple theory we have presented above.

Consider a conformally coupled massive field in a spatially-flat FRW universe. One can define the conformal vacua at $\eta_\pm$ with $\chi^{in,out}$ in terms of the positive frequency components. The probability $P_n(\mathbf{k})$ of observing n particles in mode $\mathbf{k}$ at late time is given by the modulus of the ratio of the Bogoliubov coefficients [Park76]: $P_n(\mathbf{k}) = |\beta_k/\alpha_k|^{2n} |\alpha_k|^{-2}$. One can find the average number of particles $\langle N_{\mathbf{k}} \rangle$ created in mode $\mathbf{k}$ (in a comoving volume) at late times to be $n_k \equiv \langle N_k \rangle = \sum_{n=0}^{\infty} n P_n(\mathbf{k}) = |\beta_k|^2$.

The model studied by Bernard and Duncan [BerDun77, BirDav82] has the scale factor $a(\eta)$ evolving like $a^2(\eta) = A + B \tanh \rho\eta$ which tends to constant values $a_\pm^2 \equiv A \pm B$ at asymptotic times $\eta \to \pm\infty$. Here ρ measures how fast the scale factor rises, and is the relevant parameter which enters in the temperature of thermal radiance. With this form for the scale function, α_k and β_k have analytic forms in terms of products of gamma functions. One obtains

$$|\beta_k/\alpha_k|^2 = \sinh^2(\pi\omega_-/\rho)/\sinh^2(\pi\omega_+/\rho) \tag{4.156}$$

where

$$\omega_{\pm} = (1/2)(\omega^{out} \pm \omega_{in}) \tag{4.157}$$

$$\omega_{in}^{out} = \sqrt{k^2 + m^2 a_{\pm}^2} \tag{4.158}$$

For cosmological models in which $a(+\infty) \gg a(-\infty)$, the argument of sinh is very large (i.e. $(\pi/\rho)\omega_{\pm} \gg 1$). To a good approximation this has the form $|\beta_k/\alpha_k|^2 = \exp(-2\pi\omega_{in}/\rho)$. For high momentum modes, one can recognize the Planckian distribution with temperature given by $k_B T_\eta = \hbar\rho/(2\pi a_+)$ as detected by an observer (here in the conformal vacuum) at late times.

4.6.4 Conformal stress–energy tensor

The conformal vacuum in the above section is well defined at all times and is useful to describe particle creation for fields which are nearly conformal and in spacetimes which are nearly conformally flat. We shall use the conformal wave equation (4.152) for the amplitude function χ for the $\mathbf{k}$ mode in conformal time to derive the corresponding number density and energy density of conformally invariant fields from spontaneous and stimulated particle production studied before for Minkowski space in Section 4.3. and also to illustrate the adiabatic regularization method.

The appropriate energy–momentum tensor which is conformally related to the flat space counterpart is the so-called "new, improved" one, or simply the conformal energy–momentum tensor [CaCoJa70]

$$\begin{aligned}\Lambda_{\mu\nu} = {} & \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi - \frac{1}{2}g_{\mu\nu}m^2\phi^2 \\ & + \xi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\phi^2 + \xi[g_{\mu\nu}\nabla^2(\phi^2) - \nabla_\mu\nabla_\nu(\phi^2)]\end{aligned} \tag{4.159}$$

The conformal wave equation (4.152) has the same form as the generic wave equation for Minkowski space in t time because they are conformally related. So all the results for external field problems in flat space given before are identical for conformal fields in curved spacetime upon the substitution of f by χ, t by η, and $T_{\mu\nu}$ by $\Lambda_{\mu\nu}$ plus a suitable power of the scale factor a to give the correct dimensionality.

The vacuum energy density associated with these particles is given by the expectation value of the $t-t$ component of $\Lambda_{\mu\nu}$ with respect to the conformal vacuum, i.e.

$$\rho_0^{\text{conf}} \equiv \langle 0 \mid \Lambda_{00} \mid 0\rangle = \frac{1}{a^4}\int \frac{d^3\mathbf{k}}{2(2\pi)^3}(\mid \chi_{\mathbf{k}}' \mid^2 + \omega_{\mathbf{k}}^2 \mid \chi_{\mathbf{k}} \mid^2) \tag{4.160}$$

$$= \frac{1}{a^4}\int \frac{d^3\mathbf{k}}{(2\pi)^3}(2s_{\mathbf{k}} + 1)\frac{\hbar\omega_{\mathbf{k}}}{2} \tag{4.161}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

The energy density of particles produced from an initial n particle state by stimulated production is

$$\rho_n^{\rm conf} \equiv \langle n \mid \Lambda_{00} \mid n \rangle = \frac{1}{a^4} \int \frac{d^3\mathbf{k}}{(2\pi)^3} (\mid \chi'_{\mathbf{k}} \mid^2 + \omega_{\mathbf{k}}^2 \mid \chi_{\mathbf{k}} \mid^2) \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle \tag{4.162}$$

$$= \frac{1}{a^4} \int \frac{d^3\mathbf{k}}{(2\pi)^3} (2s_{\mathbf{k}} + 1) \hbar\omega_{\mathbf{k}} \langle N_{\mathbf{k}}(t_0) \rangle \tag{4.163}$$

Combining (4.160) and (4.162), for a density matrix diagonal in the number state, the total energy density of particles created from the vacuum and from those already present in the n-particle state is given by

$$\rho^{\rm conf} = \rho_0^{\rm conf} + \rho_n^{\rm conf} = \frac{1}{a^4} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathcal{A}_{\mathbf{k}} \hbar\omega_{\mathbf{k}} (\frac{1}{2} + \langle N_{\mathbf{k}}(t_0) \rangle).$$

For a thermal density matrix μ at temperature $T = \beta^{-1}$ the magnification of the n-particle thermal state gives the finite-temperature contribution to particle creation, with energy density

$$\rho_T^{\rm conf} = \frac{1}{a^4} \int \frac{d^3\mathbf{k}}{(2\pi)^3} (2s_{\mathbf{k}} + 1) \hbar\omega_{\mathbf{k}} / (e^{\beta\hbar\omega_{\mathbf{k}}} - 1) \tag{4.164}$$

If $s_{\mathbf{k}} = 0$ the Stefan–Boltzmann relation holds for a massless conformal field in a FLRW universe

$$\rho_T^{\rm conf} = \frac{\pi^2}{30\hbar^3} T^4 \tag{4.165}$$

Thus Ta is a constant throughout the evolution of the radiation-dominated FLRW universe. $N_\gamma \sim (Ta)^3$ is proportional to the number of relativistic particles present or the entropy content of the universe [HarHu79, DeW67, Hu81]. Further discussions of finite-temperature particle creation and the related entropy generation problem can be found in [Hu82, Hu84].

4.6.5 Adiabatic regularization

To apply the adiabatic method to the regularization of the stress energy tensor in an external field or dynamical spacetime, we need to carry out a fourth-order adiabatic expansion. We study a slightly more general wave equation (4.152) for $\chi_{\mathbf{k}}(\eta_{\mathbf{k}})$ with natural frequency $\sqrt{\omega_{\mathbf{k}}^2(\eta) + Q}$ where Q is a term of second adiabatic order. In the cosmological context Q stands for either Q_ξ for a nonconformally coupled scalar field in a FLRW universe or for Q_β for a conformally coupled scalar field in an anisotropic Bianchi I universe.

Taking $n = 6$ in (4.46), we have the fourth adiabatic order positive frequency solution (we will suppress the mode index $\mathbf{k}$ in $\chi, W, \omega, \epsilon$ below)

$$\chi_{(6)} = \hbar^{1/2} \frac{e^{-i\int W_2 dt}}{\sqrt{2W_3}} \tag{4.166}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

where

$$W_3 = \omega(1 + \epsilon_2 + \epsilon_4)^{1/2} \tag{4.167}$$

Assuming that the solution χ is well-approximated by $\chi_{(6)}$ we have

$$|\chi|^2 = \hbar(2W_3)^{-1}, \qquad |\chi'|^2 = \hbar(2W_3)^{-1}\left[W_3^2 + \frac{1}{4}\left(\frac{d}{d\eta}\ln W_3\right)^2\right] \tag{4.168}$$

The adiabatic frequency corrections are given by

$$\epsilon_{2(2)} = \frac{Q}{\omega^2} - \frac{\bar{\omega}^2}{4} - \frac{\bar{\omega}'}{2\omega}, \qquad \epsilon'_{2(2)} = \frac{Q'}{\omega^2} - 2Q\frac{\bar{\omega}}{\omega} - \frac{\bar{\omega}''}{2\omega} \tag{4.169}$$

where the subscripts in parentheses denote the adiabatic order and we have defined the nonadiabaticity parameter (here, for frequency ω in conformal time η) as $\bar{\omega}_{\mathbf{k}} \equiv \omega'_{\mathbf{k}}/\omega^2_{\mathbf{k}}$. Substituting these into (4.102) and keeping terms of the same adiabatic order (as measured by the time derivatives) we get

$$\begin{aligned}
s_{k(2)} &= \frac{1}{16}\bar{\omega}^2 \\
s_{k(4)} &= \frac{1}{16}\left(-(1/2)\frac{\bar{\omega}\bar{\omega}''}{\omega^2} + \frac{1}{4}\frac{\bar{\omega}'^2}{\omega^2} + (1/2)\frac{\bar{\omega}'\bar{\omega}^2}{\omega} + \frac{3}{16}\bar{\omega}^4 \right. \\
&\qquad \left. + \frac{Q^2}{\omega^4} + Q'\frac{\bar{\omega}}{\omega^3} - Q\frac{\bar{\omega}'}{\omega^3} - 3Q\frac{\bar{\omega}^2}{\omega^2}\right)
\end{aligned} \tag{4.170}$$

The adiabatic expansion for particle production in the high-frequency range at the zeroth, second and fourth adiabatic order above matches the quartic, quadratic and logarithmic divergences in the vacuum energy density respectively. Substituting these expressions for $s_{k(div)} = s_{k(2)} + s_{k(4)}$ for each $\mathbf{k}$ mode into the vacuum energy density (4.160) we can identify the divergent vacuum energy density contributions as

$$\rho_{0(\mathrm{div})} = \frac{1}{a^4}\int \frac{d^3\mathbf{k}}{(2\pi)^3}(2s_{\mathbf{k}(div)} + 1)\frac{\hbar\omega_{\mathbf{k}}}{2} \tag{4.171}$$

Subtracting these we get the regularized vacuum energy density given by $\rho_{0(\mathrm{reg})} = \rho_0 - \rho_{0(\mathrm{div})}$. These results were obtained by [ZelSta71, Hu74, FuPaHu74].

We note again that the above adiabatic expressions give the amount of particle creation only in the high-frequency modes when $\bar{\omega}_{\mathbf{k}} \leq 1$. That is why they are suitable for the identification and removal of ultraviolet divergences in the energy–momentum tensor. Adiabatic regularization has been applied to cosmological particle creation with back-reaction [ParFul73, HuPar77, HuPar78].

4.6.6 A simple model of a cosmological phase transition

As a final example of quantum field dynamics in conformally flat universes, we shall show a simple model of the development of a cosmological phase transition through spinodal decomposition. Our discussion follows [SCHR99].

Let us consider a $\lambda\Phi^4$ theory on a spatially flat, expanding Friedmann–Lemaitre–Robertson–Walker universe. We assume the field is conformally coupled ($\xi = 1/6$) but has a bare mass m_b^2, thus breaking conformal invariance. The field equation now has an extra term $a^2(\eta)\lambda_B\Phi^3/6$ describing the self-interaction. However, at early times we may adopt the Hartree approximation

$$\Phi^3 \sim 3\left\langle\Phi^2\right\rangle(\eta)\,\Phi \tag{4.172}$$

and the wave equation becomes formally the equation for a free field with a self-consistent mass

$$m_{\rm eff}^2(\eta) = m_b^2 + \frac{\lambda_B}{2}\left\langle\Phi_{\rm HF}^2\right\rangle(\eta) \tag{4.173}$$

We are assuming of course that the initial condition is also spatially homogeneous, so that $\left\langle\Phi^2\right\rangle$ depends only on η. The "free" field $\Phi_{\rm HF}$ admits a mode expansion in terms of conformal amplitudes χ_k which obey equation (4.152) with $Q = 0$ and $m^2 = m_{\rm eff}^2(\eta)$, and boundary conditions

$$\chi_k(0) = \sqrt{\frac{\hbar}{2\omega_k(0)}} \tag{4.174}$$

$$\chi_k'(0) = -i\sqrt{\frac{\hbar\omega_k(0)}{2}} \tag{4.175}$$

We assume the expectation values

$$\left\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}\right\rangle = n_k\delta(\mathbf{k}-\mathbf{k}'); \qquad \left\langle a_{\mathbf{k}}^{\dagger 2}\right\rangle = \left\langle a_{\mathbf{k}}^2\right\rangle = 0 \tag{4.176}$$

whereby

$$\left\langle\Phi_{\rm HF}^2\right\rangle(\eta) = \frac{1}{a^2(\eta)}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\left|\chi_k(\eta)\right|^2(1+2n_k) \tag{4.177}$$

As in flat spacetime, $\left\langle\Phi_{\rm HF}^2\right\rangle(\eta)$ diverges. The theory may be rendered finite by imposing a cut-off at some physical scale $\Lambda_{\rm phys} = \Lambda(\eta)/a(\eta)$. However the resulting renormalized parameters are strongly cut-off dependent. To eliminate this dependence, let $\kappa_{\rm phys}$ be a second physical scale, large enough that for modes higher than $\kappa(\eta) = a(\eta)\kappa_{\rm phys}$ the mode functions χ_k are well approximated by adiabatic modes, but still much lower than Λ. Then we write

$$\left\langle\Phi_{\rm HF}^2\right\rangle(\eta) = \hbar\left\{\frac{\Lambda^2}{8\pi^2} - \frac{m_{\rm eff}^2(\eta)}{8\pi^2}\ln\left[\frac{\Lambda}{\kappa}\right] + \mu^2(\eta)\right\} \tag{4.178}$$

$$\mu^2(\eta) = \frac{1}{a^2(\eta)}\int^{a\Lambda}\frac{d^3\mathbf{k}}{(2\pi)^3}\left\{\frac{\left|\chi_k(\eta)\right|^2}{\hbar}(1+2n_k) - \frac{1}{2k} + \frac{a^2(\eta)\,m_{\rm eff}^2(\eta)}{4k^3}\theta(k-a\kappa)\right\} \tag{4.179}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

The point is that $\mu^2(\eta)$ is essentially cut-off independent. The gap equation now reads

$$m_{\text{eff}}^2(\eta) = m_{\text{b}}^2 + \frac{\hbar\lambda_B}{2}\left\{\frac{\Lambda^2}{8\pi^2} - \frac{m_{\text{eff}}^2(\eta)}{8\pi^2}\ln\left[\frac{\Lambda}{\kappa}\right] + \mu^2(\eta)\right\} \tag{4.180}$$

The bare mass m_{b}^2 is defined by the condition that in flat space time $(a = 1)$ and at the critical temperature T_{C}, $m_{\text{eff}}^2 = 0$. Thus

$$0 = m_{\text{b}}^2 + \frac{\hbar\lambda_B}{2}\left\{\frac{\Lambda^2}{8\pi^2} + \frac{T_{\text{C}}^2}{6}\right\} \tag{4.181}$$

where we set the Boltzmann constant $k_{\text{B}} = 1$, for simplicity. We now have the finite gap equation

$$m_{\text{eff}}^2(\eta) = \frac{\hbar\lambda}{2}\left\{\mu^2(\eta) - \frac{T_{\text{C}}^2}{6}\right\} \tag{4.182}$$

where

$$\frac{1}{\lambda} = \frac{1}{\lambda_B} + \frac{\hbar}{16\pi^2}\ln\left[\frac{\Lambda}{\kappa}\right] \tag{4.183}$$

We may now start discussing the early time evolution of the field. The central aspect of this behavior is the suppression factor $a^{-2}(\eta)$ in $\mu^2(\eta)$ (cf. equation (4.179)). Because of this factor, $m_{\text{eff}}^2(\eta)$ decreases and eventually becomes negative. Indeed, assume the initial spectrum n_k corresponds to a Planck distribution with temperature $T_0^2 \gg m_{\text{eff}}^2(0)\,, T_{\text{C}}^2$. Then, when $m_{\text{eff}}^2(\eta)$ is small we get

$$m_{\text{eff}}^2(\eta) \sim \frac{\hbar\lambda}{12}\left\{\frac{T_0^2}{a^2(\eta)} - T_{\text{C}}^2\right\} \tag{4.184}$$

and so

$$\omega_k^2(\eta) = k^2 + \frac{\hbar\lambda}{12}\left[T_0^2 - a^2(\eta)\,T_{\text{C}}^2\right] \tag{4.185}$$

If the expansion is slow enough, we may approximate this by

$$\omega_k^2(\eta) = k^2 - \frac{1}{\tau}\frac{\hbar\lambda T_0^2}{12}(\eta - \eta_{\text{C}}) \tag{4.186}$$

where η_{C} is the conformal time at which m_{eff}^2 vanishes for the first time, and $\tau^{-1} = (2aH)(\eta_{\text{C}})$ is the quench rate. $H = a'/a^2$ is the Hubble constant.

At $\eta = \eta_{\text{C}}$ the homogeneous mode becomes unstable. If m_{eff}^2 actually becomes negative, then other infrared modes become unstable as well, and the corresponding mode functions start to grow exponentially. The result is the formation of an infrared peak. Eventually, though, the approximation (4.186) becomes invalid.

To obtain an improved estimate, observe that once $\left|\left(\omega_k^2\right)'\right| \leq \left|\omega_k^3\right|$ we may approximate the mode functions by WKB wave forms. This inequality translates

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

into

$$\frac{1}{\tau}\frac{\hbar\lambda T_0^2}{12} \leq \left[\frac{1}{\tau}\frac{\hbar\lambda T_0^2}{12}\left(\eta - \eta_{\rm C}\right) - k^2\right]^{3/2} \tag{4.187}$$

When this inequality holds, we may write

$$\chi_k\left(\eta\right) \sim \frac{\sqrt{\hbar}}{\sqrt{2\left|\omega_k\left(\eta\right)\right|}} e^{S_k(\eta)} \tag{4.188}$$

where

$$S_k\left(\eta\right) = \int_{\eta_k}^{\eta}\left|\omega_k\left(\eta'\right)\right| d\eta' = \frac{2}{3}\left(\frac{12\tau}{\hbar\lambda T_0^2}\right)\left[\frac{\hbar\lambda T_0^2}{12\tau}\left(\eta - \eta_{\rm C}\right) - k^2\right]^{3/2} \tag{4.189}$$

$$\eta_k = \eta_{\rm C} + \tau\left(\frac{12k^2}{\hbar\lambda T_0^2}\right) < \eta \tag{4.190}$$

In the infrared, we may approximate

$$S_k\left(\eta\right) = S_0\left(\eta\right) - \frac{1}{2}\sigma^2\left(\eta\right) k^2 \tag{4.191}$$

$$\sigma^2\left(\eta\right) = \left(\frac{48\tau}{\hbar\lambda T_0^2}\left(\eta - \eta_{\rm C}\right)\right)^{1/2} \tag{4.192}$$

Provided $\eta_{\sigma^{-1}} < \eta$, we get

$$\mu^2\left(\eta\right) = \frac{1}{a^2\left(\eta\right)}\left[\frac{T_0^2}{6} + \frac{e^{2S_0(\eta)}T_0}{\left(2\pi\right)^2\left|\omega_0\left(\eta\right)\right|\sigma^2\left(\eta\right)}\right] \tag{4.193}$$

The infrared peak in mode space is correlated with the appearance of correlated domains in physical space, whose comoving size is $\sigma\left(\eta\right)$ and increases with time (coarse graining). We see that the exponential growth of the infrared peak counterbalances the red-shift due to the Hubble expansion. If we simply extrapolate this model, we conclude that eventually the infrared peak becomes dominant, and the effective mass is driven again to zero from below.

The actual picture is more involved. Within these domains, there is nondiagonal long-range order, and we may describe the field as a quantum field (represented by the stable modes) evolving on a nontrivial background field (which is the "square root" of the infrared peak). When the background field gets large enough, it starts to oscillate around the true equilibrium position. The quantum field then becomes a periodically driven field and, as we have seen, parametric amplification results in copious particle production from the background field.

4.7 Particle creation as squeezing

In this section we will use the language of squeezed states [CavSch85, Sch86] to treat a neutral scalar field in a dynamic background field or spacetime. This approach will shed a clearer light on two interrelated issues:

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

(a) Dependence of particle creation on the initial state. We consider in particular the number state, the coherent and the squeezed state.
(b) The relation of spontaneous and stimulated particle creation and their dependence on the initial state.

We also derive the result for the fluctuations in particle number in anticipation of its relevance to defining noise in quantum fields. Our presentation here follows [HuKaMa94].

Since the concept of squeezed state was introduced to quantum optics in the 1970s [CavSch85, Sch86, Gla05], there has been much progress in seeking its experimental realizations and theoretical implications. The language of squeezed states as a way to describe cosmological particle creation was introduced by Grishchuk and Sidorov [GriSid90]. Although the physics is not new (this was also pointed out by Albrecht *et al.* [AFJP94] in the inflationary cosmology context) and the results are largely known, the use of rotation and squeeze operators gives an alternative description which allows one to explore new avenues based on interesting ideas developed in quantum optics. Work on entropy generation in cosmological perturbations by Brandenberger and coworkers [BrMuPr92, BrMuPr93] and Gasperini and Giovannini [GasGio93, GasGioVen93] make use of coarse graining via a random phase approximation. Matacz [Mat94, LafMat93] has used the squeezed state formalism as a starting point for the study of decoherence of cosmological inhomogeneities in the coherent-state representation.

The issues of initial states and entropy generation have been discussed in restricted conditions, and the issue of spontaneous and stimulated production has only been touched upon before. For the sake of completeness, we will address these issues under a common framework, using the language of squeezed states, and present the results for different initial states (the number state, the coherent state and the squeezed state).

4.7.1 Evolutionary operator, squeezing and rotation

We now present a description of particle creation by means of the evolutionary operator U defined by

$$\tilde{a}_{\pm\mathbf{k}}(t) = U(t) a_{\pm\mathbf{k}} U^\dagger(t) \tag{4.194}$$

where $UU^\dagger = 1$. The form of U was deduced by Parker [Park69] following Kamefuchi and Umezawa [KamUme64]. In the modern language of squeezed states [CavSch85, Sch86], one can write $U = RS$ as a product of two unitary operators, the **rotation operator**

$$R(\theta) = \exp[-i\theta(a_+^\dagger a_+ + a_-^\dagger a_-)] \tag{4.195}$$

and the **two-mode squeeze operator**

$$S_2(r, \phi) = \exp[r(a_+ a_- e^{-2i\phi} - a_+^\dagger a_-^\dagger e^{2i\phi})] \tag{4.196}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

where r is the squeeze parameter with range $0 \le r < \infty$ and ϕ, θ are the rotation parameters with ranges $-\pi/2 < \phi \le \pi/2, 0 \le \theta < 4\pi$. (These parameters and U, R and S should all carry the label $\mathbf{k}$. The $\pm$ on a refer to the $\pm\mathbf{k}$ modes.) Note that

$$S_2^\dagger(r,\phi) = S_2^{-1}(r,\phi) = S_2(r,\phi+\pi/2) \tag{4.197}$$

The three real functions $(\theta_\mathbf{k}, \phi_\mathbf{k}, r_\mathbf{k})$ are related to the two complex functions $(\alpha_\mathbf{k}, \beta_\mathbf{k})$ by

$$\alpha_\mathbf{k} = e^{i\theta_\mathbf{k}} \cosh r_\mathbf{k}, \qquad \beta_\mathbf{k} = e^{i(\theta_\mathbf{k} - 2\phi_\mathbf{k})} \sinh r_\mathbf{k} \tag{4.198}$$

For mode decompositions in spatially homogeneous spacetimes leading to no mode couplings, the Bogoliubov transformation connecting the $a_\mathbf{k}$ and the $\tilde{a}_\mathbf{k}$ operators is given by equation (4.28) (for more general situations, see [Hu72]). We see that because of the linear dependence of $\tilde{a}_{+\mathbf{k}}$ on $a_{+\mathbf{k}}$ and $a^\dagger_{-\mathbf{k}}$ (but not $a^\dagger_{+\mathbf{k}}$) a two-mode squeeze operator is needed to describe particle pairs in states $\pm\mathbf{k}$.

The physical meaning of rotation and squeezing can be seen from the result of applying these operators for a single-mode harmonic oscillator as follows: (the $\mathbf{k}$th mode label is omitted below unless needed explicitly).

The Hamiltonian is

$$H_0 = \hbar\Omega \left(a^\dagger a + \frac{1}{2} \right) \tag{4.199}$$

Under rotation,

$$R|0\rangle = |0\rangle, \qquad RaR^\dagger = e^{i\theta} a \tag{4.200}$$

Also,

$$R(\theta)R(\theta') = R(\theta+\theta') \tag{4.201}$$

This implies that

$$R\hat{x}R^\dagger = (\cos\theta)\,\hat{x} - (\sin\theta)\,\hat{p} \tag{4.202}$$

$$R\hat{p}R^\dagger = (\sin\theta)\,\hat{x} + (\cos\theta)\,\hat{p} \tag{4.203}$$

where

$$a = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{M\Omega}\hat{x} + i\frac{\hat{p}}{\sqrt{M\Omega}} \right) \tag{4.204}$$

Thus the name rotation. Let $\delta a = a - \langle a \rangle$ (where $\langle\ \rangle$ denotes the expectation value with respect to any state); then the second-order noise moments of a are defined as [CavSch85, Sch86]:

$$\begin{aligned} \langle(\delta a)^2\rangle &= \langle a^2\rangle - \langle a\rangle^2 = \langle(\delta a^\dagger)^2\rangle^* \\ &= \frac{1}{2\hbar}[M\Omega\langle(\delta x)^2\rangle - (M\Omega)^{-1}\langle(\delta p)^2\rangle] + i\langle(\delta x \delta p)_{\text{sym}}\rangle \end{aligned} \tag{4.205}$$

$$\langle|\delta a|^2\rangle = \frac{1}{2}\langle\delta a \delta a^\dagger + \delta a^\dagger \delta a\rangle = \frac{1}{2\hbar}[M\Omega\langle(\delta x)^2\rangle + (M\Omega)^{-1}\langle(\delta p)^2\rangle] \tag{4.206}$$

The first quantity is the variance of a, a complex second moment, while the second is the correlation, a real second moment, which, as seen in the more familiar x, p representation, measures the mean-square uncertainty (called total noise in [CavSch85, Sch86]). Rotation preserves the number operator

$$Ra^\dagger a R^\dagger = a^\dagger a \tag{4.207}$$

It rotates the moment

$$\langle R(\delta a)^2 R^\dagger \rangle = e^{2i\theta} \langle (\delta a)^2 \rangle \tag{4.208}$$

corresponding to a redistribution between $\hat{x}, \hat{p}$, but preserves the uncertainty

$$\langle R|\delta a|^2 R^\dagger \rangle = \langle |\delta a|^2 \rangle \tag{4.209}$$

One can define a **displacement operator** as

$$D(\mu) = \exp[\mu a^\dagger - \mu^* a] \tag{4.210}$$

Note that $D^{-1}(\mu) = D^\dagger(\mu) = D(-\mu)$. The coherent state can be defined as

$$|\mu\rangle = D(\mu)|0\rangle \tag{4.211}$$

Thus

$$a|\mu\rangle = \mu|\mu\rangle \tag{4.212}$$

and

$$Da^\dagger a D^\dagger = a^\dagger a - (\mu a^\dagger + \mu^* a) + |\mu|^2 \tag{4.213}$$

Under displacement,

$$D(\mu) a D^\dagger(\mu) = a - \mu \tag{4.214}$$

The displacement operation also preserves the uncertainty

$$\langle D|\delta a|^2 D^\dagger \rangle = \langle |\delta a|^2 \rangle \tag{4.215}$$

The **single-mode squeeze operator** is defined as

$$S_1(r, \phi) = \exp\left[\frac{r}{2}(a^2 e^{-2i\phi} - a^{\dagger 2} e^{2i\phi})\right] \tag{4.216}$$

If we construct a Gaussian state in the position basis, with initially the same width σ_0 as that of the ground state of such an ordinary harmonic oscillator, displaced by some arbitrary amount and with a phase proportional to x, we find this to be an eigenstate of the lowering operator, and is called a coherent state. Suppose we locate the point (x, p) in phase space and draw an ellipse about this point, the lengths of whose axes are the uncertainties $\Delta x^2, \Delta p^2$. Then as the oscillator evolves this uncertainty ellipse revolves about the origin with angular speed Ω. A squeezed state is again such a state, but with an arbitrary initial width σ. We find that as the oscillator evolves the uncertainty ellipse again revolves about the origin, but its axes change length and it can also rotate about its own center. It turns out that the squeeze parameter r is related to the width of such a state:

$$r = \ln \frac{\sigma_0}{\sigma}, \quad \sigma_0 \equiv \sqrt{\frac{\hbar}{2M\Omega}} \tag{4.217}$$

Hence a coherent state has $r = 0$, or zero squeezing. A Gaussian that initially has a width smaller than σ_0 will evolve to a squeezed state with some $r > 0$. A squeezed state is formed by squeezing a coherent state,

$$|\sigma\rangle_\mu = S_1(r, \phi)|\mu\rangle \tag{4.218}$$

or,

$$|\sigma\rangle_\mu = |r, \phi, \mu\rangle = S_1(r, \phi)D(\mu)|0\rangle \tag{4.219}$$

Call $a_{S1} = S_1 a S_1^\dagger$. Then

$$a_{S1}|\sigma\rangle = \mu|\sigma\rangle \tag{4.220}$$

and

$$a_{S1} = S_1 a S_1^\dagger = a \cosh r + e^{2i\phi} a^\dagger \sinh r \tag{4.221}$$

Thus a squeezed state in the Fock space of a becomes a coherent state in the Fock space of $a_{S1}$with the same eigenvalue. From this we see the result of S_1 acting on $\hat{x}$ and $\hat{p}$:

$$S_1 \hat{x} S_1^\dagger = (\cosh r + \cos 2\phi \sinh r)\hat{x} + (\sin 2\phi \sinh r)\,(\hat{p}/(M\Omega)) \tag{4.222}$$

$$S_1 \hat{p} S_1^\dagger = (\cosh r - \cos 2\phi \sinh r)\hat{p} + (\sin 2\phi \sinh r)(M\Omega)\hat{x} \tag{4.223}$$

For $\phi = \pi/2$, these give

$$S_1 \hat{x} S_1^\dagger = e^{-r}\hat{x}, \quad S_1 \hat{p} S_1^\dagger = e^{r}\hat{p} \tag{4.224}$$

Hence the name squeezing. Two successive squeezes with the same rotation parameter result in one squeeze with the squeeze parameter as the sum of the two parameters:

$$S_1(r, \phi)S_1(r', \phi) = S_1(r + r', \phi) \tag{4.225}$$

The expectation value of squeezing the number operator is

$$\langle S_1^\dagger a^\dagger a S_1\rangle = \sinh^2 r + (1 + 2\sinh^2 r)\langle a^\dagger a\rangle + \sinh 2r \mathrm{Re}[e^{-2i\phi}\langle a^2\rangle] \tag{4.226}$$

and that of the correlation is

$$\langle S_1^\dagger |\delta a|^2 S_1\rangle = \cosh 2r\langle|\delta a|^2\rangle + \sinh 2r \mathrm{Re}[e^{-2i\phi}\langle(\delta a)^2\rangle] \tag{4.227}$$

which for the vacuum and coherent states is always greater than or equal to the original value.

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

The two-mode squeeze operator S_2 defined in (4.196) is more suitable for the description of cosmological particle creation. One can show that the *out* state is generated from the *in* state by including contributions from all $\mathbf{k}$ modes,

$$|out\rangle = RS|in\rangle \tag{4.228}$$

where

$$S = \Pi_{\mathbf{k}=0}^{\infty} S_2(r_{\mathbf{k}}, \phi_{\mathbf{k}}) \tag{4.229}$$

In general

$$\langle out|F(\tilde{a}_\pm, \tilde{a}_\pm^\dagger)|out\rangle = \langle in|F(a_\pm, a_\pm^\dagger)|in\rangle \tag{4.230}$$

where F is an arbitrary analytic function. The $|in\rangle$ state can be a number state, a coherent state or a squeezed state. If the initial state is a vacuum state, $|in\rangle = |0in\rangle$, then

$$|0out\rangle = S(r, \phi - \frac{\theta}{2})|0in\rangle \tag{4.231}$$

where

$$S(r, \phi - \theta) = \exp\{\Sigma_{\mathbf{k}} r_{\mathbf{k}} [e^{-2i(\phi_{\mathbf{k}} - \theta_{\mathbf{k}})} a_{\mathbf{k}} a_{-\mathbf{k}} - e^{2i(\phi_{\mathbf{k}} - \theta_{\mathbf{k}})} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger]\} \tag{4.232}$$

The squeeze parameter $\sinh^2 r_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2$ measures the number of particles created. Rotation does not play a role. Thus, as observed by Grishchuk and Sidorov [GriSid90], cosmological particle creation amounts to squeezing the vacuum. The same can be said about Hawking radiation [Haw75]. See [HuKaMa94].

4.7.2 Dynamics of the squeezing parameters

So far we have used the language of squeezed states to describe the integrated effect of the dynamics, as in equation (4.194). We will show now that for a linear system the dynamics itself may be described in terms of the evolution of the squeezing parameters r, ϕ and θ as functions of time.

Let us begin with a general quadratic Lagrangian. This Lagrangian has time-dependent mass and frequency, and we will also allow it to have a time-dependent cross-term, denoted $2\mathcal{E}(t)$:

$$L = \frac{M(t)}{2}\left[\dot{x}^2 + 2\mathcal{E}(t)\dot{x}x - \Omega^2(t)x^2\right] \tag{4.233}$$

We perform a Legendre transformation to obtain the Hamiltonian, and switch to creation–destruction operators

$$a = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\kappa}\hat{x} + i\frac{\hat{p}}{\sqrt{\kappa}}\right) \tag{4.234}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

where κ is an arbitrary positive constant related to the frequency. The result is [HuMat94]

$$\hbar^{-1}H(t) = g(t)\frac{a^2}{2} + g^*(t)\frac{a^{\dagger 2}}{2} + h(t)(a^\dagger a + 1/2) \tag{4.235}$$

$$\begin{aligned} g &= \frac{1}{2}\left[\frac{M}{\kappa}(\Omega^2 + \mathcal{E}^2) - \frac{\kappa}{M} + 2i\mathcal{E}\right] \\ h &= \frac{1}{2}\left[\frac{M}{\kappa}(\Omega^2 + \mathcal{E}^2) + \frac{\kappa}{M}\right] \end{aligned} \tag{4.236}$$

The value of κ can be chosen so that at the initial time $g(t_i) = 0$. Thus if $\mathcal{E} = 0$ we will usually have $\kappa = M(t_i)\Omega(t_i)$.

The evolution operator $U = SR$ may be written as the product of a single-mode squeeze operator S and a rotation operator R, which in turn are parameterized in terms of a squeeze parameter r and angles θ and ϕ as in equations (4.195) and (4.216). Acting on the destruction operator, U induces a Bogoliubov transformation as in equation (4.194), with Bogoliubov coefficients given in equation (4.198). Their equations of motion are

$$\begin{aligned} \dot\alpha &= -ih\alpha - ig^*\beta \\ \dot\beta &= ig\alpha + ih\beta \\ \alpha(t_i) &= 1, \quad \beta(t_i) = 0 \end{aligned} \tag{4.237}$$

with g, h as defined in equation (4.236).

A quantity of much importance turns out to be the sum of the Bogoliubov coefficients, $\chi \equiv \alpha + \beta$. It follows from equations (4.237) that χ satisfies the classical equation of motion for the system:

$$\ddot\chi + \frac{\dot M}{M}\dot\chi + \left(\Omega^2 + \dot{\mathcal{E}} + \frac{\dot M\mathcal{E}}{M}\right)\chi = 0 \tag{4.238}$$

with initial conditions

$$\chi(t_i) = 1; \quad \dot\chi(t_i) = \frac{-i\kappa}{M(t_i)} - \mathcal{E}(t_i) \tag{4.239}$$

With this result, the usual task of finding the Bogoliubov coefficients α, β from two coupled first-order differential equations is reduced to that of solving one second-order equation for χ. We have two equations

$$\begin{aligned} \chi &= \alpha + \beta \\ \dot\chi &= i(g - h)\alpha + i(h - g^*)\beta \end{aligned} \tag{4.240}$$

so, solving for α, β using equation (4.236):

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \frac{1}{2}\left(1 \pm \frac{i\mathcal{E}M}{\kappa}\right)\chi \pm \frac{iM}{2\kappa}\dot\chi \tag{4.241}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

Equivalently, we can follow the behavior of r, ϕ, θ by writing equation (4.237) in terms of the squeeze parameter, with $g \equiv |g|e^{i\delta}$:

$$\begin{aligned}\dot{r} &= |g|\sin(2\phi+\delta)\\ \dot{\phi} &= -h + |g|\coth 2r\cos(2\phi+\delta)\\ \dot{\theta} &= h - |g|\tanh r\cos(2\phi+\delta)\end{aligned} \tag{4.242}$$

As an example, consider an inverted oscillator, where the coefficients in the Lagrangean are time-independent and $\Omega^2 < 0$. The variable χ blows up, and so does the squeeze parameter $r \to \ln(2|\alpha|)$. If we set $r \to \infty$ then the equations for ϕ, θ become

$$\dot{\theta} = -\dot{\phi} = h - |g|\cos(2\phi+\delta) \tag{4.243}$$

or

$$t = \int_{\phi_0}^{\phi} \frac{d\varphi}{|g|\cos(2\varphi+\delta) - h} \tag{4.244}$$

The integral may be solved analytically but we do not need the result in what follows. Simply observe that $\Omega^2 < 0$ implies $|g| > h$, and so as $t \to \infty$, ϕ must approach a zero of $|g|\cos(2\varphi+\delta) - h$, so that the integral increases without bound. This makes $\dot{\theta} \to 0$ too. Therefore for late times ϕ and θ approach constant values, while r increases.

4.7.3 Number, coherence and initial states

We will show in this section that the number of particles produced depends very much on the initial state chosen. The number operator for a particle pair in mode k is given by

$$N = a_+^\dagger a_+ + a_-^\dagger a_- \tag{4.245}$$

Note that the subscripts $\pm$ here denote a particle pair in states $\pm\mathbf{k}$ whereas in the charged particle case $+, -$ denote particle and antiparticle states respectively. For the charged particle case since at the end we assume the number of positive and negative charged particles is the same, it gives the same expression as a neutral particle there. However, here since we count the two states as distinct, we should have twice the amount for vacuum particle production.

The expectation value of the number operator with respect to the $|out\rangle$ vacuum for a general initial state is

$$\begin{aligned}\tilde{N} = \langle N\rangle_t = \langle S_2^\dagger R^\dagger N R S_2\rangle = 2|\beta|^2 + (1+2|\beta|^2)\langle N\rangle\\ -2|\alpha||\beta|(e^{2i\phi}\langle a_+^\dagger a_-^\dagger\rangle + e^{-2i\phi}\langle a_+ a_-\rangle)\end{aligned} \tag{4.246}$$

Comparing this expression with (4.103) or (4.108), the only difference of a factor of 2 for the first $|\beta|^2$ term comes from the spontaneous creation of particles in the $\pm\mathbf{k}$ modes. The net change in the particle number from the initial to the final state is

$$\delta N \equiv \langle N\rangle_t - \langle N\rangle = 2|\beta|^2[1+\langle N\rangle] - 2|\beta||\alpha|\{e^{2i\phi}\langle a_+^\dagger a_-^\dagger\rangle + e^{-2i\phi}\langle a_+ a_-\rangle\} \tag{4.247}$$

Here, the first two terms in the square brackets are respectively the spontaneous and stimulated emissions and the last term in the curly brackets is the interference term. The difference between spontaneous and stimulated creation of particles in cosmology was explained first by Parker [Park69] and explored in more detail by Hu and Kandrup [HuKan87]. Note that since there is no θ dependence, rotation has no effect. If $r_{\mathbf{k}} \neq 0$ for some $\mathbf{k}$ both spontaneous and stimulated contributions are positive. The interference term can be negative for states which give nonzero $\langle a_+ a_-\rangle$. Only when this term is non-zero can δN be negative.

We will calculate the change in particle number for some specific initial states.

(a) **Number state**
For an initial number state $|n\rangle = |n_+, n_-\rangle$

$$\delta N = 2|\beta|^2(1+n_+ + n_-) \tag{4.248}$$

We see that the number of particles will always increase.

(b) **Coherent state**
For an initial coherent state

$$|\mu\rangle = D(\mu_+)D(\mu_-)|0,0\rangle \tag{4.249}$$

we find that

$$\delta N = 2|\beta|^2[1+\langle N_+\rangle + \langle N_-\rangle] - 4|\beta||\alpha|\sqrt{\langle N_+\rangle\langle N_-\rangle}\cos(2\phi - \zeta_+ - \zeta_-) \tag{4.250}$$

where

$$\mu_+ = \sqrt{\langle N_+\rangle}e^{i\zeta_+}, \quad \mu_- = \sqrt{\langle N_-\rangle}e^{i\zeta_-} \tag{4.251}$$

Note the existence of the interference term which can give a negative contribution. It depends not only on the squeeze parameters $|\beta|$ and ϕ, but also on the particles present and the phase of the initial coherent state. Conditions favorable to a decrease in δN are $\cos(2\phi - \zeta_+ - \zeta_-) = 1$ and $\langle N_+\rangle = \langle N_-\rangle = \langle N\rangle/2$. In this case we find δN is negative if

$$\langle N\rangle > \frac{|\beta|}{|\alpha| - |\beta|} \tag{4.252}$$

(c) **Single-mode squeezed vacuum state**
For an initial one-mode squeezed state

$$|\sigma\rangle_1 = S_{1+}(r_+,\phi_+)S_{1-}(r_-,\phi_-)|0,0\rangle \tag{4.253}$$

generated by squeezing the vacuum with $S_{1\pm}$ for the $\pm\mathbf{k}$ modes, we get

$$\delta N = 2|\beta|^2(1 + \langle N_+\rangle + \langle N_-\rangle) \tag{4.254}$$

Once again particle number will always increase.

(d) **Two-mode squeezed vacuum state**
For an initial two-mode squeezed vacuum

$$|\sigma\rangle_2 = S_2(r_0, \phi_0)|0, 0\rangle \tag{4.255}$$

where S_2 is defined earlier,

$$\delta N = 2|\beta|^2[1 + \langle N\rangle] + 2|\beta||\alpha|\sqrt{\langle N\rangle(2 + \langle N\rangle)}\cos 2(\phi - \phi_0) \tag{4.256}$$

The cosine factor shows that particle number can decrease given the right phase relations. It can be shown that for $\cos 2(\phi - \phi_0) = -1$ particle number would decrease ($\delta N \leq 0$) if $r_0 \geq r/2$. If the phase information is randomized the cosine factor averages to zero and there is a net increase in particle number. Since a squeezed state is the end result of squeezing a vacuum via particle creation, one might naively expect to see a monotonic increase in number. Our result shows that this is true only if the phase information is lost in the squeezed state to begin with.

In summary we can make the following observations:

(a) Rotation R in the evolution operator $U = RS$ does not influence particle creation.
(b) For an initial number state or single-mode squeezed vacuum we find a net increase in the number of particles.
(c) For an initial coherent state and two-mode squeezed vacuum, particle number can increase or decrease. A net increase can nevertheless be obtained by suitable choices of $S_2(r, \phi)$ and $S_2(r_0, \phi_0)$.
(d) If random phase is assumed for the initial state the interference term can be averaged out to zero and there will be a net increase in number of particles.

Coherence can persist

A measure of the coherence of the system is given by the uncertainty (called variance in [Hu72, Mol67, BroCar79, HuPav86])

$$|\delta a|^2 = \frac{1}{2}(\delta a \delta a^\dagger + \delta a^\dagger \delta a) \tag{4.257}$$

where $\delta a = a - \langle a\rangle$. The expectation value of the uncertainty with respect to a state $|\psi\rangle$ is thus,

$$\langle\psi||\delta a|^2|\psi\rangle = \langle\psi|a^\dagger a|\psi\rangle - |\langle\psi|a|\psi\rangle|^2 + \frac{1}{2} \tag{4.258}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

The expectation value of the uncertainty with respect to a transformed state $|\psi\rangle_t \equiv RS|\psi\rangle$ is given by

$$\langle\psi||\delta a|^2|\psi\rangle_t = \cosh 2r\langle\psi||\delta a|^2|\psi\rangle - 2\sinh 2r\mathrm{Re}[e^{-2i\phi}\langle\psi|\delta a_+\delta a_-|\psi\rangle] \qquad (4.259)$$

where $|\delta a|^2 \equiv |\delta a_+|^2 + |\delta a_-|^2$. For an initial number state, $|\psi\rangle = |n\rangle$,

$$\langle n||\delta a|^2|n\rangle_t = 2\left(\frac{1}{2} + |\beta|^2\right)\langle n||\delta a|^2|n\rangle \geq \langle n||\delta a|^2|n\rangle \qquad (4.260)$$

For a coherent state, $|\psi\rangle = |\mu\rangle$

$$\langle\mu||\delta a|^2|\mu\rangle_t = 2\left(\frac{1}{2} + |\beta|^2\right)\langle\mu||\delta a|^2|\mu\rangle \geq \langle\mu||\delta a|^2|\mu\rangle \qquad (4.261)$$

where the first term corresponds to the vacuum fluctuation and the second term (whose sum over all modes is equivalent to $\mathrm{Tr}(v^\dagger_{\mathbf{k}} v_{\mathbf{k}})$ in [Hu72, HuPav86]) measures the mixing of the positive and negative frequency components of different modes. This result was first derived in [Hu72], and discussed further in [HuPav86]. Notice that it is always greater than the original value $\langle|\delta a|^2\rangle_\mu$.

For a squeezed state, $|\psi\rangle = |\sigma\rangle = S_2(r_0, \phi_0)|\mu\rangle$

$$\langle\sigma||\delta a|^2|\sigma\rangle_t = \cosh 2r\langle\sigma||\delta a|^2|\sigma\rangle - 2\sinh 2r\mathrm{Re}[e^{-2i\phi}\langle\sigma|\delta a_+\delta a_-|\sigma\rangle] \qquad (4.262)$$

which can be smaller than the initial value.

Notice that of the three states we discussed, only the squeezed state can allow for a decrease in the uncertainty, i.e. an increase in the coherence as the system evolves. In addition, even though the total number and the total uncertainty of the initial state of the two modes change with particle creation, their difference remains a constant. This is because cosmological particle creation is described by the two-mode squeezed operator which satisfies the relations: $\langle\psi|S^\dagger(a^\dagger_+ a_+ - a^\dagger_- a_-)S|\psi\rangle = \langle\psi|a^\dagger_+ a_+ - a^\dagger_- a_-|\psi\rangle$,

$$\langle\psi|S^\dagger(|\delta a_+|^2 - |\delta a_-|^2)S|\psi\rangle = \langle\psi|(|\delta a_+|^2 - |\delta a_-|^2)|\psi\rangle \qquad (4.263)$$

4.7.4 Fluctuations in number

Spontaneous particle creation can be viewed as the parametric amplification of vacuum fluctuations (or squeezing the vacuum). Particle number is an interesting quantity as it measures the degree to which the vacuum is excited. The fluctuation in particle number is another interesting quantity, as it can be related to the noise of the quantum field and the susceptibility of the vacuum. This is similar in nature to the energy fluctuation (measured by the heat capacity at constant volume) of a system being related to the thermodynamic stability of a canonical system, or the number fluctuation (measured by the compressibility at constant pressure) of a system being related to the thermodynamic stability of a grand canonical system. In gravity, we know that the number fluctuation of a self-gravitating system can be used as a measure of its heat capacity (negative)

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

[LynBel77]; and those associated with particle creation from a black hole can be used in a linear-response theory description as a measure of the susceptibility of spacetime [CanSci77, Mot86]. We expect that this quantity associated with cosmological particle creation may provide some important information about quantum noise and vacuum instability.

Define $\delta_i O \equiv [\langle O^2 \rangle - \langle O \rangle^2]$ as the variance or mean-square fluctuations of the variable O with respect to the initial state $|\ \rangle$, and the corresponding quantity $\delta_f O$ as that with respect to the final state $|\ )$. Consider the difference between the final and the initial number fluctuation of both the $\pm$ kinds,

$$\delta N = (\delta_f N_+ + \delta_f N_-) - (\delta_i N_+ + \delta_i N_-) \tag{4.264}$$

Using the expressions given above, we obtain

$$\begin{aligned}\delta N = 2|\alpha|^2|\beta|^2[\delta N_+ + \delta N_- + \delta L + \partial(N_+ N_-)]_i \\ -(|\alpha|^3|\beta| + |\alpha||\beta|^3)[\partial(N_+ L) + \partial(N_- L)]_i\end{aligned} \tag{4.265}$$

where the subscript i refers to expectation values with respect to the initial states $|\ \rangle$, the symbol ∂ denotes

$$\partial(PQ) \equiv [\langle PQ \rangle + \langle QP \rangle - 2\langle P \rangle \langle Q \rangle] \tag{4.266}$$

and

$$L = e^{2i\phi} a_+^\dagger a_-^\dagger + e^{-2i\phi} a_- a_+ \tag{4.267}$$

Now for an initial number state $|n\rangle = |n_+, n_-\rangle$,

$$\delta N = 2|\alpha|^2|\beta|^2(1 + n_+ + n_- + 2n_+ n_-) \tag{4.268}$$

we see that the number fluctuations will always increase. For an initial coherent state $|\mu\rangle = D(\mu_+)D(\mu_-)|0,0\rangle$, where $\mu_\pm = \sqrt{\langle N_\pm \rangle} e^{i\zeta_\pm}$,

$$\begin{aligned}\delta N = 2|\alpha|^2|\beta|^2[1 + 2(\langle N_+ \rangle + \langle N_- \rangle)] \\ - 4\sqrt{\langle N_+ \rangle \langle N_- \rangle}(|\alpha|^3|\beta| + |\alpha||\beta|^3)\cos(2\phi - \zeta_+ - \zeta_-)\end{aligned} \tag{4.269}$$

We find that under the conditions $\cos(2\phi - \zeta_+ - \zeta_-) = 1$ and $\langle N_+ \rangle = \langle N_- \rangle = \langle N \rangle / 2$

$$\langle N \rangle > \frac{|\beta||\alpha|}{|\alpha|^2 + |\beta|^2 - |\beta||\alpha|} \tag{4.270}$$

δN can be negative. In the weak particle creation limit $|\beta| \to 0, |\alpha| \to 1$ we find that this expression is equivalent to (4.252). In the strong particle creation limit we see that (4.252) diverges but in (4.270) $\langle N \rangle \to 1$. Clearly conditions for a decrease in number fluctuations are not the same as those for a decrease in the number.

For a single-mode squeezed state $|\sigma\rangle_1 = S_{1+}(r_+, \phi_+)S_{1-}(r_-, \phi_-)|0,0\rangle$

$$\begin{aligned}\delta N = 2|\alpha|^2|\beta|^2[(1 + \langle N_+ \rangle + \langle N_- \rangle)^2 + \langle N_+ \rangle(1 + \langle N_+ \rangle) + \langle N_- \rangle(1 + \langle N_- \rangle) \\ - 2\sqrt{\langle N_+ \rangle(1 + \langle N_+ \rangle)\langle N_- \rangle(1 + \langle N_- \rangle)}\cos 2(2\phi - \phi_+ - \phi_-)]\end{aligned} \tag{4.271}$$

From this it can be shown that, like the change in number, the change in the number fluctuations will always be positive for an initial single-mode squeezed vacuum.

For a two-mode squeezed state $|\sigma\rangle_2 = S_2(r_0, \phi_0)|0, 0\rangle$

$$\delta N = |\alpha|^2|\beta|^2\{2(1 + \langle N\rangle)^2 + \langle N\rangle(2 + \langle N\rangle)[1 + \cos 4(\phi - \phi_0)]\} \tag{4.272}$$
$$+ 2(|\alpha|^3|\beta| + |\beta|^3|\alpha|)(1 + \langle N\rangle)\sqrt{\langle N\rangle(2 + \langle N\rangle)} \cos 2(\phi - \phi_0) \tag{4.273}$$

Note that there is no definite relation between N and δN. For large $N \gg 1$ or small $|\beta| \ll 1$, $\delta N \leq 0$. The result obtained here for particle number fluctuations is relevant to issues of noise and fluctuation of quantum fields, and in turn, the dissipation and instability of condensates, background fields and spacetimes [HuSin95, HuMat96, HuMat95].

4.8 Squeezed quantum open systems

In this last section we discuss a squeezed quantum system interacting with an environment. From the examples given in this chapter we see that this encompasses a rather broad spectrum of systems with time-dependent background fields or spacetimes.

This theory was developed in the influence functional formalism by Hu and Matacz [HuMat94] extending the work on quantum Brownian motion by Hu, Paz and Zhang [HuPaZh92, HuPaZh93a], and Caldeira and Leggett [CalLeg83a] to oscillators with time-dependent frequencies. From this oscillator model it is an easy step to extend to quantum fields, which was done in [Zha90, Hu94b]. We shall treat open systems of quantum fields in the next chapter.

Our discussion here follows Koks *et al.* [KoMaHu97] based on the work of [HuMat94] which considers a squeezed (time-dependent, parametric) quantum open system coupled to a bath at temperature T with a time-dependent coupling constant. The results here are useful for calculating the entropy and uncertainty functions as well as for fluctuations and coherence, a topic to be discussed in Chapter 9.

4.8.1 Dissipation and noise kernels

For a parametric oscillator system interacting with a bath of many parametric oscillators at temperature T described by the Lagrangian given by equation (3.133) in Chapter 3 one can calculate the dissipation and noise kernels in closed forms ((equation (2.19) of [HuMat94]) in terms of the squeezed state parameterization (r, θ, ϕ) introduced in the previous section and the Bogoliubov coefficients (α, β) representation of the mode functions. In the case of a squeezed bath when the cross-term ($\varepsilon_n = 0$) is absent and the mass of the bath oscillator is a constant ($m_n = 1$) these expressions are in a manageable form. Note that the functions $\chi_n(t) = \alpha_n(t) + \beta_n(t)$ obey the equations (cf. equation (4.238)):

$$\ddot{\chi}_n(t) + \omega_n^2(t)\chi_n(t) = 0 \tag{4.274}$$

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

with initial conditions (compare to (4.239))

$$\chi_n(t_i) = 1; \quad \dot{\chi}_n(t_i) = -i\kappa_n \tag{4.275}$$

The bath canonical variables then admit a simple representation

$$q_n(s) = \frac{1}{2}\left\{[\chi_n(s) + \chi_n^*(s)]\, q_n(t_i) + \frac{i}{\kappa_n}\,[\chi_n(s) - \chi_n^*(s)]\, \dot{q}_n(t_i)\right\} \tag{4.276}$$

The initial state of the bath is a squeezed thermal state. It has the form

$$\hat{\rho}_b(t_i) = \prod_n \hat{S}_n(r(n), \phi(n))\hat{\rho}_{\text{th}}\hat{S}_n^\dagger(r(n), \phi(n)) \tag{4.277}$$

where $\hat{\rho}_{\text{th}}$ is a thermal density matrix of temperature T and $\hat{S}(r, \phi)$ is a squeeze operator defined in equation (4.216).

In this still rather general class of problems, the noise and dissipation kernels can be found from equations (3.62), where the relevant expectation values are computed with the help of (4.223)

$$\mathbf{D}(s, s') = 2\int_0^\infty d\omega\, I(\omega, s, s')\text{Im}[\chi_\omega(s)\chi_\omega^*(s')] \tag{4.278}$$

$$\mathbf{N}(s, s') = \int_0^\infty d\omega\, I(\omega, s, s') \coth\left(\frac{\hbar\omega(t_i)}{2k_B T}\right)\left\{\cosh 2r(\omega)\text{Re}[\chi_\omega(s)\chi_\omega^*(s')]\right.$$
$$\left. -\frac{1}{2}\sinh 2r(\omega)\left[e^{-2i\phi(\omega)}\chi_\omega^*(s)\chi_\omega^*(s') + e^{2i\phi(\omega)}\chi_\omega(s)\chi_\omega(s')\right]\right\} \tag{4.279}$$

We have adopted the convention that if f_n is a quantity defined for each mode of the bath, then we call $f(\omega) = f_\omega \equiv f_n$ evaluated at the mode that satisfies $\omega = \omega_n(t_i)$. $I(\omega, s, s')$ is the spectral density defined by

$$I(\omega, s, s') = \sum_n \delta(\omega - \omega_n(t_i))\frac{c_n(s)c_n(s')}{2\kappa_n} \tag{4.280}$$

It contains information about the environmental mode density and coupling strength as a function of frequency. Different environments are classified according to the functional form of the spectral density $I(\omega)$. On physical grounds, one expects the spectral density to go to zero for very high frequencies. Let us introduce a certain cut-off frequency Λ (a property of the environment) such that $I(\omega) \to 0$ for $\omega > \Lambda$. The environment is classified as ohmic if in the physical range of frequencies ($\omega < \Lambda$) the spectral density is such that $I(\omega) \sim \omega$, as supra-ohmic if $I(\omega) \sim \omega^n, n > 1$ or as sub-ohmic if $n < 1$. The most studied ohmic case corresponds to an environment which induces a dissipative force linear in the velocity of the system. Also, by considering the continuum limit of the coupling constant, it can be shown that this constant's independence of n also leads to an ohmic environment.

Note that the dissipation kernel is independent of the bath's initial state. More generally, the noise and dissipation kernels are built out of symmetric and antisymmetric combinations of identical Bogoliubov factors. Thus the two kernels are intimately linked. For the case when the bath is a standard harmonic oscillator this inter-relationship can be written as a generalized fluctuation–dissipation relation [HuPaZh93a].

4.8.2 $u_1 \to v_2$ functions

In these last two subsections we present the explicit forms of the u, v and a, b functions for this squeezed quantum system. Recall that these are the functions first appearing in Chapter 3 in the derivation of the propagator for the reduced density matrix which determine the coefficients of the master equation. First consider equation (3.138). We treat the integral of a delta function and its derivative in the following way: use a smooth step function (i.e. $\theta(0) \equiv 1/2$) to write ($x_1 > x_0$)[1]

$$\int_{x_0}^{x_1} f(x)\delta(x-a)\,dx \equiv \quad f(a)\,\theta(x_1-a)\,\theta(a-x_0) \tag{4.281}$$

$$\int_{x_0}^{x_1} f(x)\delta'(x-a)\,dx \equiv -f'(a)\,\theta(x_1-a)\,\theta(a-x_0) \tag{4.282}$$

Hence equation (3.138) together with equation (3.142) becomes (with u being either u_1 or u_2)

$$\ddot{u}(s) + \left(\frac{\dot{M}}{M} + \frac{2\gamma_0 c^2}{M}\right)\dot{u} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} + \frac{2\gamma_0 c\dot{c}}{M}\right)u = 0 \tag{4.283}$$

Now define $\tilde{u}$ by

$$\tilde{u} \equiv u \exp\left[\gamma_0 \int_{t_i}^{s} \frac{c^2(s')}{M(s')} ds'\right] \tag{4.284}$$

in which case it follows that

$$\ddot{\tilde{u}} + \frac{\dot{M}}{M}\dot{\tilde{u}} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} - \frac{\gamma_0^2 c^4}{M^2}\right)\tilde{u} = 0 \tag{4.285}$$

Comparing with (4.238), we recognize this as just the equation of motion of an oscillator with mass M, cross-term $\mathcal{E}$ and an effective frequency

$$\Omega_{\text{eff}}^2 \equiv \Omega^2 - \frac{\gamma_0^2 c^4}{M^2} \tag{4.286}$$

Hence we know a solution for $\tilde{u}(s)$ – it is the sum χ of the Bogoliubov coefficients for this new system. So we write (with g_1, g_2 constants to be determined)

$$u(s) = \exp\left[-\gamma_0 \int_{t_i}^{s} \frac{c^2}{M} ds'\right] [g_1\chi(s) + g_2\chi^*(s)] \tag{4.287}$$

[1] These relations can easily be proved by checking the five cases individually, of $a < x_0$, $a = x_0$, $x_0 < a < x_1$, etc. Note that treating the delta function in this "smoothed" way eliminates the need for the frequency renormalization in [PaHaZu93]. This smoothing essentially just defines $\int_0^\infty \delta(x)dx = 1/2$ (see, e.g. [NeuHil27] for a discussion of this).

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

By including the boundary conditions for u_1 and u_2 we obtain

$$u_1(s) = \exp\left[-\gamma_0 \int_{t_i}^{s} \frac{c^2}{M} ds'\right] \frac{\mathrm{Im}[\chi(t)\chi^*(s)]}{\mathrm{Im}\chi(t)}$$

$$u_2(s) = \exp\left[\gamma_0 \int_{s}^{t} \frac{c^2}{M} ds'\right] \frac{\mathrm{Im}\chi(s)}{\mathrm{Im}\chi(t)} \tag{4.288}$$

Using the propagator formalism in the language of squeezed states with the Bogoliubov coefficients will be very useful for relating the entropy of a field mode to its squeeze parameter r.

Proceeding in the same way, equation (3.139) and (3.39) becomes

$$\ddot{v}(s) + \left(\frac{\dot{M}}{M} - \frac{2\gamma_0 c^2}{M}\right)\dot{v} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} - \frac{2\gamma_0 c\dot{c}}{M}\right) v = 0 \tag{4.289}$$

Now write

$$\tilde{v} \equiv v \exp\left[-\gamma_0 \int_{t_i}^{s} \frac{c^2}{M} ds'\right] \tag{4.290}$$

and just as for the case of u we have

$$\ddot{\tilde{v}} + \frac{\dot{M}}{M}\dot{\tilde{v}} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} - \frac{\gamma_0^2 c^4}{M^2}\right)\tilde{v} = 0 \tag{4.291}$$

So now v_1 and v_2 can also be written as combinations of χ and χ^*. Including the boundary conditions we eventually obtain

$$v_1(s) = \exp\left[\gamma_0 \int_{t_i}^{s} \frac{c^2}{M} ds'\right] \frac{\mathrm{Im}[\chi(t)\chi^*(s)]}{\mathrm{Im}\chi(t)}$$

$$v_2(s) = \exp\left[-\gamma_0 \int_{s}^{t} \frac{c^2}{M} ds'\right] \frac{\mathrm{Im}\chi(s)}{\mathrm{Im}\chi(t)} \tag{4.292}$$

4.8.3 $a_{11} \to b_4$ functions

To facilitate our calculations we introduce dimensionless parameters for time

$$z \equiv \kappa t, \quad \sigma \equiv \kappa s$$
$$\chi(\tau) \equiv \chi(t), \quad \text{etc.} \tag{4.293}$$

and a carat will denote division by κ, e.g. $\hat{\gamma}_0 = \gamma_0/\kappa$. Note that t is the Lagrangian time.

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press

Now we have all the necessary ingredients to calculate the propagator. Making use of equation (3.136) and equation (3.137) we obtain

$$
\begin{aligned}
a_{11}(z, z_i) = \frac{1}{2\kappa^2} \int_{z_i}^{z} d\sigma \int_{z_i}^{z} d\sigma' \, \exp\left(\hat{\gamma}_0 \int_{z_i}^{\sigma} \frac{c^2}{M} d\sigma''\right) \frac{\mathrm{Im}[\chi(z)\chi^*(\sigma)]}{\mathrm{Im}\chi(z)} \, N(\sigma, \sigma') \\
\times \exp\left(\hat{\gamma}_0 \int_{z_i}^{\sigma'} \frac{c^2}{M} d\sigma''\right) \frac{\mathrm{Im}[\chi(z)\chi^*(\sigma')]}{\mathrm{Im}\chi(z)} \\
a_{12} = \frac{1}{\kappa^2} \int_{z_i}^{z} d\sigma \int_{z_i}^{z} d\sigma' \, \exp\left(\hat{\gamma}_0 \int_{z_i}^{\sigma} \frac{c^2}{M} d\sigma''\right) \frac{\mathrm{Im}[\chi(z)\chi^*(\sigma)]}{\mathrm{Im}\chi(z)} \, N(\sigma, \sigma') \\
\times \exp\left(-\hat{\gamma}_0 \int_{\sigma'}^{z} \frac{c^2}{M} d\sigma''\right) \frac{\mathrm{Im}\chi(\sigma')}{\mathrm{Im}\chi(z)} \\
a_{22} = \frac{1}{2\kappa^2} \int_{z_i}^{z} d\sigma \int_{z_i}^{z} d\sigma' \, \exp\left(-\hat{\gamma}_0 \int_{\sigma}^{z} \frac{c^2}{M} d\sigma''\right) \frac{\mathrm{Im}\chi(\sigma)}{\mathrm{Im}\chi(z)} \, N(\sigma, \sigma') \\
\times \exp\left(-\hat{\gamma}_0 \int_{\sigma'}^{z} \frac{c^2}{M} d\sigma''\right) \frac{\mathrm{Im}\chi(\sigma')}{\mathrm{Im}\chi(z)} \\
b_1(z, z_i) = -\hat{\gamma}_0 \, \kappa c^2(z) + \kappa M(z) \frac{\mathrm{Im}\chi'(z)}{\mathrm{Im}\chi(z)} + M(z)\mathcal{E}(z) \\
b_{\{{2 \atop 3}\}} = \frac{\mp\kappa}{\mathrm{Im}\chi(z)} \exp\left(\pm\hat{\gamma}_0 \int_{z_i}^{z} \frac{c^2}{M} d\sigma\right) \\
b_4 = -\hat{\gamma}_0 \, \kappa c^2(z_i) + \kappa \, \frac{\mathrm{Re}\chi(z)}{\mathrm{Im}\chi(z)} + M(z_i)\mathcal{E}(z_i)
\end{aligned}
\tag{4.294}
$$

These coefficients will be useful for calculating the entropy generation in a squeezed open quantum system in Chapter 9.

https://doi.org/10.1017/9781009290036.007 Published online by Cambridge University Press