ON ARCS IN A FINITE PROJECTIVE PLANE

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1. Introduction and summary. The aim of this paper is to generalize and unify results of B. Qvist, B. Segre, M. Sce, and others concerning arcs in a finite projective plane. The method consists of applying completely elementary combinatorial arguments.

To the usual axioms for a projective plane we add the condition that the number of points be finite. Thus there exists an integer $n \geqslant 2$, called the *order* of the plane, such that the number of points and the number of lines equal $n^2 + n + 1$ and the number of points on a line and the number of lines through a point equal $n + 1$. In the following, n will always denote the order of a finite plane. Desarguesian planes of order n, formed by the analytic geometry with coefficients from the Galois field of order n, are examples of finite projective planes. We shall not assume that our planes are Desarguesian, however.

A set K of $k > 3$ points in a finite projective plane such that no three are collinear will be called a *k-arc*. A line containing exactly one point of K will be called a *tangent* of K, a line containing two points of K will be called a *secant* of K, and a line that does not contain any points of K will be called an *exterior line* of K. An $(n + 1)$-arc in a plane of order n is called an *oval*. We define t by the equation $n + 2 = k + t$. Thus $t = (n + 1) - (k - 1)$ is the number of tangents of K at any point of K.

Clearly a k-arc K can have at most $n + 2$ points since each of the $n + 1$ lines through an arbitrary point of K can contain at most one other point of K. Such an arc can, of course, have no tangents. Since our emphasis will be on incidence relations involving the tangents of K, we shall assume that $3 < k < n + 2$ or equivalently $0 < t < n - 1$ unless otherwise specified.

Immediately we have that there are exactly kt tangents, $\binom{k}{2}$ secants, and $\binom{n}{2} + \binom{t}{2}$ exterior lines.

Suppose that a point off K lies on a tangents, b secants, and c exterior lines. Then $a + b + c = n + 1$ and $a + 2b = k$.

Given a k-arc K, we distinguish the points of the plane as follows. First there are the points of K itself. A point off K which lies on at most one tangent will be called an *interior point* of K. A point off K which lies on at least two tangents will be called an *exterior point* of K. We note that if k is even, a point off K must lie on an even number of tangents, while if k is odd a point off K must lie on an odd number of tangents.

We shall further distinguish the exterior points as follows. If a point off K lies

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on exactly two (k even) or three (k odd) tangents, we shall call the point an *ordinary exterior point.* If a point off K lies on more than three tangents, we shall call the point an *extra exterior point.* A point lying on k tangents will be called a *completion point.* An arc is said to be *complete* if every point of the plane lies on a secant of the arc.

Define e_i to be the number of points off K which lie on exactly i tangents. Thus if k is even $e_{2j+1} = 0$, and if k is odd $e_{2j} = 0$ (for all j). Finally we say a k-arc is *uniform* of *index* v if every extra exterior point lies on exactly v tangents.

In §2 we derive some elementary incidence theorems and incidence equations which will be used throughout. The most important of these is the set of diophantine equations (8), concerning incidence with respect to the tangents of an arc. These equations are equivalent to the set of equations in (**9**, p. 291). There the emphasis is on incidence relations with regard to the secants.

In §3 we show that the properties of the ovals follow easily from the incidence equations and generalize some results of B. Qvist. The Hughes plane of order 9 is used to give an example of two distinct ovals in a plane of odd order having $\frac{1}{2}(n + 1)$ points in common, proving that a result of Qvist is the best possible.

In §4 it is proved that except for some extreme values of k, a k-arc K is complete if it is of uniform index. Specifically if $k \neq 4$ or n and K is uniform of even index or if $5 < k < n - 1$ and K is uniform of odd index, then K is complete. Counterexamples are given to show that these excepted values of k must necessarily be excluded.

§5 deals with necessary and sufficient conditions that an n-arc in a plane of order n be complete. B. Segre has proved that every n-arc in a finite Desarguesian plane is contained in an oval. This result cannot be extended to the general finite plane. A counterexample is given by exhibiting complete n-arcs in the Hughes plane of order 9.

It is shown that for an n-arc N in a plane of even order each of the following is equivalent to N being contained in an oval:

(i) N is uniform of index n.
(ii) There exists a tangent with at most one extra exterior point.
(iii) Every tangent contains at most one extra exterior point.
(iv) There exist at most two extra exterior points.
(v) No secant contains an extra exterior point.
(vi) There exist $n - 1$ collinear interior points.
(vii) The number of interior points is at most $n - 1$.
(viii) The number of interior points is exactly $n - 1$.
(ix) The number of exterior points is at least $n^2 - n + 2$.
(x) The number of exterior points is exactly $n^2 - n + 2$.

It is similarly shown that for an n-arc N in a plane of odd order each of the following is equivalent to N being contained in an oval:

(i) N is uniform of index n.
(ii) The number of interior points is at most $\frac{1}{2}n(n + 1)$.
(iii) The number of interior points is exactly $\frac{1}{2}n(n + 1)$.

(iv) There exist n collinear interior points.
(v) No secant contains an extra exterior point.
(vi) Every secant contains exactly $\frac{1}{2}(n-3)$ ordinary exterior points.
(vii) Every secant contains exactly $\frac{1}{2}(n+1)$ interior points.
(viii) Every exterior line but one has exactly $\frac{1}{2}(n+3)$ interior points.
(ix) Every exterior line but one has exactly $\frac{1}{2}(n-1)$ ordinary exterior points.
(x) At each point of N, one tangent contains exactly one interior point and the other tangent contains exactly $\frac{1}{2}(n+1)$ interior points.

This paper is an augmentation of the second chapter of the author's dissertation "On Arcs in the Finite Projective Planes" (University of Michigan, 1964).

2. Elementary incidence theorems. Let L be a secant of a k-arc K in a plane of order n. Let m be the minimum and M the maximum of the number of secants through a point of L off K. Thus each such point lies on at least $m-1$ and at most $M-1$ secants different from L. There are $n-1$ points on L off K. Also, the number of secants intersecting L off K is

$$\binom{k}{2} - 2(k-1) + 1.$$

Hence,

$$(n-1)(m-1) \leqslant \binom{k}{2} - 2(k-1) + 1 \leqslant (n-1)(M-1)$$

or

$$(1) \qquad m \leqslant z = 1 + \left[\binom{k}{2} - 2(k-1) + 1\right] \Big/ (n-1) \leqslant M.$$

We write z in three different forms, each of which will be of use below.

$$(2) \qquad z = \frac{k}{2} - \frac{t(n-t)}{2(n-1)} = \frac{k-1}{2} - \left[\frac{t(n-t)}{2(n-1)} - \frac{1}{2}\right] = \frac{n-2t+1}{2} + \frac{1}{2} + \frac{t(t-1)}{2(n-1)}.$$

Now if a point lies on z secants, then it lies on $k - 2z$ tangents. This fact, together with (2), gives the following theorem.

(3) Theorem. *Let K be a k-arc in a plane of order n. Every secant contains a point off K which lies on at most*

$$\frac{k}{2} - \frac{t(n-t)}{2(n-1)}$$

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secants and hence on at least $t - [t(t-1)/(n-1)]$ *tangents. Every secant contains a point off* K *which lies on at least*

$$\frac{k}{2} - \frac{t(n-t)}{2(n+1)}$$

secants and hence on at most $t - [t(t-1)/(n-1)]$ *tangents.*

We remark that this theorem is the best possible in the sense that the upper and lower limits as stated in the theorem can actually be achieved. This can be observed as follows. Let P be an exterior point of an oval in the plane of order 7. Form a 6-arc K by deleting from the oval the two points on a secant through P. Let L be a secant through P which does not contain the exterior point of the oval formed by the intersection of the tangents to the oval at the deleted points. Then every point on L off K lies on exactly two tangents and exactly two secants of K. Note that L is a secant without an interior point.

(4) COROLLARY. *Let* K *be a* k*-arc in a plane of order* n*. If* k *is even and*

$$t(n-t)/[2(n-1)] < 1$$

or if k *is odd and* $t(n-t)/[3(n-1)] < 1$, *then every secant contains an interior point.*

Proof. Any point off K can lie at most $\frac{1}{2}k$ secants if k is even and at most $\frac{1}{2}(k-1)$ secants if k is odd. With the notation as above, $z \leqslant M$. By the first expression for z in (2), if k is even and $t(n-t)/[2(n-1)] < 1$, then $M = \frac{1}{2}k$. Likewise by the second expression for z in (2), if k is odd and

$$\frac{t(n-t)}{2(n-1)} - \frac{1}{2} < 1,$$

then $M = \frac{1}{2}(k-1)$. Therefore every secant has at least one point that does not lie on more than one tangent. Such a point is an exterior point.

(5) COROLLARY **(6)**. *Let* K *be a* k*-arc in a plane of order* n*. If* n *is even and* $n > \frac{1}{2}t(t-1) + 1$, *then on every secant there is at least one point off* K *which lies on at least* t *tangents. If* n *is odd and* $n > t(t-1) + 1$, *then on every secant there is at least one point off* K *which lies on at least* $t+1$ *tangents.*

Proof. It follows from (1) and the third expression for z in (2) that if n is even and $t(t-1)/[2(n-1)] < 1$, then

$$m \leqslant \tfrac{1}{2}(n-2t+1) - \tfrac{1}{2}$$

(an integer). Similarly, if n is odd and $t(t-1)/n < 1$, then $m \leqslant \frac{1}{2}(n-2t+1)$ (an integer). A point on at most m secants lies on at least $k - 2m$ tangents. The corollary follows from the identity

$$k - 2[\tfrac{1}{2}(n-2t+1) + \tfrac{1}{2}] = t + 1 - 2(\tfrac{1}{2}).$$

By exactly the same procedure as in Theorem (3), the following can easily be proved. The proof is omitted.

(6) THEOREM. *Let K be a k-arc in a plane of order n. Every tangent contains a point off K which lies on at least $(k-1)(k-2)/2n$ secants and hence on at most $t+1+[t(t-1)/n]$ tangents. Every tangent contains a point off K which lies on at most $(k-1)(k-2)/2n$ secants and hence on at least $t+1+[t(t-1)/n]$ tangents. Every exterior line contains a point off K which lies on at least*

$$k(k-1)/[2(n+1)]$$

secants and hence on at most $t-[t(t-1)/(n+1)]$ tangents. Every exterior line contains a point off K which lies on at most $k(k-1)/[2(n+1)]$ secants and hence on at least $t-[t(t-1)/(n+1)]$ tangents.

Since a point on a tangent lies on at most $(k-1)(k-2)/2n<1$ secants, we have

(7) COROLLARY. *Let K be a k-arc in a plane of order n. If $n > \binom{k-1}{2}$, then K is not complete.*

We now consider incidence in the whole plane. There are $n^2+n+1-k$ points off a k-arc K in a plane of order n. The number of points off K which lie on exactly i tangents is e_i. The configuration of a point and a line through the point is called a *flag*. We consider the number of flags where the point is a point off K and the line is a tangent. Each tangent has exactly n points off K, and thus there are ktn such flags. A point off K lying on exactly i tangents accounts for exactly i such flags and $\binom{i}{2}$ pairs of distinct tangents intersecting off K. The total number of pairs of distinct tangents is $\binom{kt}{2}$, and $k\binom{t}{2}$ of these have intersection on K. Thus we have the following set of Diophantine equations for a k-arc in a plane of order n:

$$\sum_{i=0}^{k} e_i = n^2+n+1-k,$$

$$\sum_{i=0}^{k} ie_i = ktn, \tag{8}$$

$$\sum_{i=0}^{k} i(i-1)e_i = t^2k(k-1).$$

We have to distinguish between even and odd values of k. If k is even, then

$$\sum_{j=0}^{\frac{1}{2}k} e_{2j} = n^2+n+1-k,$$

$$\sum_{j=0}^{\frac{1}{2}k} je_{2j} = \tfrac{1}{2}ktn, \tag{9}$$

$$\sum_{j=0}^{\frac{1}{2}k} j(j-1)e_{2j} = \tfrac{1}{4}t(t-1)k(k-2).$$

If k is odd, then

$$\sum_{j=0}^{\frac{1}{2}(k-1)} e_{2j+1} = n^2 + n + 1 - k,$$

$$\sum_{j=0}^{\frac{1}{2}(k-1)} je_{2j+1} = \tfrac{1}{2}(k-1)(nt - n + 1), \tag{10}$$

$$\sum_{j=0}^{\frac{1}{2}(k-1)} j(j-1)e_{2j+1} = \tfrac{1}{4}(t^2 - 3t + 3)(k+1)(k-3).$$

We note the existence of extra exterior points for any k-arc except possibly the extreme case $t = 1$. This follows from the third equations of (9) and (10). In the next section we shall examine the case $t = 1$, i.e., the ovals.

3. Ovals. Ovals are the $(n + 1)$-arcs of a plane of order n. In **(5)** B. Qvist has determined the elementary properties of the ovals in a finite projective plane. We shall first show that the elementary properties of the ovals are an immediate consequence of the equations (8).

(11) THEOREM **(5)**. *In a plane of even order n all the tangents to an oval are concurrent and hence every oval can be uniquely completed to an $(n + 2)$-arc.*

Proof. Let $t = 1$ with $k = n + 1$ odd. From (10) we have

$$\sum_{j=0}^{\frac{1}{2}n} j(n - 2j)e_{2j+1} = 0.$$

Thus $e_{2j+1} = 0$ if $j \neq 0$ or $j \neq \frac{1}{2}n$. Then

$$\tfrac{1}{2}ne_{n+1} = \sum_{j=0}^{\frac{1}{2}n} je_{2j+1} = \tfrac{1}{2}n.$$

Therefore $e_{n+1} = 1$, $e_1 = n^2 - 1$, and all other e_i are zero. Thus the oval has exactly one exterior point which lies on $n + 1$ tangents.

(12) LEMMA. *If $k = n + 1$ is even, then $e_{2j} = 0$ when $j > 1$.*

Proof. With $t = 1$ and $k = n + 1$, (9) gives

$$\sum_{j=0}^{\frac{1}{2}(n+1)} j(j-1)e_{2j} = 0.$$

Hence $e_{2j} = 0$ if $j > 1$.

(13) THEOREM **(1; 5)**. *A k-arc in a plane of odd order has at most $n + 1$ points.*

Proof. The result follows from the lemma, as in particular $e_{n+1} = 0$. Thus an oval never has a completion point.

(14) THEOREM **(5)**. *In a plane of odd order every point off an oval lies on either exactly two tangents or none at all.*

Proof. The theorem is an immediate consequence of Lemma (12) and the fact that a point off an oval must lie on an even number of tangents.

(15) Corollary. *An oval in a plane of odd order n has exactly $\frac{1}{2}n(n+1)$ exterior points and $\frac{1}{2}n(n-1)$ interior points. Further, every secant contains $\frac{1}{2}(n-1)$ exterior points and $\frac{1}{2}(n-1)$ interior points. An exterior line carries $\frac{1}{2}(n+1)$ exterior points and $\frac{1}{2}(n+1)$ interior points.*

Proof. Every exterior point lies on exactly two tangents.

(16) Theorem. *Suppose V is an oval and K is a k-arc in a plane of odd order n. If K and V have more than $\frac{1}{2}(n+3)$ points in common, then V contains K. If K and V have more than $\frac{1}{2}(n+1)$ points in common and $n > 4t+1$, then V contains K.*

Proof. Suppose K is not contained in V but they have s points in common. From a point on K off V there are s lines to the common points. Say m of these are tangents to V. Thus we account for $m + 2(s-m) \leqslant n+1$ points of V. Since m is either zero or two, it then follows that

$$2s \leqslant n + 1 + m \leqslant n + 3.$$

Thus $s \leqslant \frac{1}{2}(n+3)$, which proves the first part of the theorem.

For the second part, assume K and V have exactly $\frac{1}{2}(n+3)$ points in common. From each point on K off V we have two tangents to V tangent at a common point. Each such tangent contains two points of K. Hence

$$2[k - \tfrac{1}{2}(n+3)] \leqslant \tfrac{1}{2}(n+3),$$

which reduces to $4k \leqslant 3(n+3)$ or $n \leqslant 4t+1$.

(17) Corollary **(5)**. *If n is odd, $n > 5$, and two ovals in a plane of order n have more than half their points in common, then the ovals coincide.*

Proof. Let $t = 1$ in the second part of the theorem.

The statements of Theorem (16) and Corollary (17) are the best possible in the sense indicated in the following theorem.

(18) Theorem. *There exist planes of odd order n containing arcs which are subsets of no oval but which have $\frac{1}{2}(n+3)$ points in common with an oval. There exist planes of odd order $n > 7$ containing two distinct ovals with $\frac{1}{2}(n+1)$ points in common.*

In the proof of this theorem we shall use an important result due to B. Segre, which we state without proof.

(19) Theorem **(8)**. *In a Desarguesian plane of odd order, every oval is an irreducible conic.*

The converse of Segre's theorem is well known.

Proof of Theorem (18). A very simple construction demonstrates the first statement. Let Q be an exterior point of an oval in a finite Desarguesian plane of odd order n. Let the tangents from Q intersect the oval at P_1 and P_2 (Theorem (14)). From each of the $\frac{1}{2}(n-1)$ secants through Q, select one of the two points in common with the oval. Then these $\frac{1}{2}(n-1)$ points together with P_1, P_2, and Q form a $\frac{1}{2}(n+5)$-arc which has $\frac{1}{2}(n+3)$ points in common with the oval. If $n \geqslant 7$, so that $\frac{1}{2}(n+3) \geqslant 5$, the point Q does not lie on any non-degenerate conic containing the remaining $\frac{1}{2}(n+3)$ points of the $\frac{1}{2}(n+5)$-arc. The result follows from Segre's theorem.

We remark that L. Lombardo-Radice **(4)** has shown that in the above situation, if $n \equiv 3 \pmod 4$, if the oval is taken to be $z^2 = xy$, and if the exterior point is taken to be the origin $(0, 0, 1)$ and the points on the secants through the origin are chosen so that the coordinates are all squares, then the resulting $\frac{1}{2}(n+5)$-arc is actually complete.

We have yet to demonstrate the second statement of Theorem (18). By Corollary (17) and Theorem (19), the desired construction is impossible in a Desarguesian plane. Therefore we produce a non-Desarguesian plane of order 9 which will be of use:

Points: $A_i, B_i, C_i, D_i, E_i, F_i, G_i$, where $i = 0, 1, 2, \ldots, 12$.

Lines and incidence given by:

$$
\begin{aligned}
&L_1: A_0, A_1, A_3, A_9, B_0, C_0, D_0, E_0, F_0, G_0;\\
&L_2: A_0, B_1, B_8, D_3, D_{11}, E_2, E_5, E_6, G_7, G_9;\\
&L_3: A_0, C_1, C_8, E_7, E_9, F_3, F_{11}, G_2, G_5, G_6;\\
&L_4: A_0, B_7, B_9, D_1, D_8, F_2, F_5, F_6, G_3, G_{11};\\
&L_5: A_0, B_2, B_5, B_6, C_3, C_{11}, E_1, E_8, F_7, F_9;\\
&L_6: A_0, C_7, C_9, D_2, D_5, D_6, E_3, E_{11}, F_1, F_8;\\
&L_7: A_0, B_3, B_{11}, C_2, C_5, C_6, D_7, D_9, G_1, G_8;\\
&L_sA^m: \text{add } m \text{ to subscripts of } L_s \text{ and reduce (mod 13)},\\
&\qquad\qquad s = 1, 2, \ldots, 7 \text{ and } m = 1, 2, \ldots, 12.
\end{aligned}
$$

There are 91 points and 91 lines. This is the plane introduced by O. Veblen and J. Wedderburn in 1907 (**10**; cf. **2**, p. 411). It is also the Hughes plane of order 9, and D. R. Hughes **(3)** has determined that

$$V_1 = \{A_0, A_1, A_6, A_7, B_2, C_8, C_9, D_8, D_9, E_2\}$$

is an oval. Let M be the collineation defined by

$$PM = P[(BC)(DF)(EG)]A^6,$$

for every point P; cf. **(3)**. Then

$$V_2 = V_1 M = \{A_6, A_7, A_{12}, A_0, C_8, B_1, B_2, F_1, F_2, G_8\}$$

is an oval. A_0, A_6, A_7, B_2, and C_8 are common to V_1 and V_2. That is, V_1 and V_2 intersect in exactly $\frac{1}{2}(n+1)$ points. This completes the proof of Theorem (18).

The following theorem is a slight generalization of Qvist's theorem: if two $(n + 2)$-arcs have more than half of their points in common, then they coincide.

(20) THEOREM. *Let C be an $(n + 2)$-arc and K a k-arc in a plane of order n. If C and K have more than $\frac{1}{2}(n + 2)$ points in common, then C contains K.*

Proof. Suppose K is not contained in C, but they have s points in common. Consider the lines from a point on K off C through the s common points. Each such line is a secant of C. Hence $2s \leqslant n + 2$. The theorem follows.

We note that two ovals in a plane of even order may have as many as n points in common and still be distinct. Such ovals would, of course, be contained in the same $(n + 2)$-arc.

4. On complete arcs. Examples of ovals in non-Desarguesian planes have been given by D. R. Hughes **(3)** and by A. Wagner **(11)**. It is the ovals which naturally lead to the study of k-arcs. The most important arcs are those which are complete. Since every point of the plane lies on a secant of a complete arc, such an arc is maximal in the sense that no other arc can contain it. Hence it is desirable to obtain necessary and sufficient conditions that an arc be complete. Such conditions seem in general to be elusive.

For the smallest values of k, we have the following:

(21) THEOREM. *Let n be the order of a plane. Then*
(i) *If $n > 3$, a 4-arc is not complete.*
(ii) *A 5-arc is never complete.*
(iii) *If $n > 10$, a 6-arc is not complete.*
(iv) *If $n > 13$, a 7-arc is not complete.*

Proof. Equations (8) give the following:
(i) For a 4-arc, $e_0 = 3$, $e_2 = 6(n - 2)$, and $e_4 = (n - 2)(n - 3)$.
(ii) For a 5-arc, $e_1 = 15$, $e_3 = 10(n - 4)$, and $e_5 = n^2 - 9n + 21$.
(iii) For a complete 6-arc, $e_0 = n^2 - 14n + 55$, $e_2 = 3(n - 4)(10 - n)$, and $e_4 = 3(n - 4)(n - 5)$.
(iv) For a complete 7-arc, $e_1 = n^2 - 20n + 120$, $e_3 = -3(n^2 - 20n + 85)$, and $e_5 = 3(n^2 - 13n + 43)$.

The statements of the theorem follow from the fact that $e_i \geqslant 0$.

It is trivial to observe that every point on a tangent of a complete arc must lie on at least one secant. Hence for a complete arc, $n \leqslant \frac{1}{2}(k - 1)(k - 2)$. This inequality is equivalent to $t \leqslant \frac{1}{2}(k - 2)(k - 3)$ and to $(n + t) \leqslant (n - t)^2$. If $n > 3$ is a prime power then the equalities may be ignored. However this result is still meager. In fact for $n > \frac{1}{2}(k - 1)(k - 2)$ there must be at least t completion points, since there is one on every tangent by Theorem (6). An improvement is due to M. Sce **(7)**. A k-arc is complete if and only if $e_k = 0$. Sce has proved the following inequalities:

$$e_k \leqslant n^2 - \tfrac{1}{2}(k+1)(k-2)n + \tfrac{1}{2}(k-1)(k-2)[\tfrac{1}{4}k(k-3)+1],$$
$$e_k \geqslant n^2 - \tfrac{1}{2}(k+1)(k-2)n + \tfrac{1}{2}(k-1)(k-2)^2.$$

Some results on complete arcs have been obtained by B. Segre, M. Sce, L. Lunelli, and R. Cruciani in the case where the plane is Desarguesian; cf. **(9**, Chapter 17**)**. Suppose a plane is Desarguesian of odd order n. Then

$$n \geqslant 16t^2 + t - 37, \qquad t \neq 1,$$

implies that a k-arc is not complete and hence can be completed to a unique oval. Let S_n designate the Desarguesian plane of order n. There does not exist a complete 7-arc in S_{13}. However, there do exist complete arcs of the following types: 6-arc in S_7, 6-arc in S_9, 7-arc in S_9, 8-arc in S_9, 10-arc in S_{11}, 12-arc in S_{13}, and 14-arc in S_{17}. Now suppose the plane is Desarguesian of even order n. There exist complete 6-arcs in S_8 but no complete 7-arcs or 8-arcs in S_8. If $t = 1, 2, 3,$ or 4 then every k-arc is incomplete and can be expanded to an $(n+2)$-arc except when $t = 4$ and $n = 8$. Let $n = 2^h$, then for $h = 1$, 2, or 3, every $(n+2)$-arc is a completed conic. However, for $h = 4$, 5 or $h \geqslant 7$, there exist $(n+2)$-arcs which are not obtained by the completion of a conic. Finally, let x be the greatest integer in $\tfrac{1}{2}(t-1)$. Then n even and $n \geqslant t^2 - t - x$, $t \neq 0$, implies that a k-arc is not complete and is contained in one and only one $(n+2)$-arc.

At this point we introduce some new incidence equations. Let T, S, and R, respectively, be a tangent, secant, and exterior line of a k-arc K in a plane of order n. Let t_i be the number of points off K on T which lie on exactly i tangents. Let s_i be the number of points off K on S which lie on exactly i tangents. Let r_i be the number of points off K on R which lie on exactly i tangents. Then counting incidences, the following are immediate:

$$\text{(22)}\quad \begin{aligned} &\sum_{i=0}^{k} t_i = n, && \sum_{i=0}^{k} (i-1)t_i = t(k-1),\\ &\sum_{\substack{\text{all}\\ \text{tangents}}} t_j = je_j && (j = 0, 1, \dots, k). \end{aligned}$$

$$\text{(23)}\quad \begin{aligned} &\sum_{i=0}^{k} s_i = n-1, && \sum_{i=0}^{k} is_i = t(k-2),\\ &\sum_{\substack{\text{all}\\ \text{secants}}} s_j = \tfrac{1}{2}(k-j)e_j && (j = 0, 1, \dots, k). \end{aligned}$$

$$\text{(24)}\quad \begin{aligned} &\sum_{i=0}^{k} r_i = n+1, && \sum_{i=0}^{k} ir_i = tk,\\ &\sum_{\substack{\text{all}\\ \text{exterior}\\ \text{lines}}} r_j = \tfrac{1}{2}(n+t-j)e_j && (j = 0, 1, \dots, k). \end{aligned}$$

The system (23) is equivalent to one given by B. Segre **(9)**. Now the systems (22) and (24) as well as (8) and (23) should be considered in studying k-arcs. As a simple example of this, which will be of use in the proof of the next theorem, consider the case $k = 6$, $t = 3$, $n = 7$, $e_0 = 0$, $e_2 = 45$, $e_6 = 6$, and all other $e_i = 0$. It is easy to check that these values satisfy (8). Also $t_2 = 5$ and $t_6 = 2$ satisfy (22), while $s_2 = 6$ and $s_6 = 0$ satisfy (23). Thus we have consistency in (8), (22), and (23). However, (24) becomes $r_0 + r_2 + r_6 = 8$ and $r_2 + 3r_6 = 9$. But since $e_0 = 0 = r_0$, the two equations are inconsistent (no non-negative integral solution). Thus it is (24) that demonstrates that the example given is impossible.

(25) THEOREM. *Let K be a k-arc in a plane of order n. If $k \neq 4$, $k \neq n$, and K is uniform of even index, then K is complete.*

Proof. Suppose K is not complete and uniform of even index. Then K is uniform of index $2r = k$. It follows from (9) that

$$e_0 = \tfrac{1}{2}[(2 - t)k^2 + (5t - 8)k + (6 - 4t)], \qquad e_2 = \tfrac{1}{2}tk(k - 1),$$

and $e_k = t(t - 1)$ with all other $e_i = 0$. Thus $t \neq 0$ and $t \neq 1$. Further, assume $k \neq 4$ and $k \neq n$. Then $k \geqslant 6$ and $t \geqslant 3$. However, in order that $e_0 \geqslant 0$, it is necessary to have $k \leqslant (4t - 6)/(t - 2)$. But $(4t - 6)/(t - 2) \leqslant 6$, with equality if and only if $t = 3$. Thus

$$6 \leqslant k \leqslant (4t - 6)/(t - 2) \leqslant 6.$$

So $k = 6$ and $t = 3$, which in turn yields $n = 7$, $e_0 = 0$, $e_2 = 45$, and $e_6 = 6$. However, it has been shown above that these values are impossible. The theorem follows.

It is easy to see that the values $k = 4$ and $k = n$ must necessarily be excluded from the last theorem. By Theorem (21), for $n > 3$ a 4-arc is never complete and hence always uniform of index 4. For the case $n = k$, consider an n-arc which is a subset of an oval in a plane of even order. Then $e_0 = n - 1$, $e_2 = n(n - 1)$, and $e_k = 2$ with all other $e_i = 0$. Thus the n-arc is not complete but is uniform of even index.

Similarly there exist k-arcs which are not complete but are uniform of odd index for the extreme values of k. An oval in a plane of even order is uniform of odd index but never complete, by Theorem (11). For an n-arc which is a subset of an oval in a plane of odd order we have $e_1 = \frac{1}{2}n(n + 1)$, $e_3 = \frac{1}{2}n(n - 1)$, and $e_k = 1$ with all other $e_i = 0$, and hence the n-arc is not complete but is uniform of odd index. Likewise for an $(n - 1)$-arc which is a subset of an oval in a plane of even order we have $e_1 = 3(n - 1)$, $e_3 = (n - 1)(n - 2)$, and $e_k = 3$ with all other $e_i = 0$, and hence the $(n - 1)$-arc is uniform of odd index but is not complete. Also a 5-arc is necessarily uniform of odd index but is never complete by Theorem (21). Therefore the values $k = n + 1$, $k = n$, $k = n - 1$, and $k = 5$ must necessarily be excluded from the following theorem.

(26) THEOREM. *Let K be a k-arc in a plane of order n. If $5 < k < n - 1$ and K is uniform of odd index, then K is complete.*

Proof. Suppose K is not complete and uniform of odd index. Then K is uniform of index $k = 2r + 1$. It follows from (10) that

$$e_1 = \tfrac{1}{2}k[(3-t)k + (5t-9)], \qquad e_3 = \tfrac{1}{2}(t-1)k(k-1), \qquad e_k = t^2 - 3t + 3,$$

with all other $e_i = 0$. Assume $5 < k < n - 1$ or equivalently $n - 3 > t > 3$. $e_1 \geqslant 0$ implies that $k \leqslant (5t-9)/(t-3)$. For $t > 6$, we have

$$7 \leqslant k \leqslant (5t-9)/(t-3) < 7,$$

which is impossible. Thus $4 \leqslant t \leqslant 6$.

If $t = 4$, then $2e_1 = k(11 - k)$, and so $k \leqslant 11$. If $t = 5$, then $e_1 = k(8 - k)$, and so $k = 7$. If $t = 6$, then $2e_1 = 3k(7 - k)$, and so $k = 7$. Thus there are five cases to consider:

Case	t	k	n	e_1
(i)	4	7	9	14
(ii)	4	9	11	9
(iii)	4	11	13	0
(iv)	5	7	10	7
(v)	6	7	11	0

We shall show that the values are inconsistent with (22) in each case.

Case (i). The first two equations of (22) becomes

$$t_1 + t_3 + t_7 = 9, \qquad t_3 + 3t_7 = 12.$$

For a given tangent, there are only three possibilities:

$$t_1 = 5, \quad t_3 = 0, \quad t_7 = 4;$$
$$t_1 = 3, \quad t_3 = 3, \quad t_7 = 3;$$
$$t_1 = 1, \quad t_3 = 6, \quad t_7 = 2.$$

Suppose there exist x, y, and z of these, respectively. Then $x + y + z = 28$ and by the third equation of (22) for $j = 1$, $5x + 3y + z = 14$. But these two equations are not solvable in non-negative integers. Case (i) is impossible.

Case (ii). The first two equations of (22) become

$$t_1 + t_3 + t_9 = 11, \qquad t_3 + 4t_9 = 16.$$

For a given tangent, there are only three possibilities:

$$t_1 = 7, \quad t_3 = 0, \quad t_9 = 4;$$
$$t_1 = 4, \quad t_3 = 4, \quad t_9 = 3;$$
$$t_1 = 1, \quad t_3 = 8, \quad t_9 = 2.$$

Suppose there are x, y, and z of these, respectively. Then $x + y + z = 36$ and by the third equation of (22) for $j = 1$, $7x + 4y + z = 9$. It follows that Case (ii) is impossible.

Case (iii). Since $e_1 = 0$, then $t_1 = 0$. But then the first two equations of (22) are $t_3 + t_{11} = 13$ and $t_3 + 5t_{11} = 20$. It follows that Case (iii) is impossible.

Case (iv). The first two equations of (22) become

$$t_1 + t_3 + t_7 = 10, \qquad t_3 + 3t_7 = 15.$$

For a given tangent, there are only three possibilities:

$$\begin{array}{lll} t_1 = 5, & t_3 = 0, & t_7 = 5; \\ t_1 = 3, & t_3 = 3, & t_7 = 4; \\ t_1 = 1, & t_3 = 6, & t_7 = 3. \end{array}$$

Suppose there are x, y, and z of these, respectively. Then $x + y + z = 35$ and by the third equation of (22) for $j = 1$, $5x + 3y + z = 7$. It follows that Case (iv) is impossible.

Case (v). Since $e_1 = 0$, then $t_1 = 0$. But then the first two equations of (22) become $t_3 + t_7 = 11$ and $t_3 + 3t_7 = 18$. It follows that Case (v) is impossible. This completes the proof of Theorem (26).

5. n–arcs. It is known **(9)**, that every n-arc in a finite Desarguesian plane has a completion point, i.e., is contained in an oval. In this section we investigate the necessary and sufficient conditions that an n-arc be a subset of an oval when the plane is not necessarily Desarguesian and conclude with examples of n-arcs which, indeed, are not contained in any oval.

First, formulas (9) and (10) are restated for the case $k = n$, $t = 2$ in such a way that e_0, e_1, e_2, e_3, and e_n are given in terms of the remaining e_i.

For an n-arc in a plane of even order n:

$$\begin{aligned} e_0 &= n - 1 + \frac{1}{n} \sum_{j=2}^{\frac{1}{2}(n-2)} (j-1)(n-2j)e_{2j}, \\ e_2 &= n(n-1) - \frac{1}{n-2} \sum_{j=2}^{\frac{1}{2}(n-2)} j(n-2j)e_{2j}, \qquad (27) \\ e_n &= 2 - \frac{4}{n(n-2)} \sum_{j=2}^{\frac{1}{2}(n-2)} j(j-1)e_{2j}. \end{aligned}$$

For an n-arc in a plane of odd order n:

$$\begin{aligned} e_1 &= \tfrac{1}{2}n(n+1) + \frac{1}{n-1} \sum_{j=2}^{\frac{1}{2}(n-3)} (j-1)(n-2j-1)e_{2j+1}, \\ e_3 &= \tfrac{1}{2}n(n-1) - \frac{1}{n-3} \sum_{j=2}^{\frac{1}{2}(n-3)} j(n-2j-1)e_{2j+1}, \qquad (28) \\ e_n &= 1 - \frac{4}{(n-1)(n-3)} \sum_{j=2}^{\frac{1}{2}(n-3)} j(j-1)e_{2j+1}. \end{aligned}$$

(29) LEMMA. *Let N be an n-arc in a plane of even order n. Then:*
(i) *Every secant contains at least one interior point.*
(ii) *Every point off N lies on at least one exterior line.*
(iii) *Every point off N on a tangent is an exterior point.*
(iv) *Every tangent contains at least one extra exterior point.*
(v) *If x is the number of ordinary exterior points on a tangent, then $\frac{1}{2}n < x < n$.*

Proof. (i) follows from Theorem (3) with $t = 2$. (ii) and (iii) are trivial.

If each of the n points off N on a tangent were an ordinary exterior point, then $(2 - 1)n = 2(n - 1) =$ the number of tangents intersecting a given tangent off N. The equality is impossible for $n > 2$. (iv) follows.

Suppose a tangent has x ordinary exterior points. By (iv), $x < n$. Also if a point lies on more than two tangents, then it lies on at least four. Thus

$$(2 - 1)x + (4 - 1)(n - x) \leqslant 2(n - 1)$$

or $x \geqslant \frac{1}{2}(n + 2)$, proving (v).

(30) THEOREM. *If N is an n-arc in a plane of even order n, then the following statements are equivalent:*
(a) *N is a subset of an oval.*
(b) *N is a subset of a unique $(n + 2)$-arc.*
(c) *N is uniform of index n.*
(d) *There exists a point on n tangents.*
(e) *There exists a point off N on at most one exterior line.*
(f) *There exist at most two extra exterior points.*
(g) *Every tangent contains at most one extra exterior point.*
(h) *There exists a tangent with at most one extra exterior point.*
(i) *Every secant contains at most one interior point.*
(j) *Every secant contains at least $n - 2$ exterior points.*
(k) *No secant contains an extra exterior point.*
(l) *There exist $n - 1$ collinear interior points.*
(m) *There exist exactly (at most) $n - 1$ interior points.*
(n) *There exist exactly (at least) $n^2 - n + 2$ exterior points.*
(o) *There exist exactly (at least) $n(n - 1)$ ordinary exterior points.*

Proof. By Theorem (11), all the tangents to an oval in a plane of even order are concurrent. By (27), $e_n \leqslant 2$. If $e_n \neq 2$, then $e_n = 0$. So (a) implies $e_n = 2$. It is easy to check that (a) implies all the remaining properties listed in the theorem. It will now be shown that each of the properties implies (a).

That each of (b), (c), and (d) implies (a) is trivial. That (e) implies (d) follows easily. For if a point lies on only one exterior line, then the remaining n lines through this point must each intersect the n-arc in exactly one point.

(f) implies (g): By Lemma (29), every tangent contains at least one extra exterior point. Let P be a point of N. Let L_1 and L_2 be the tangents at P. Let E_1 and E_2 be extra exterior points on L_1 and L_2 respectively. Suppose E_1 and E_2 are the only extra exterior points of N. (That $E_1 = E_2$ is impossible.) Then

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every tangent of N contains at most one extra exterior point, for suppose a tangent at Q on N contained both E_1 and E_2. Then the other tangent at Q contains no extra exterior point, which is impossible.

(g) implies (h) trivially.

(h) implies (d): Suppose a tangent has just one extra exterior point, say, on y tangents. There is at least one by the lemma. Then the remaining points on the tangent which are off N are ordinary exterior points. Counting the tangents intersecting this tangent off N, we have

$$(2-1)(n-1)+(y-1)(1)=2(n-1) \text{ or } y=n.$$

(i) implies (j) trivially.

(j) implies (k): A secant is intersected off N by $2(n-2)$ tangents. Hence if a secant has at least $n-2$ exterior points, it has exactly $n-2$ ordinary exterior points and one interior point. Therefore (j) implies no extra exterior point lies on a secant.

(k) implies (d): Suppose no secant has an extra exterior point. But there exist extra exterior points by Lemma (29). Therefore there exists an extra exterior point which does not lie on a secant and hence must lie on n tangents.

(l) implies (d): Suppose L is a line with $n-1$ interior points. The remaining two points on L are either both on N or both off N as L cannot be a tangent. Suppose, first, both are points of N. Then L is a secant. Counting secants, we have

$$1+2(n-2)+(n-1)(\tfrac{1}{2}n-1)=\tfrac{1}{2}n(n-1)+(n-2),$$

which is impossible if $n>2$ as the total number of secants is $\frac{1}{2}n(n-1)$. Hence L is an exterior line. Through the $n-1$ interior points of L pass $(n-1)(\frac{1}{2}n)$ secants. But this accounts for all the secants. The remaining two points on L do not lie on any secants. Since these points are off N, they must each lie on n tangents.

Each of (m), (n), and (o) implies (c) since (27) gives

$$e_0 \geqslant n-1, \qquad e_2+e_n \leqslant n(n-1)+2, \qquad \text{and } e_2 \leqslant n(n-1)$$

with equalities if and only if $e_1=0$ for $2<i<n$. This completes the proof of the theorem.

(31) Theorem. *A complete n-arc in a plane of order n contains an $(n-1)$-arc with exactly one completion point.*

Proof. Since e_{n-2} is the number of points on only one secant of an n-arc, it is sufficient to have $n>2e_{n-2}$. We may assume that $n>8$, as otherwise the plane is Desarguesian. Let $k=n$, $t=2$, and $e_n=0$ in the third equation of (27):

$$e_{n-2}=\frac{2n}{n-4}-\frac{4}{(n-2)(n-4)}\sum_{j=2}^{\frac{1}{2}(n-4)} j(j-1)e_{2j}<4.$$

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Let $k = n$, $t = 2$, and $e_n = 0$ in the third equation of (28):

$$e_{n-2} = \frac{n-1}{n-5} - \frac{4}{(n-3)(n-5)} \sum_{j=2}^{\frac{1}{2}(n-5)} j(j-1)e_{2j+1} \leqslant 2.$$

Therefore, whether n is even or odd, $4 > e_{n-2}$. Hence $n > 8 > 2e_{n-2}$, and the theorem follows.

(32) THEOREM. *If N is an n-arc in a plane of odd order n, then the following statements are equivalent:*

(a) *N can be (uniquely) completed to form an oval.*
(b) *N is uniform of index n.*
(c) *There exist exactly (at most) $\frac{1}{2}n(n+1)$ interior points.*
(d) *There exist exactly (at least) $\frac{1}{2}n(n-1)$ ordinary exterior points.*
(e) *There exists a point on n tangents.*
(f) *There exists a point off N on at most one exterior line.*
(g) *There exist n collinear interior points.*
(h) *No secant contains an extra exterior point.*
(i) *Every secant contains exactly $\frac{1}{2}(n-3)$ ordinary exterior points.*
(j) *Every secant contains exactly $\frac{1}{2}(n+1)$ interior points.*
(k) *Every exterior line except one has exactly $\frac{1}{2}(n+3)$ interior points.*
(l) *Every exterior line except one has exactly $\frac{1}{2}(n-1)$ ordinary exterior points.*
(m) *At each point of N, one tangent contains exactly one interior point and the other tangent contains exactly $\frac{1}{2}(n-1)$ interior points.*
(n) *At each point of N, one tangent contains one interior point and $n-1$ ordinary exterior points while the other tangent contains one extra exterior point, $\frac{1}{2}(n-1)$ ordinary exterior points, and $\frac{1}{2}(n+1)$ interior points.*

Proof. It is easy to check that (a) implies each of the remaining properties listed in the theorem. It will now be shown that each of the properties implies (a).

It follows from the definitions, Theorem (16), and equations (28) that each of the properties (b), (c), (d), (e), and (f) implies (a).

(g) implies (e): Suppose there exist n collinear interior points on line L. None of these points lies on N, and each lies on exactly one tangent. The remaining point, P, on L cannot be a point on N as then L would be a tangent which is not intersected off N by any other tangent. Hence L is an exterior line. Thus L is intersected by $2n$ tangents, and the point P lies on $n = 2n - n$ tangents.

(h) implies (e): Since $t = 2$, N has at least one extra exterior point. If no secant contains an extra exterior point, then an extra exterior point lies on n tangents and one exterior line.

(i) implies (j): Suppose a secant contains exactly $\frac{1}{2}(n-3)$ ordinary exterior points. Since the remaining $\frac{1}{2}(n+1)$ points on the secant off N each lie on at least one tangent, we have accounted for

$$(3)\tfrac{1}{2}(n-3) + (1)\tfrac{1}{2}(n+1) = 2(n-2)$$

tangents intersecting the secant off N. So none of the $\frac{1}{2}(n+1)$ points lies on more than one tangent.

(j) implies (h): Suppose a secant has exactly $\frac{1}{2}(n+1)$ interior points. Each of the remaining $\frac{1}{2}(n-3)$ points off N on the secant lies on at least three tangents. Then none can lie on more than three tangents either. Thus no point off N on a secant lies on more than three tangents.

(k) implies (l): This follows from the type of argument as used to show (i) implies (j), using the identity

$$(1)\tfrac{1}{2}(n+3) + (3)\tfrac{1}{2}(n-1) = 2n.$$

(l) implies (d): Suppose every exterior line contains exactly $\frac{1}{2}(n-1)$ ordinary exterior points with the exception of one exterior line which contains v ordinary exterior points. Each ordinary exterior point lies on $\frac{1}{2}(n-1)$ of the $\frac{1}{2}n(n-1)+1$ exterior lines. Thus

$$\frac{v + [\frac{1}{2}(n-1)][\frac{1}{2}n(n-1)]}{\frac{1}{2}(n-1)} = \frac{2v}{n-1} + \tfrac{1}{2}n(n-1) = e_3.$$

By (28), $e_3 \leqslant \frac{1}{2}n(n-1)$. Hence $v = 0$ and $e_3 = \frac{1}{2}n(n-1)$.

(m) implies (c): Two tangents do not intersect at an interior point. We have n interior points from the n tangents which contain one interior point each and $n(\frac{1}{2}(n-1))$ interior points from the n tangents which contain $\frac{1}{2}(n-1)$ interior points each. Hence.

$$e_1 = n + n(\tfrac{1}{2}(n-1)) = \tfrac{1}{2}n(n+1).$$

(n) implies (m) trivially.

This completes the proof of Theorem (32).

For some examples of complete arcs we return to the Hughes plane of order $n = 9$, where the notation is that used in the proof of Theorem (18).

Let N_1 and N_2 be the following 9-arcs:

$$N_1 = \{A_1, B_0, C_9, D_8, D_9, E_2, F_2, G_5, G_8\},$$
$$N_2 = \{A_1, B_0, D_8, E_2, F_2, F_7, G_5, G_7, G_8\}.$$

N_1 and N_2 are complete n-arcs. They are not equivalent under collineations of the plane; cf. **(12)**. The values of the e_i for both arcs are: $e_1 = 48$, $e_3 = 28$, $e_5 = 6$, and $e_7 = 0$. Thus both arcs are uniform of index 5.

In the same plane, let

$$K_1 = \{A_1, B_0, B_1, B_2, C_9, D_8, D_9, E_2\},$$
$$K_2 = \{A_1, B_0, B_1, C_9, D_8, D_9, E_2, G_8\},$$
$$K_3 = \{A_1, B_0, D_8, E_2, F_2, F_{11}, G_5, G_8\},$$
$$K_4 = \{A_1, A_6, B_0, C_9, D_8, D_9, E_2\},$$
$$K_5 = \{A_1, B_0, C_9, D_8, D_9, E_2, G_{11}\}.$$

If $S = \{A_1, B_0, C_9, D_8, D_9, E_2\}$, then S is a 6-arc which is not complete. A_6 added to S gives K_4, and G_{11} added to S gives K_5. K_4 and K_5 are two complete 7-arcs with 6 points in common. B_1 and B_2 added to S gives K_1, while B_1 and B_8 added to S gives K_2. K_1 and K_2 are two complete 8-arcs with 7 points in common. F_2, G_5, and G_8 added to S gives the complete n-arc N_1.

The complete 7-arc K_4 has six points in common with each of the complete arcs V_1, N_1, K_2, and K_5, which are, respectively, a 10-arc, a 9-arc, an 8-arc, and a 7-arc.

The complete 8-arc K_3 has 7 points in common with each of the 9-arcs N_1 and N_2.

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