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ON THE LOCATION OF CRITICAL POINTS OF POLYNOMIALS

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Abstract

Let all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \leq 1$ and a be a given complex number. In this paper we study the location of the zeros of higher derivatives of the polynomial $(z - z)P(z)$ and obtain certain generalizations of some results of Rahman and Rubinstein. We shall also extend a result of Goodman, Rahman and Ratti for the zeros of the polar derivative of the polynomial $P(z)$ given $P(1) = 0$.

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Let all the zeros of a polynomial $P(z)$ of degree n lie in the closed unit disk $|z| \leq 1$. It was asked by Rahman [3], given a complex number a what is the radius of the smallest disk centred at a containing at least one zero of the polynomial $((z - a)P(z))'$? He has answered the question by showing that one and only one zero of $((z - a)P(z))'$ lies in

$$(1) \qquad |z - a| \leq \frac{|a| + 1}{n + 1}$$

provided $|a| > (n + 2)/n$. The remaining $(n - 1)$ zeros of $((z - a)P(z))'$ lie in $|z| \leq 1$.

Here we first obtain an extension of this result for the zeros of the higher derivatives of the polynomial $(z - a)P(z)$ and thereby give an independent proof of (1) as well. We prove

THEOREM 1. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \leq 1$ and $F(z) = (z - a)P(z)$, then $F^{(k)}(z)$, $1 \leq k \leq n$, has one and only one zero in*

$$|z - a| \leq \frac{k(|a| + 1)}{n + 1} \tag{2}$$

provided $|a| > (n + k + 1)/(n - k + 1)$. The remaining $n - k$ zeros of $F^{(k)}(z)$ lie in $|z| \leq 1$. The example $P(z) = (z + e^{i\theta})^n$ where $\theta = \arg a$ shows that the result is best possible.

For the proof of this theorem we need the following lemma, which is the Coincidence Theorem of Walsh [2, page 62].

LEMMA. *Let $G(z_1, z_2, \ldots, z_n)$ be a symmetric n-linear form of total degree n in $z_1, z_2, \ldots, z_n$ and let C be a circle containing the n points $w_1, w_2, \ldots, w_n$. Then there exists at least one point α belonging to C such that*

$$G(\alpha, \alpha, \ldots, \alpha) = G(w_1, w_2, \ldots, w_n).$$

PROOF OF THEOREM 1. We have $F(z) = (z - a)P(z)$, so that

$$F^{(k)}(z) = (z - a)P^{(k)}(z) + kP^{(k-1)}(z), \qquad k = 1, 2, \ldots, n. \tag{3}$$

Clearly $F^{(k)}(z)$ is a polynomial of degree $n - k + 1$. Since all the zeros of the polynomial $P(z)$ lie in $|z| \leq 1$, it follows by the Gauss-Lucas Theorem that all the zeros of the polynomial $P^{(k-1)}(z)$ of degree $n - k + 1$ also lie in $|z| \leq 1$. Therefore, if $w_1, w_2, \ldots, w_{n-k+1}$ are the zeros of $P^{(k-1)}(z)$, then $|w_j| \leq 1$, $j = 1, 2, \ldots, n - k + 1$. If now w is any zero of $F^{(k)}(z)$, then from (3) we have

$$(w - a)P^{(k)}(w) + kP^{(k-1)}(w) = 0. \tag{4}$$

This is an equation which is linear and symmetric in the zeros of $P^{(k-1)}(z)$, that is in $w_1, w_2, \ldots, w_{n-k+1}$. Hence an application of the lemma above shows that w will also satisfy the equation obtained by substituting into equation (4)

$$P^{(k-1)}(z) = (z - \alpha)^{n-k+1},$$

where α is a suitably chosen point in $|z| \leq 1$. That is, w satisfies the equation

$$(n - k + 1)(w - a)(w - \alpha)^{n-k} + k(w - \alpha)^{n-k+1} = 0,$$

or equivalently

$$(w - \alpha)^{n-k}\{(n - k + 1)(w - a) + k(w - \alpha)\} = 0.$$

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Thus w has the values

$$w = \alpha \quad \text{or} \quad w = \frac{(n-k+1)a + k\alpha}{n+1}.$$

Since $|\alpha| \leqslant 1$, it follows that all the zeros of $F^{(k)}(z)$ lies in the union of the two circles

$$(5) \qquad |z| \leqslant 1 \quad \text{and} \quad \left| z - \frac{(n-k+1)a}{n+1} \right| \leqslant \frac{k}{n+1},$$

and hence also lie in the union of the two circles

$$(6) \qquad |z| \leqslant 1 \quad \text{and} \quad |z - a| \leqslant \frac{k(|a|+1)}{n+1}.$$

Since $|a| > (n+k+1)/(n-k+1)$, it follows that the closed interiors of the two circles defined by (6) have no point in common and we show that $F^{(k)}(z)$ has one and only one zero in $|z-a| \leqslant k(|a|+1)/(n+1)$. Since

$$\frac{zP^{(k)}(z)}{P^{(k-1)}(z)} = \sum_{j=1}^{n-k+1} \frac{z}{z - w_j}$$

and $|w_j| \leqslant 1, j = 1, 2, \ldots, n-k+1$, we have

$$\operatorname{Re} \frac{e^{i\theta}P^{(k)}(e^{i\theta})}{P^{(k-1)}(e^{i\theta})} = \sum_{j=1}^{n-k+1} \operatorname{Re} \frac{e^{i\theta}}{e^{i\theta} - w_j} \geqslant \sum_{j=1}^{n-k+1} \frac{1}{2} = \frac{n-k+1}{2},$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than the zeros of $P^{(k-1)}(z)$. This implies

$$|e^{i\theta}P^{(k)}(e^{i\theta}) - (n-k+1)P^{(k-1)}(e^{i\theta})| \leqslant |P^{(k)}(e^{i\theta})|$$

for points $e^{i\theta}$ other than the zeros of $P^{(k-1)}(z)$. Since this inequality is trivially true for points $e^{i\theta}$ which are the zeros of $P^{(k-1)}(z)$, it follows that

$$(7) \qquad |zP^{(k)}(z) - (n-k+1)P^{(k-1)}(z)| \leqslant |P^{(k)}(z)| \qquad \text{for } |z| = 1.$$

Now the degree of the polynomial $zP^{(k)}(z) - (n-k+1)P^{(k-1)}(z)$ is at most equal to the degree of the polynomial $P^{(k)}(z)$ and $P^{(k)}(z)$ does not vanish in $|z| > 1$, we conclude with the help of the Maximum Modulus Principle that the inequality (7) holds for $|z| > 1$ also. Thus

$$\left| z - \frac{(n-k+1)P^{(k-1)}(z)}{P^{(k)}(z)} \right| \leqslant 1 \quad \text{for } |z| > 1.$$

We write

$$\delta(z) = z - \frac{(n-k+1)P^{(k-1)}(z)}{P^{(k)}(z)},$$

then $\delta(z)$ is an analytic function defined for all $|z|>1$ and $|\delta(z)|\leqslant 1$ for $|z|>1$. If $|a|>1$, then

$$\frac{P^{(k)}(z)}{P^{(k-1)}(z)} = \frac{(n-k+1)\beta(z)}{(z-a)\beta(z)-1}$$

where $\beta(z) = 1/(\delta(z)-a)$ is analytic in $|z|>1$ and

(8) $$\frac{1}{|a|+1} \leqslant |\beta(z)| \leqslant \frac{1}{|a|-1}.$$

Since for $|z|>1$

$$\frac{(z-a)P^{(k)}(z)+kP^{(k-1)}(z)}{P^{(k-1)}(z)} = \frac{(n+1)(z-a)\beta(z)-k}{(z-a)\beta(z)-1},$$

the zeros of $F^{(k)}(z) = (z-a)P^{(k)}(z)+kP^{(k-1)}(z)$ in $|z|>1$ are the same as the zeros of $(n+1)(z-a)\beta(z)-k$. Now if

$$|a| > \frac{n+k+1}{n-k+1} \quad \text{and} \quad \frac{k(|a|+1)}{n+1} < |z-a| < |a|-1,$$

then from (8) we have

$$k < |(n+1)(z-a)\beta(z)|.$$

Applying Rouche's theorem we conclude that $(n+1)(z-a)\beta(z)-k$ and $(n+1)(z-a)\beta(z)$ have the same number of zeros in $|z-a|\leqslant k(|a|+1)/(n+1)$, namely one. Now it easily follows from (6) that the remaining $n-k$ zeros of $F^{(k)}(z)$ lie in $|z|\leqslant 1$. This completes the proof of the theorem.

Remark. Since $|a|>(n+k+1)/(n-k+1)$, it can be easily seen that the two circles defined by (5) have no point in common. Hence it follows by the similar reasoning as above that, in fact, $F^{(k)}(z)$ has one and only one zero in the circle

$$\left|z - \frac{(n-k+1)a}{n+1}\right| \leqslant \frac{k}{n+1}.$$

Next we prove the following result which is valid for $|a|\geqslant 1$. For simplicity we assume that a is real and $a\geqslant 1$.

THEOREM 2. *If all the zeros of a polynomial* $P(z)$ *of degree* n *lie in* $|z| \leq 1$ *and* $F(z) = (z-a)^r P(z)$, $a \geq 1$, *then* $F^{(k)}(z)$, $1 \leq k \leq n+r-1$, *has at least one zero in both the circles*

$$\left|z - a + \left(\frac{a+1}{2}\right)\left(1 - \frac{r}{k+1}\right)\right| \leq \left(\frac{a+1}{2}\right)\left(1 - \frac{r}{k+1}\right) \tag{9}$$

and

$$|z-a| \leq (a+1)\left(1 - \frac{r}{k+1}\right). \tag{10}$$

PROOF. First we observe that the circle defined by (9) is contained in the circle defined by (10). So to prove the theorem it suffices to show that $F^{(k)}(z)$ has a zero in the circle defined by (9).

We assume $k \geq r$. Since $F(z) = (z-a)^r P(z)$, it is easy to see that

$$\frac{F^{(k+1)}(a)}{F^{(k)}(a)} = \frac{k+1}{k-r+1} \frac{P^{(k-r+1)}(a)}{P^{(k-r)}(a)}. \tag{11}$$

Now $F^{(k)}(z)$ and $P^{(k-r)}(z)$ are polynomials of degree $n+r-k$ and therefore, if $\alpha_1, \alpha_2, \ldots, \alpha_{n+r-k}$ are the zeros of $F^{(k)}(z)$ and $\beta_1, \beta_2, \ldots, \beta_{n+r-k}$ are those of $P^{(k-r)}(z)$, then from (11) we have

$$\sum_{j=1}^{n+r-k} \frac{1}{a-\alpha_j} = \frac{k+1}{k-r+1} \sum_{j=1}^{n+r-k} \frac{1}{a-\beta_j}.$$

Since by the Gauss-Lucas theorem all the zeros of $P^{(k-r)}(z)$ lie in $|z| \leq 1$, therefore $|\beta_j| \leq 1$ for all $j = 1, 2, \ldots, n+r-k$ and thus

$$\operatorname{Re} \frac{1}{a-\beta_j} \geq \frac{1}{a+1} \quad \text{for all } j = 1, 2, \ldots, n+r-k.$$

Now

$$\sum_{j=1}^{n+r-k} \operatorname{Re} \frac{1}{a-\alpha_j} = \frac{k+1}{k-r+1} \sum_{j=1}^{n+r-k} \operatorname{Re} \frac{1}{a-\beta_j} \geq \frac{(k+1)(n+r-k)}{(k-r+1)(a+1)},$$

and therefore,

$$\operatorname{Re}\frac{1}{a-\alpha} \equiv \max_{1\leqslant j\leqslant n+r-k} \operatorname{Re}\frac{1}{a-\alpha_j}$$

$$\geqslant \frac{1}{n+r-k}\sum_{j=1}^{n+r-k} \operatorname{Re}\frac{1}{a-\alpha_j}$$

$$\geqslant \frac{(k+1)}{(a+1)(k-r+1)}.$$

This implies

$$\left|\alpha - a + \left(\frac{a+1}{2}\right)\left(1-\frac{r}{k+1}\right)\right| \leqslant \left(\frac{a+1}{2}\right)\left(1-\frac{r}{k+1}\right),$$

which is equivalent to (9) and the theorem is established.

The following corollary is obtained from Theorem 2 by taking $r = k$.

COROLLARY 1. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \leqslant 1$ and $F(z) = (z-a)^k P(z)$, $a \geqslant 1$, then at least one zero of $F^{(k)}(z)$ lies in both the circles*

$$\left|z - a + \frac{a+1}{2(k+1)}\right| \leqslant \frac{a+1}{2(k+1)},$$

and

$$|z-a| \leqslant \frac{a+1}{k+1}.$$

In particular when $k = 1$ and $a = 1$, then the Corollary 1 states that if all the zeros of a polynomial $F(z) = (z-1)P(z)$ lie in $|z| \leqslant 1$, then $F'(z)$ has at least one zero in both the circles

(12) $$|z - \tfrac{1}{2}| \leqslant \tfrac{1}{2}$$

and

(13) $$|z - a| \leqslant 1.$$

The result (12) is due to Goodman, Rahman and Ratti [1] and (13) was proved by Rubinstein [4].

Finally we prove the following result which extends (12) for the zeros of the polar derivative with respect to $\alpha \geqslant 1$ of the polynomial $P(z)$ having all its zeros in the closed unit disk.

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Theorem 3. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \leqslant 1$ and $P(1) = 0$, then the polynomial $nP(z) + (\alpha - z)P'(z)$ has at least one zero in the circle*

$$\left| z - \frac{1}{2} - \frac{1}{2\alpha} \right| \leqslant \frac{1}{2} - \frac{1}{2\alpha} \quad \textit{where } \alpha \geqslant 1. \tag{14}$$

Proof. We write

$$G(z) = nP(z) + (\alpha - z)P'(z) \quad \text{and} \quad P(z) = (z-1)Q(z),$$

then

$$\frac{G'(z)}{G(z)} = \frac{(n-1)P'(z) + (\alpha - z)P''(z)}{nP(z) + (\alpha - z)P'(z)}. \tag{15}$$

The case $\alpha = 1$ is trivial, so we assume $\alpha > 1$. If $z = 1$ is a multiple zero of $P(z)$ then $z = 1$ is also a zero of $P'(z)$ and therefore $z = 1$ is a zero of $G(z)$. Since 1 lies in (14), the assertion is true in this case. Hence we assume that $z = 1$ is a simple zero of $P(z)$. Now from (15) we have

$$\frac{G'(1)}{G(1)} = \frac{n-1}{\alpha - 1} + \frac{P''(1)}{P'(1)} = \frac{n-1}{\alpha - 1} + \frac{2Q'(1)}{Q(1)}. \tag{16}$$

If $w_1, w_2, \ldots, w_{n-1}$ are the zeros of $G(z)$ and $z_1, z_2, \ldots, z_{n-1}$ are those of $Q(z)$, then from (16) we have

$$\sum_{j=1}^{n-1} \frac{1}{1 - w_j} = \frac{n-1}{\alpha - 1} + 2 \sum_{j=1}^{n-1} \frac{1}{1 - z_j}.$$

Since $|z_j| \leqslant 1$, so that

$$\operatorname{Re} \frac{1}{1 - z_j} \geqslant \frac{1}{2} \quad \text{for all } j = 1, 2, \ldots, n-1,$$

and therefore

$$\sum_{j=1}^{n-1} \operatorname{Re} \frac{1}{1 - w_j} = \frac{n-1}{\alpha - 1} + 2 \sum_{j=1}^{n-1} \operatorname{Re} \frac{1}{1 - z_j} \geqslant \frac{n-1}{\alpha - 1} + n - 1 = \frac{(n-1)\alpha}{\alpha - 1}.$$

Hence

$$\operatorname{Re} \frac{1}{1 - w} \equiv \operatorname*{Max}_{1 \leqslant j \leqslant n-1} \operatorname{Re} \frac{1}{1 - w_j} \geqslant \frac{1}{n-1} \sum_{j=1}^{n-1} \operatorname{Re} \frac{1}{1 - w_j} \geqslant \frac{\alpha}{\alpha - 1}.$$

This implies

$$\left|w - \frac{1}{2} - \frac{1}{2\alpha}\right| \leqslant \frac{1}{2} - \frac{1}{2\alpha},$$

which is equivalent to (14) and the theorem is proved.

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