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ON MAXIMUM MATCHINGS IN CUBIC GRAPHS WITH A BOUNDED NUMBER OF BRIDGE-COVERING PATHS

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It is proved that if G is a connected cubic graph of order p all of whose bridges lie on r edge-disjoint paths of G, then every maximum matching of G contains at least $p/2 - \lfloor 2r/3 \rfloor$ edges. Moreover, this result is shown to be best possible.

1. Introduction and historical background

A *matching* in a graph G is a set of pairwise nonadjacent (independent) edges of G. A matching with maximum cardinality is a *maximum matching*. If G has order p, then a matching of cardinality $p/2$ is called a *perfect matching*. Graphs with perfect matchings were characterized by Tutte [5].

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THEOREM A. (Tutte). *A graph* G *has a perfect matching if and only if for every proper subset* S *of* $V(G)$ *, the number of odd components of* $G-S$ *does not exceed* $|S|$.

Much research has centred around the determination of regular graphs that contain perfect matchings. A well known result on this subject is due to Petersen [4].

THEOREM B. (Petersen). *Every cubic graph with at most two bridges contains a perfect matching.*

This result cannot be improved, in general, since cubic graphs having three bridges but no perfect matchings exist. The graph of Figure 1 is the unique smallest such graph.

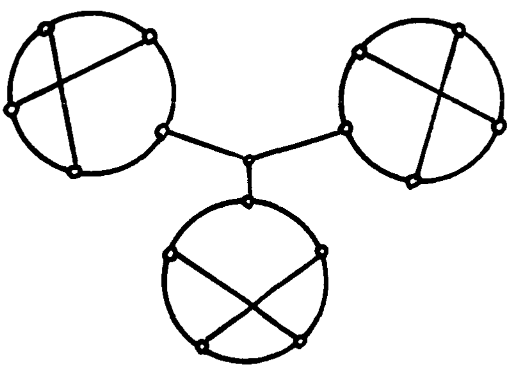

Figure 1

Note that the three bridges of the graph of Figure 1 do not lie on a single path. Indeed, since this graph has no perfect matching, this property is necessary, by a result of Errera [3].

THEOREM C. (Errera). *If all the bridges of a connected cubic graph* G *lie on a single path of* G *, then* G *has a perfect matching.*

The goal of this paper is to provide a generalisation of Theorem C by establishing a lower bound on the cardinality of a maximum matching in a connected cubic graph all of whose bridges lie on a specified number of edge-disjoint paths. Towards this end we state the following generalisation (see [1]) of the aforementioned theorem of Tutte.

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THEOREM D. *Let* G *be a cubic graph of order* p *and let* ℓ *be an integer with* $0 \le \ell \le p/2$. *Then every maximum matching of* G *has at least* $(p - 2\ell)/2$ *edges if and only if for every proper subset* S *of* $V(G)$, *the number of odd components of* $G - S$ *does not exceed* $|S| + 2\ell$.

2. The main result

We are now prepared to present a bound on the number of edges in a maximum matching in a connected cubic graph G in terms of the number of paths containing the bridges of G .

THEOREM 1. *If the bridges of a connected cubic graph* G *lie on* r *edge-disjoint paths of* G , *then each maximum matching of* G *contains at least* $p/2 - \lfloor 2r/3 \rfloor$ edges.

Proof. Suppose, to the contrary, that G contains a maximum matching M with fewer that $p/2 - \lfloor 2r/3 \rfloor$ edges. By Theorem D there exists a proper subset S of $V(G)$ such that the number n of odd components of $G - S$ exceeds $|S| + 2\lfloor 2r/3 \rfloor$. Let $|S| = k$. Since p is even, n and k are of the same parity, so that

$$n \ge k + 2\lfloor 2r/3 \rfloor + 2.. \qquad \text{...........}(*)$$

Denote the odd components of $G - S$ by $G_1, G_2, \ldots, G_n$. Since G is connected, every component $G_i (1 \le i \le n)$ contains at least one vertex that is adjacent to some vertex of S . Suppose, without loss of generality, that $G_1, G_2, \ldots, G_t$ denote the odd components of $G - S$ for which there exists exactly one edge e_i joining a vertex in G_i $(1 \le i \le t)$ to a vertex of S . For $i = t + 1, t + 2, \ldots, n$, then, there are at least three edges joining vertices of G_i to vertices of S ; otherwise, for some $j (t + 1 \le j \le n)$, vertices of G_j are joined to vertices of S by exactly two edges, implying that G_j has an odd number of odd vertices, which is not possible.

Let $P_1, P_2, \ldots, P_r$ denote r edge-disjoint paths of G which contain all the bridges of G . Then for every $i (1 \le i \le r)$, at most two bridges of G that lie on P_i are in the set $\{e_1, e_2, \ldots, e_t\}$. Hence $t \le 2r$. Since at least $t + 3(n - t) = 3n - 2t$ edges join vertices of

$V(G_1) \cup V(G_2) \cup \ldots \cup V(G_n)$ to vertices of S it follows that $3n - 4r \le 3n - 2t \le 3k$. Therefore, $3(n - k) \le 4r$ so that by (*), $3(2\lfloor 2r/3 \rfloor + 2) \le 4r$, that is $3\lfloor 2r/3 \rfloor + 3 \le 2r$. However,

$$2r + 1 = 3((2r - 2)/3) + 3 \le 3\lfloor 2r/3 \rfloor + 3 \le 2r ,$$

which gives a contradiction. □

Another bound (see [2]) for the number of edges in a maximum matching in a connected cubic graph G depends only on the number of bridges in G.

THEOREM E. *Every maximum matching in a connected cubic graph of order* p *with fewer than* $3(\ell + 1)$ *bridges* $(\ell \ge 0)$ *has at least* $(p - 2\ell)/2$ *edges.*

If the bridges of a connected cubic graph lie on sufficiently few paths, then the bound provided in Theorem 1 on the number of edges in a maximum matching is an improvement on the bound provided in Theorem E. A specific statement of this improved result is given next.

COROLLARY 1. *Let* G *be a connected cubic graph of order* p *having* m *bridges, and let* $\ell \ge 0$ *be an integer such that* $3\ell \le m < 3(\ell + 1)$. *If these bridges lie on* r *edge-disjoint paths, where* $\lfloor 2r/3 \rfloor < \ell$, *then the number of edges in a maximum matching of* G *is at least* $p/2 - \lfloor 2r/3 \rfloor$.

The result in Corollary 1 can be shown to be sharp, which we do next. Since the case $\ell = 0$ corresponds to the existence of at most 2 bridges in a connected cubic graph, and sharpness is already known, we consider $\ell \ge 1$ to be given, and choose the maximum r with $r \equiv 0 \pmod 3$, say $r = 3s$, such that $\lfloor 2r/3 \rfloor < \ell$. Then

$$r = \begin{cases} (3\ell - 6)/2 & \text{if } \ell \text{ is even ,} \\ (3\ell - 3)/2 & \text{if } \ell \text{ is odd .} \end{cases}$$

We show that there exists a connected cubic graph G of order p having $m = 3\ell + j$ bridges $(j = 0, 1, 2)$ all of which lie on r edge-disjoint paths but no fewer, such that each maximum matching contains $p/2 - \lfloor 2r/3 \rfloor$ edges.

We begin by constructing a graph P_n^* $(n \ge 1)$, consisting of graphs $H_1, H_2, \ldots, H_n$, where $H_i (1 \le i \le n - 1)$ is obtained by deleting an edge of K_4 and H_n is obtained by subdividing an edge of K_4. Denote the two vertices of degree 2 in $H_i (1 \leq i \leq n-1)$ by u_i and v_i and the vertex of degree 2 in H_n by u_n. Then P_n^* is produced by joining v_i and $u_{i+1} (1 \leq i \leq n-1)$. Observe that each $P_n^* (n \geq 1)$ has odd order. Let the graph H be the $12s$-cycle $w_1, w_2, \ldots, w_{12s}, w_1$ to which we add $2s$ new vertices $x_1, x_2, \ldots, x_{2s}$, where x_i is joined to w_{6i-5}, w_{6i-3} and w_{6i-1} $(1 \leq i \leq 2s)$. Consider next the graphs $G_1, G_2, \ldots, G_{6s-1}$, each isomorphic to P_1^*, and the graph G_{6s}, where

$$G_{6s} = \begin{cases} P_{7+j}^* & \text{if } \ell \text{ is even }, \\ P_{4+j}^* & \text{if } \ell \text{ is odd }. \end{cases}$$

The desired graph G is now produced by joining w_{2i} to the vertex u_1 in $G_i (1 \leq i \leq 6s)$ by an edge e_i. Figure 2 illustrates the graph G for $\ell = 3$, $r = 3$, $s = 1$, $m = 9$ and $j = 0$.

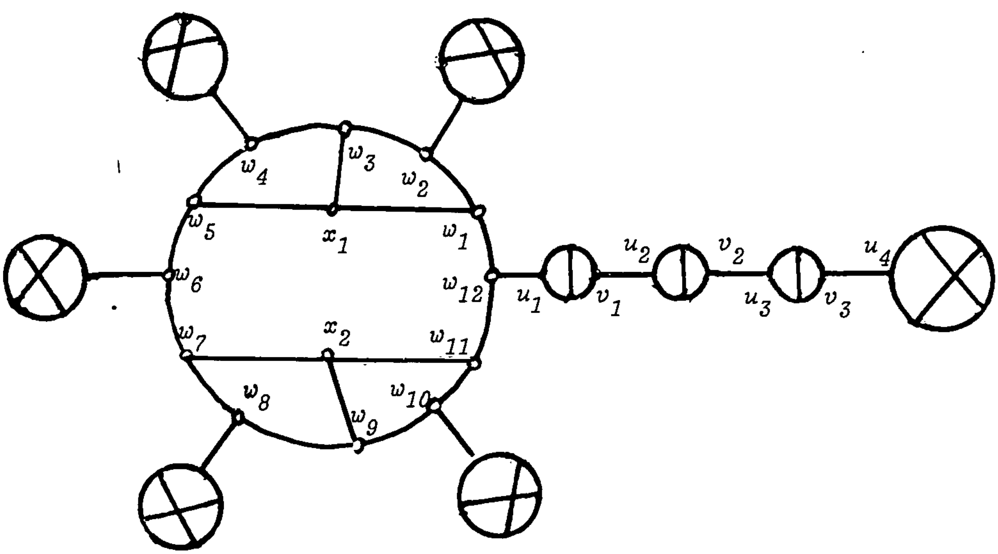

Figure 2

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Clearly G is connected, cubic, and each edge $e_i (1 \le i \le 6s)$ is a bridge of G. Further, since G_{6s} contains $6 + j$ or $3 + j$ bridges, depending on whether ℓ is even or odd, respectively, it follows that G contains exactly $6s + 6 + j$ or $6s + 3 + j$ bridges, according to whether ℓ is even or odd. Since the bridges of G lie on $r = 3s$ edge-disjoint paths, Corollary 1 implies that every maximum matching of G contains at least $p/2 - \lfloor 2r/3 \rfloor$ edges.

It remains to be shown that every maximum matching of G contains at most $p/2 - \lfloor 2r/3 \rfloor$ edges. We use Theorem D to prove this statement. Let $S = \{w_{2i} \mid 1 \le i \le 6s\} \cup \{x_1, x_2, \ldots, x_{2s}\}$. Then $|S| = 8s$, and

$$G - S = \begin{cases} 6sK_1 \cup (6s - 1)P_1^* \cup P_{7+j}^* & \text{if } \ell \text{ is even}, \\ 6sK_1 \cup (6s - 1)P_1^* \cup P_{4+j}^* & \text{if } \ell \text{ is odd}. \end{cases}$$

Therefore, $G - S$ contains $12s = |S| + 4s$ odd components. Theorem D now implies that every maximum matching of G contains at most $p/2 - 2s = p/2 - \lfloor 2r/3 \rfloor$ edges. Hence every maximum matching of G contains exactly $p/2 - \lfloor 2r/3 \rfloor$ edges.

The cases where $r \equiv 1 \pmod 3$ or $r \equiv 2 \pmod 3$ can be handled in a similar manner. If $r \equiv 1 \pmod 3$, the maximum r with $\lfloor 2r/3 \rfloor < \ell$ is given by

$$r = \begin{cases} (3\ell - 4)/2 & \text{if } \ell \text{ is even}, \\ (3\ell - 1)/2 & \text{if } \ell \text{ is odd}. \end{cases}$$

Further, the maximum r for $r \equiv 2 \pmod 3$ and $\lfloor 2r/3 \rfloor < \ell$ satisfies

$$r = \begin{cases} (3\ell - 2)/2 & \text{if } \ell \text{ is even}, \\ (3\ell + 1)/2 & \text{if } \ell \text{ is odd}. \end{cases}$$

Then using a construction similar to the one described for $r \equiv 0 \pmod 3$ we can show, for the above choices of r, that there is a graph G having $m + j$ bridges ($j = 0, 1, 2$ and $3\ell \le m < 3(\ell + 1)$) all of which lie on r edge-disjoint paths and where every maximum matching of G has $p/2 - \lfloor 2r/3 \rfloor$ edges. Consequently, the result stated in Corollary 1 is the best possible.

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