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Iterates of meromorphic functions on escaping Fatou components

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In this paper, we prove that the ratio of the modulus of the iterates of two points in an escaping Fatou component could be bounded even if the orbit of the component contains a sequence of annuli whose moduli tend to infinity, and this cannot happen when the maximal modulus of the meromorphic function is uniformly large enough. In this way we extend certain related results for entire functions to meromorphic functions with infinitely many poles.

Keywords: Escaping sets; meromorphic functions; Fatou set; Julia set

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1. Introduction and main results

Let f be a meromorphic function that is not a Möbius transformation and let f^n, $n \in \mathbb{N}$, denote the nth iterate of f. The Fatou set $F(f)$ of f is defined as the set of points, $z \in \hat{\mathbb{C}}$, such that $\{f^n\}_{n\in\mathbb{N}}$ is well-defined and forms a normal family in some neighbourhood of z. Therefore, $F(f)$ is an open set. Since $F(f)$ is completely invariant under f, i.e. $f(z) \in F(f)$ if and only if $z \in F(f)$, for a component U of $F(f)$, $f(U)$ is contained in certain component of $F(f)$. We write U_n for the component of $F(f)$ such that $f^n(U) \subseteq U_n$. The complement $J(f) = \hat{\mathbb{C}} \setminus F(f)$ of $F(f)$ is called the Julia set of f. Then $\bigcup_{n=0}^{\infty} f^{-n}(\infty) \subset J(f)$ and if $\bigcup_{n=0}^{\infty} f^{-n}(\infty)$ contains three distinct points in the extended complex plane $\hat{\mathbb{C}}$, we have $\overline{\bigcup_{n=0}^{\infty} f^{-n}(\infty)} = J(f)$. Thus every f^n is analytic in $F(f)$ for $n \geqslant 1$.

In this paper, we investigate Fatou components U of f with $f^n|_U \to \infty\,(n \to \infty)$.

For convenience, we call such U an escaping Fatou component of f. We first determine what kinds of Fatou components will be escaping to ∞ under iterates.

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A periodic Fatou component W of f with period $p \geqslant 1$, i.e. $f^p(W) \subseteq W = W_p$ and p is the minimal positive integer such that the inclusion holds, is known as a Baker domain if $f^{np}|_W \to a\,(n \to \infty)$, but f^p is not defined at a. Then ∞ must be the limit value of f^{np+j} in W for some $0 \leqslant j \leqslant p-1$ and so W_j is unbounded. If an escaping Fatou component U of f is periodic, i.e. for some positive integer p, $f^p(U) \subseteq U = U_p$, then U is a Baker domain of f. The periodic Fatou components can be classified into five possible types: attracting domains, parabolic domains, Siegel discs, Herman rings and Baker domains (see [**7**], theorem 6).

A Fatou component U is called a wandering domain of f, if for any pair of positive integers $m \neq n$, $U_m \neq U_n$ and actually $U_m \cap U_n = \emptyset$. We know that the escaping Fatou component U is either a wandering domain, a Baker domain or the preimage of a Baker domain. Many examples of such Baker domains and wandering domains have been revealed; see [**20**] for a survey about Baker domains and [**2**, **3**, **6**, **8**, **10**, **11**, **13**, **14**, **16**–**18**, **21**, **23**] for studies on wandering domains. In addition, the existence of wandering domains with special geometric or dynamical behaviours and/or for special classes of meromorphic functions continues to be an important topic and to attract much interest.

Let U be an escaping Fatou component of a meromorphic function f. Generally for any two distinct points a and b in U there exist $M > 1$ and a positive integer N such that for $\forall\ n \geqslant N$,

$$|f^n(b)|^{1/M} \leqslant |f^n(a)| \leqslant |f^n(b)|^M \tag{1.1}$$

and if $\hat{\mathbb{C}} \setminus U$ contains an unbounded component, we have a more precise inequality

$$M^{-1}|f^n(a)| \leqslant |f^n(b)| \leqslant M|f^n(a)|, \quad \forall\ n \geqslant N. \tag{1.2}$$

(see [**1**], lemma 5 or [**7**], lemma 7). Therefore, if U is an unbounded or simply connected wandering domain, then (1.2) holds. As we know, an escaping Baker domain is unbounded and may be either simply connected or multiply connected. However, it was shown in [**27**] and [**19**] that inequality (1.2) holds for Baker domains. Therefore, an escaping Fatou component U that does not satisfy (1.2) must be bounded, multiply connected and wandering.

The first entire function with multiply connected Fatou component was constructed by Baker [**2**–**4**]. Every multiply connected Fatou component U of a transcendental entire function is a bounded escaping wandering domain [**5**, **24**] and for all sufficiently large n, $f^n(U)$ contains a round annulus centred at the origin with the modulus tending to ∞ as $n \to \infty$ [**9**, **26**]. However, this is not the case for meromorphic functions with poles. A meromorphic function may have Herman rings, multiply connected attracting domains, parabolic domains or Baker domains; see [**12**] for examples of meromorphic functions with only one pole that have an invariant multiply connected Fatou component. A meromorphic function was constructed by Baker *et al.* [**6**] to have a multiply connected wandering domain U of preassigned connectivity such that the limit set of $\{f^n|_U\}$ is an infinite set including ∞ (a planar region E is said to have connectivity m if $\hat{\mathbb{C}} \setminus E$ has m components). Such a wandering domain of a transcendental entire function must be simply connected and a corresponding example has been constructed by Eremenko and Lyubich [**13**]. Examples of meromorphic functions with finitely or infinitely many poles, which have simply, doubly or infinitely connected wandering domains and which have connectivity-changing wandering domains under iterates, can be found in [**21**].

There exists a meromorphic function with bounded, multiply connected, escaping and wandering domains separating the origin and ∞ where (1.2) holds (see example 5.2 of [**28**]). As we know, there also exist escaping wandering domains such that (1.1) holds while (1.2) does not hold. In fact, an entire function does not satisfy (1.2) in its multiply connected Fatou components; however, (1.1) is the best possible (see theorem 1.1 in [**9**]). Furthermore, if (1.2) does not hold, then U contains two points a and b such that

$$|f^{n_k}(a)|/|f^{n_k}(b)| \to \infty \quad (k \to \infty). \tag{1.3}$$

We have shown in [**28**] that, when (1.3) holds, $\bigcup_{n=0}^{\infty} f^n(U)$ contains a sequence of annuli $A(r_m, R_m)$ with $R_m/r_m \to \infty (m \to \infty)$ and $r_m \to \infty (m \to \infty)$; we call such a sequence of annuli *an infinite modulus annulus sequence*. Here and henceforth, for $R > r > 0$ we denote the round annulus $\{z : r < |z| < R\}$ by $A(r, R)$. In addition, for all sufficiently large n_k, $U_{n_k} = f^{n_k}(U)$ separates the origin and ∞. A condition was given in [**28**] to ensure that for all sufficiently large n, $U_n = f^n(U)$ surrounds the origin and contains a large round annulus centred at the origin which forms an infinite modulus annulus sequence. Also, we can construct meromorphic functions which have a wandering domain U such that $f^{2k}(U)$ surrounds the origin and contains a large round annulus centred at the origin, and (1.3) holds for $n_k = 2k$, but $f^{2k-1}(U)$ does not surround the origin, by suitably modifying the construction of example 5.4 in [**28**]. Here we omit the details which are routine but lengthy.

This leads us to the following:

Question $\mathcal{A}$: For an escaping Fatou component U of f, if $\bigcup_{n=0}^{\infty} f^n(U)$ contains an infinite modulus annulus sequence, is there some pair of points a and b in U such that (1.2) does not hold?

An equivalent statement of question $\mathcal{A}$ is that if (1.2) holds in an escaping Fatou component U of f, then whether $\bigcup_{n=0}^{\infty} f^n(U)$ may contain an infinite modulus annulus sequence? It turns out that there exist wandering domains whose orbit contains an infinite modulus annulus sequence and (1.2) holds. This is the first result of this paper.

THEOREM 1.1. *There exists a transcendental meromorphic function f which has an escaping wandering domain U such that for all sufficiently large n, $f^n(U) \supset A(r_n, R_n)$ with $R_n/r_n \to \infty (n \to \infty)$ and $r_n \to \infty (n \to \infty)$, but for any two points $a, b \in U$, there exists an $M > 1$ such that for all sufficiently large n,*

$$M^{-1}|f^n(a)| \leqslant |f^n(b)| \leqslant M|f^n(a)|.$$

However, the answer to question $\mathcal{A}$ is positive for transcendental entire functions. Indeed, (1.2) does not hold for an entire function in its multiply connected Fatou components, as we mentioned earlier (see theorem 1.1 in [**9**]). In this paper, we generalize this result to meromorphic functions with infinitely many poles.

In order to clearly state our second result of this paper, we introduce some notations. From now on, we let $M(r, f)$ ($\hat{m}(r, f)$, respectively) denote the maximal (minimal, respectively) modulus of f on the circle $\{|z| = r\}$, i.e.

$$M(r, f) = \max\{|f(z)| : \ |z| = r\}, \quad \hat{m}(r, f) = \min\{|f(z)| : \ |z| = r\}.$$

If f has a pole on $\{|z| = r\}$, then $M(r, f) = +\infty$. Then $T(r, f)$ refers to the Nevanlinna characteristic function of f; $m(r, f)$ denotes the proximity function of f; $N(r, f)$ is the integrated counting function of f in Nevanlinna theory (their definitions are presented in § 2; see [**15**, **25**]).

For an escaping Fatou component U and $z_0 \in U$, define

$$h_n(z) := \frac{\log |f^n(z)|}{\log |f^n(z_0)|}, \quad \forall\ z \in U,$$
$$\overline{h}(z) := \limsup_{n\to\infty} h_n(z), \quad \underline{h}(z) := \liminf_{n\to\infty} h_n(z),\ \forall\ z \in U.$$

In view of (1.1) we easily see that $\overline{h}(z)$ and $\underline{h}(z)$ exist on U. In [**9**], Bergweiler *et al.* first introduced and proved that $h(z) := \overline{h}(z) = \underline{h}(z)$ on a multiply connected Fatou component U of an entire function f and they applied h successfully to describe the geometric structure of U. This shows that h is a significant function.

THEOREM 1.2. *Assume that for some $C \geqslant 1$, $c > 0$ and $r_0 > 0$, we have*

$$\log M(Cr, f) \geqslant \left(1 - \frac{c}{\log r}\right) T(r, f), \quad \forall\ r > r_0. \tag{1.4}$$

Then there exist constants $D > 1$, $d > 1$ and $R_0 > 0$, which depend on C, c and r_0, such that for a Fatou component U of f, if for some $m > 0$ and some $r \geqslant R_0$, we have

$$A(D^{-d}r, D^d r) \subseteq f^m(U), \tag{1.5}$$

then U is an escaping wandering domain, $h(z) := \overline{h}(z) = \underline{h}(z)$ exists on U and $h(z)$ is a non-constant positive harmonic function on U. Therefore for any point $z_0 \in U$ and any neighbourhood V of z_0 in U, we have for some constant $0 < \alpha < 1$ and all sufficiently large n

$$f^n(V) \supset A(|f^n(z_0)|^{1-\alpha}, |f^n(z_0)|^{1+\alpha}).$$

Moreover, there exist two points a and b in V and $\tau > 1$ such that

$$|f^n(a)|^\tau \leqslant |f^n(b)|$$

and

$$A_n = A(r_n, R_n) \subseteq f^n(V)$$

with $R_n \geqslant r_n^\tau \to \infty\, (n \to \infty)$ and $A_{n+1} \subseteq f(A_n)$.

We can give effective conditions to determine D, d and R_0 in theorem 1.2. We also note that the conditions that D, d and R_0 satisfy are chosen to guarantee the conclusion of lemma 3.2 to hold. Therefore from the proof of theorem 1.1 in [**28**], a rough calculation shows that D, d and R_0 satisfying the following inequalities are sufficient for theorem 1.2:

$$D > 2C\mathrm{e}^{c+1}, \quad \frac{\pi}{d\cos(\pi/2d)}\left(1+\frac{\pi}{2\log D}\right) < \frac{1}{6}, \tag{1.6}$$

$R_0 \geqslant D^d r_0$ and

$$\frac{T(r,f)}{\log r} > d\log D, \quad \forall\, r \geqslant R_0. \tag{1.7}$$

For instance, we have (1.6) by taking $C = c = 1$, $D = \mathrm{e}^3$ and $d = 10\pi$.

Since for a transcendental meromorphic function f,

$$\frac{T(r,f)}{\log r} \to \infty \ (r \to \infty),$$

it follows that if $\bigcup_{n=0}^{\infty} f^n(U)$ contains an infinite modulus annulus sequence for an escaping Fatou component U of f, then we have (1.5), where D, d and R_0 are chosen such that (1.6) and (1.7) hold.

We give some remarks on (1.4). Theorem 1.1 tells us that condition (1.4) cannot be removed in theorem 1.2. In example 5.3 of [**28**], we constructed a meromorphic function which has an escaping wandering domain U in which there exist two points a and b such that $|f^n(b)|/|f^n(a)| \to \infty\,(n \to \infty)$ but $h_U(z) \equiv 1$ on U.

If f has at most finitely many poles, then (1.4) holds immediately. Hence, theorem 1.2 extends the related results in [**9**] to meromorphic functions with infinitely many poles. If

$$T(r,f) \geqslant N(r,f)\log r, \quad \forall\, r \geqslant r_0, \tag{1.8}$$

then we have

$$m(r,f) = T(r,f) - N(r,f) \geqslant \left(1 - \frac{1}{\log r}\right)T(r,f), \quad \forall\, r \geqslant r_0.$$

Also, (1.8) implies (1.4) and that the Nevanlinna deficiency $\delta(\infty, f)$ of f of poles is 1. It is easy to find a meromorphic function f satisfying (1.4) and $\delta(\infty, f) < 1$. Consider the function $f(z) = (\mathrm{e}^z - 1)/(\mathrm{e}^{-z} + 1)$. Then

$$m(r,f) \sim \frac{r}{\pi}, \quad N(r,f) \sim \frac{r}{\pi}\ (r \to \infty), \quad \delta(\infty, f) = \frac{1}{2}$$

and

$$\log M(r,f) \geqslant \log\frac{\mathrm{e}^r - 1}{\mathrm{e}^{-r} + 1} \sim r \sim \frac{\pi}{2}T(r,f)\ (r \to \infty).$$

Then we obtain (1.4) for $C = 1$.

Finally, we mention that theorem 1.2 is still true even if (1.4) is satisfied for all r possibly outside a set with finite logarithmic measure. Indeed, for a set E with finite logarithmic measure, for any $d > 1$, we have $(r, dr) \setminus E \neq \emptyset$ for all sufficiently large r. Then in the proof of theorem 1.2, we can choose available r outside E.

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The existence of large annuli in the orbit of an escaping Fatou component guarantees the stability of the annuli in the sense that the suitable large annuli still exist under relatively small changes to the function in question.

THEOREM 1.3. *Let f be a meromorphic function satisfying the conditions in theorem 1.2. If f has an escaping Fatou component U such that $\bigcup_{n=0}^{\infty} f^n(U)$ contains an infinite modulus annulus sequence, then for any meromorphic function g which is analytic on $\bigcup_{n=N}^{\infty} f^n(U)$ for some N and satisfies*

$$M(r, f-g) \leqslant M(r, f)^{\delta}, \text{ for } \{z : |z| = r\} \subset \bigcup_{n=N}^{\infty} f^n(U), \quad \delta \in (0,1), \tag{1.9}$$

g has an escaping Fatou component V on which the results in theorem 1.2 also hold for g.

Theorem 1.3 was proved in [**9**] for the case where f and g are entire. Theorem 1.3 gives us a strong motivation to find a meromorphic function f with $\delta(\infty, f) = 0$ which has an escaping wandering domain U satisfying the properties stated in theorem 1.2. This is stated as follows.

THEOREM 1.4. *There exists a meromorphic function f with $\delta(\infty, f) = 0$ which has a wandering domain W such that the results in theorem 1.2 hold for f and W.*

Furthermore, in terms of theorem 1.3, we can find a meromorphic function to which theorem 2 applies, but it does not satisfy (1.4) in a set of r with infinite logarithmic measure. Indeed, we can make a modification of f in $\mathbb{C} \setminus \bigcup_{n=N}^{\infty} f^n(U)$ such that (1.4) does not hold.

The organization of this paper is as follows. In § 2, we collect some preliminary results that will be needed in the proofs of theorems. The proofs of theorems 1.1–1.4 are given in § 3. We will make further discussions or remarks after the proofs of theorems.

2. Preliminary lemmas

Firstly we need a covering lemma which comes from the hyperbolic metric. Let Ω be a hyperbolic domain in the extended complex plane $\hat{\mathbb{C}}$, that is, $\hat{\mathbb{C}} \setminus \Omega$ contains at least three points. Then there exists the hyperbolic metric $\lambda_\Omega(z)|\mathrm{d}z|$ with Gaussian curvature -1 on Ω, where $\lambda_\Omega(z)$ is the hyperbolic density of Ω at $z \in \Omega$. In particular, we have

$$\lambda_D(z) = \frac{1}{|z| \log |z|}, \quad D = \{z : |z| > 1\}$$

and [**28**, p. 791]

$$\lambda_A(z) = \frac{\pi}{|z| \operatorname{mod}(A) \sin(\pi \log(R/|z|)/\operatorname{mod}(A))}, \quad \forall\, z \in A = A(r, R), \tag{2.1}$$

where $A(r, R) = \{z : r < |z| < R\}$ and $\mathrm{mod}(A) = \log R/r$ is the modulus of A. The hyperbolic distance $d_\Omega(u, v)$ of two points $u, v \in \Omega$ is defined by

$$d_\Omega(u,v) = \inf_\alpha \int_\alpha \lambda_\Omega(z)\,|\mathrm{d}z|\,,$$

where the infimum is taken over all piecewise smooth paths α in Ω joining u and v.

LEMMA 2.1 [28]. *Let f be analytic on a hyperbolic domain U with $0 \notin f(U)$. If there exist two distinct points z_1 and z_2 in U such that $|f(z_1)| > \mathrm{e}^{\kappa\delta}|f(z_2)|$, where $\delta = d_U(z_1, z_2)$ and $\kappa = \Gamma(1/4)^4/(4\pi^2) \approx 4.37688$, then there exists a point $\hat{z} \in U$ such that $|f(z_2)| \leqslant |f(\hat{z})| \leqslant |f(z_1)|$ and*

$$f(U) \supset A\left(\mathrm{e}^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{1/\delta}|f(\hat{z})|,\ \mathrm{e}^{-\kappa}\left(\frac{|f(z_1)|}{|f(z_2)|}\right)^{1/\delta}|f(\hat{z})|\right). \tag{2.2}$$

If $|f(z_1)| \geqslant \exp(\kappa\delta/(1-\delta))|f(z_2)|$ and $0 < \delta < 1$, then

$$f(U) \supset A(|f(z_2)|, |f(z_1)|). \tag{2.3}$$

In particular, for $\delta \leqslant 1/6$ and $|f(z_1)| \geqslant e|f(z_2)|$, we have (2.3).

In order to cover an effective annulus, we are forced to calculate carefully the hyperbolic distance in an annulus.

LEMMA 2.2. *Set $A = A(r, R)$ with $0 < r < R$. Then for any two points $z_1, z_2 \in A$ with $|z_2| \leqslant |z_1|$ we have*

$$\max\left\{\frac{\pi}{\mathrm{mod}(A)}\log\frac{|z_1|}{|z_2|}, \log\frac{\log(R/|z_2|)}{\log(R/|z_1|)}\right\} \leqslant d_A(z_1, z_2)$$
$$\leqslant \frac{\pi^2}{\mathrm{mod}(A)\hat{K}_0} + \frac{\pi}{\mathrm{mod}(A)\hat{K}_1}\log\frac{|z_1|}{|z_2|},$$

where

$$\hat{K}_0 = \max_{j=1,2}\sin(\pi\log(R/|z_j|)/\mathrm{mod}(A)), \quad \hat{K}_1 = \min_{j=1,2}\sin(\pi\log(R/|z_j|)/\mathrm{mod}(A)).$$

In particular, for z_1, z_2 with $R/|z_1| = (R/r)^\sigma$, $R/|z_2| = (R/r)^\tau$ and $0 < \sigma \leqslant \tau < 1$, we have

$$\max\left\{(\tau-\sigma)\pi, \log\frac{\tau}{\sigma}\right\} \leqslant d_A(z_1, z_2) \leqslant \frac{\pi^2}{\hat{K}_0\,\mathrm{mod}(A)} + \frac{(\tau-\sigma)\pi}{\hat{K}_1}, \tag{2.4}$$

with $\hat{K}_0 = \max\{\sin(\sigma\pi), \sin(\tau\pi)\}$ and $\hat{K}_1 = \min\{\sin(\sigma\pi), \sin(\tau\pi)\}$.

Proof. We prove the first inequality. In view of (2.2), for all $z \in A$, we have

$$\lambda_A(z) = \frac{\pi}{|z| \mathrm{mod}(A) \sin(\pi \log(R/|z|)/\mathrm{mod}(A))} \geqslant \frac{1}{|z| \log(R/|z|)}$$

and

$$\lambda_A(z) \geqslant \frac{\pi}{|z| \mathrm{mod}(A)}.$$

Therefore

$$d_A(z_1, z_2) \geqslant \int_\gamma \frac{|\mathrm{d}z|}{|z| \log(R/|z|)} \geqslant \log \frac{\log(R/|z_2|)}{\log(R/|z_1|)}$$

and

$$d_A(z_1, z_2) \geqslant \frac{\pi}{\mathrm{mod}(A)} \log \frac{|z_1|}{|z_2|},$$

where γ is the geodesic curve in A connecting z_1 and z_2.

Next we prove the second inequality. For all z with $|z_2| \leqslant |z| \leqslant |z_1|$ we have

$$\sin(\pi \log(R/|z|)/\mathrm{mod}(A)) \geqslant \min_{j=1,2} \sin(\pi \log(R/|z_j|)/\mathrm{mod}(A)) = \hat{K}_1.$$

Without loss of generality, assume that $\hat{K}_0$ attains the maximal value at z_2, as the same argument applies to the case when z_2 is replaced by z_1. Then

$$\begin{aligned} d_A(z_1, z_2) &\leqslant \frac{\pi}{\mathrm{mod}(A)\hat{K}_0} \int_{\gamma_0} \frac{|\mathrm{d}z|}{|z|} + \frac{\pi}{\mathrm{mod}(A)\hat{K}_1} \int_{\gamma_1} \frac{|\mathrm{d}z|}{|z|} \\ &\leqslant \frac{\pi^2}{\mathrm{mod}(A)\hat{K}_0} + \frac{\pi}{\mathrm{mod}(A)\hat{K}_1} \log \frac{|z_1|}{|z_2|}, \end{aligned}$$

where γ_0 is the shortest arc from $|z_2|\mathrm{e}^{i \arg z_1}$ to z_2 and $\gamma_1 = \{z = t\mathrm{e}^{i \arg z_1} : \ |z_1| \leqslant t \leqslant |z_2|\}$. □

Note that the bounds of $d_A(z_1, z_2)$ depend only on $\mathrm{mod}(A)$, but otherwise are independent of the size of R and r.

The next lemma is proved in [**28**] using the density-decreasing property of the hyperbolic metric.

Lemma 2.3 [**28**]. *Let $h(z)$ be analytic on the annulus $B = A(r, R)$ with $0 < r < R < +\infty$ such that $|h(z)| > 1$ on B. Then*

$$\begin{aligned} \log \hat{m}(\rho, h) &\geqslant \exp\left(-\frac{\pi^2}{2} \max\left\{\frac{1}{\log(R/\rho)}, \frac{1}{\log(\rho/r)}\right\}\right) \log M(\rho, h) \\ &\geqslant \min\left\{\frac{\log(\rho/r) - \pi}{\log(\rho/r) + \pi}, \frac{\log(R/\rho) - \pi}{\log(R/\rho) + \pi}\right\} \log M(\rho, h), \end{aligned} \tag{2.5}$$

where $\rho \in (r, R)$ and $\hat{m}(\rho, h) = \min\{|h(z)| : |z| = \rho\}$.

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Secondly we need some basic notations and results from Nevanlinna theory of meromorphic functions [15, 25]. Set $\log^+ x = \log\max\{1, x\}$. Let f be a meromorphic function. Define

$$m(r, f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\mathrm{d}\theta,$$
$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t}\mathrm{d}t + n(0, f)\log r,$$

where $n(t, f)$ denotes the number of poles of f counted according to their multiplicities in $\{z : |z| < t\}$; sometimes we write $n(t, \infty)$ for $n(t, f)$ and $n(t, 0)$ for $n(t, 1/f)$ when f is clear in the context, and the Nevanlinna characteristic of f by

$$T(r, f) := m(r, f) + N(r, f).$$

Then the deficiency of poles in terms of the Nevanlinna characteristic of f is given by

$$\delta(\infty, f) = \liminf_{r\to\infty} \frac{m(r, f)}{T(r, f)} = 1 - \limsup_{r\to\infty} \frac{N(r, f)}{T(r, f)}.$$

In the following lemma we summarize some results on the Nevanlinna characteristic that will be used later; this lemma also holds for $\log M(r, f)$ when f is transcendental entire (see theorem 2.2 of [9]).

Lemma 2.4. *Let f be a transcendental meromorphic function. Then*

(1) $T(r, f)/\log r \to \infty (r \to \infty)$;

(2) *There exists $r_0 > 0$ such that for $r_0 \leqslant r < R$,*

$$T(R, f) \geqslant \frac{\log R}{\log r} T(r, f); \tag{2.6}$$

Since $T(r, f)$ is a logarithmic convex function, the result (2) in lemma 2.4 follows from (1) in lemma 2.4 of [28].

3. Proofs of theorems 1.1–1.4

To prove theorem 1.1, we first recall Runge theorem.

Runge theorem (cf. [22]). Let W be a compact set on the complex plane and let $f(z)$ be analytic on W. Assume that E is a set which intersects every component of $\mathbb{C} \setminus W$. Then for any $\varepsilon > 0$, there exists a rational function $R(z)$ such that all poles of $R(z)$ lie in E and

$$|f(z) - R(z)| < \varepsilon, \ \forall z \in W.$$

Now we proceed to the proof of theorem 1.1.

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Take four sequences $\{r_n\}$, $\{R_n\}$, $\{r'_n\}$ and $\{R'_n\}$ such that $10 < r_n < r'_n < R'_n < R_n$, $2 \leqslant r'_n/r_n \leqslant 3$, $2 \leqslant R_n/R'_n \leqslant 3$, $9R_n < r_{n+1}$ and $R_n/r_n = R'_{n+1}/r'_{n+1}$. Thus

$$\frac{R_n}{r_n} = \frac{R'_{n+1}}{r'_{n+1}} \leqslant \frac{1}{4}\frac{R_{n+1}}{r_{n+1}}, \quad \frac{R_n}{r_n} \geqslant 4^{n-1}\frac{R_1}{r_1} \to \infty \quad (n \to \infty).$$

Define

$$T_n(z) = \frac{r'_{n+1}}{r_n} z: \ A(r_n, R_n) \to A(r'_{n+1}, R'_{n+1}).$$

Take a_n and b_n such that $b_n > R_n + n$, $b_n/R_n \to 1 (n \to \infty)$ and $a_n < r_n - n$, $a_n/r_n \to 1$ $(n \to \infty)$. Set $A_n = A(r_n, R_n)$, $C_n = \{z : |z| = b_n \text{ or } a_n\}$ and $B_n = B(0, r_n/4)$ with $r_1 > 4$.

Choose a sequence of positive numbers $\{\varepsilon_n\}$ such that $\varepsilon_{n+1} < (1/2)\varepsilon_n$ and $\varepsilon_1 < 1/2$. In view of the Runge's theorem, we have a rational function $f_1(z)$ such that

$$|f_1(z) - T_1(z)| < \varepsilon_1, \forall\ z \in A_1;\ |f_1(z)| < \varepsilon_1, \quad \forall\ z \in B_1$$
$$|f_1(z)| < \varepsilon_1, \quad \forall\ z \in C_1$$

and inductively, we have rational function $f_{n+1}(z)$ such that

$$\left|\sum_{k=1}^{n+1} f_k(z) - T_{n+1}(z)\right| < \varepsilon_{n+1}, \forall\ z \in A_{n+1};\ |f_{n+1}(z)| < \varepsilon_{n+1}, \quad \forall\ z \in B_{n+1}$$

and

$$\left|\sum_{k=1}^{n+1} f_k(z)\right| < \varepsilon_{n+1}, \quad \forall\ z \in C_{n+1}.$$

Write $f(z) = \sum_{n=1}^{\infty} f_n(z)$. Since this series is uniformly convergent on any compact subset of $\mathbb{C}$, $f(z)$ is a meromorphic function on $\mathbb{C}$.

For $z \in B_1$, we have

$$|f(z)| \leqslant \sum_{n=1}^{\infty} |f_n(z)| < \sum_{n=1}^{\infty} \varepsilon_n < 1,$$

that is to say, $f(B_1) \subset B(0, 1) \subset B_1$ and so B_1 is contained in an invariant Fatou component of f. For $z \in C_n$, we have, by noting that $C_n \subset B_m$ for $m > n$,

$$|f(z)| \leqslant \left|\sum_{k=1}^{n} f_k(z)\right| + \sum_{k=n+1}^{\infty} |f_k(z)| < \varepsilon_n + \sum_{k=n+1}^{\infty} |f_k(z)| < \sum_{k=n}^{\infty} \varepsilon_k < 1$$

and so $f(C_n) \subset B(0, 1)$ and C_n is contained in a preperiodic Fatou component of f. Since for $z \in A_n = A(r_n, R_n) \subset B_{n+1}$,

$$|f(z) - T_n(z)| \leqslant \left|\sum_{k=1}^{n} f_k(z) - T_n(z)\right| + \left|\sum_{k=n+1}^{\infty} f_k(z)\right| < \sum_{k=n}^{\infty} \varepsilon_k = \varepsilon'_n \text{ (say)}, \quad (3.1)$$

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we have

$$f(A(r_n, R_n)) \subset A\left(r'_{n+1} - \varepsilon'_n,\ R'_{n+1} + \varepsilon'_n\right) \subset A(r_{n+1}, R_{n+1}). \tag{3.2}$$

Therefore, A_n is contained in a wandering domain U_n of f and $f : U_n \to U_{n+1}$ is proper. Since f is univalent in A_n by Rouché's theorem, U_n is not doubly connected. Each of U_n has no isolated boundary points. If U_n is finitely connected, then in view of the Riemann–Hurwitz theorem, f is univalent in U_n, but the modulus of annulus A_m tends to infinity as $m \to \infty$, which contradicts the conformal invariance of annulus modulus. Therefore, U_n must be infinitely connected.

For a point $a \in A_n$, it follows from (3.1) that

$$\frac{r'_{n+1}}{r_n}|a| - \varepsilon'_n \leqslant |f(a)| \leqslant \frac{r'_{n+1}}{r_n}|a| + \varepsilon'_n$$

and

$$\frac{r'_{n+1} - \varepsilon'_n}{r_n} \leqslant \frac{|f(a)|}{|a|} \leqslant \frac{r'_{n+1} + \varepsilon'_n}{r_n}.$$

Inductively, from (3.2) we have

$$\prod_{k=1}^{m} \frac{r'_{n+k} - \varepsilon'_{n+k-1}}{r_{n+k-1}} \leqslant \frac{|f^m(a)|}{|a|} \leqslant \prod_{k=1}^{m} \frac{r'_{n+k} + \varepsilon'_{n+k-1}}{r_{n+k-1}}.$$

We note that

$$\prod_{k=1}^{\infty} \frac{r'_{n+k} + \varepsilon'_{n+k-1}}{r'_{n+k} - \varepsilon'_{n+k-1}} = \prod_{k=1}^{\infty} \left(1 + \frac{2\varepsilon'_{n+k-1}}{r'_{n+k} - \varepsilon'_{n+k-1}}\right)$$
$$< \exp \sum_{k=1}^{\infty} \frac{2\varepsilon'_{n+k-1}}{r'_{n+k} - \varepsilon'_{n+k-1}} < \mathrm{e}^2.$$

Thus for two points a and b in A_n, we have

$$\frac{|f^m(a)|}{|f^m(b)|} \leqslant \frac{|a|}{|b|} \prod_{k=1}^{m} \frac{r'_{n+k} + \varepsilon'_{n+k-1}}{r'_{n+k} - \varepsilon'_{n+k-1}} < \frac{|a|}{|b|}\mathrm{e}^2, \quad \forall m \in \mathbb{N}. \tag{3.3}$$

Now we need to treat two cases:

Case (I): $|f^m(b)| \leqslant R'_m < R_m \leqslant |f^m(a)|$;

Case (II): $|f^m(b)| \leqslant r_m < r'_m \leqslant |f^m(a)|$.

Below we only prove that case (I) would be impossible. The same argument applies to case (II) as well. Set $E_m = A(a_m, b_m)$. In view of the principle of hyperbolic metric (Schwarz–Pick lemma), we can obtain that

$$\begin{aligned}
d_U(a,b) &\geqslant d_{U_m}(f^m(a), f^m(b))\\
&\geqslant d_{E_m}(f^m(b), f^m(a))\\
&\geqslant d_{E_m}(R'_m, R_m)\\
&= \int_{R'_m}^{R_m} \lambda_{E_m}(z)|\mathrm{d}z|\\
&= \int_{R'_m}^{R_m} \frac{\pi}{t\ \mathrm{mod}(E_m)\sin(\pi\log(b_m/t)/\mathrm{mod}(E_m))}\mathrm{d}t\\
&\geqslant \int_{R'_m}^{R_m} \frac{\mathrm{d}t}{t\log(b_m/t)}\\
&= \log\frac{\log(b_m/R'_m)}{\log(b_m/R_m}\\
&= \log\left(1+\frac{\log(R_m/R'_m)}{\log(b_m/R_m)}\right)\\
&\geqslant \log\left(1+\frac{\log 2}{\log(b_m/R_m)}\right) \to \infty \quad (m\to\infty).
\end{aligned}$$

A contradiction is derived. Then for all sufficiently large $m > 0$, $f^m(a)$ and $f^m(b)$ lie in A_m, $A(a_m, r'_m)$ or $A(R'_m, b_m)$ simultaneously.

By noting that $2 \leqslant r'_n/a_n \leqslant 4$ and $2 \leqslant b_n/R'_n \leqslant 4$ together with (3.3), we deduce that for any pair of points a, $b \in U$ and all sufficiently large n, we have

$$\frac{|f^n(a)|}{|f^n(b)|} \leqslant \max\left\{4, \frac{|a|}{|b|}\mathrm{e}^2\right\}.$$

For the proof of theorem 1.2, we first establish two lemmas.

Lemma 3.1. *Let f be a meromorphic function on $\{z: |z| \leqslant R\}$ and f be analytic on $\overline{A}(r, R)$ with $0 < r < R/4$. Then for $r+1 < r' < R' \leqslant R$ and $|z| = r'$, we have*

$$\log^+|f(z)| \leqslant \left(\frac{R'+r'}{R'-r'} + \left(\log\frac{R}{r}\right)^{-1}\log\frac{R'}{r'-r}\right)T(R,f). \tag{3.4}$$

Proof. In view of the Poisson formula, for a point $z \in D = \{z: |z| < R'\}$ with $f(z) \neq 0, \infty$, we have

$$\log|f(z)| = \frac{1}{2\pi}\int_{\partial D}\log|f(\zeta)|\frac{\partial G_D(\zeta,z)}{\partial\vec{n}}\mathrm{d}s - \sum_{a_m\in D} G_D(a_m,z) + \sum_{b_n\in D} G_D(b_n,z)$$

so that

$$\log|f(z)| \leqslant m(D,z,f) + N(D,z,f), \tag{3.5}$$

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where

$$m(D,z,f)=\frac{1}{2\pi}\int_{\partial D}\log^+|f(\zeta)|\frac{\partial G_D(\zeta,z)}{\partial\vec{n}}\mathrm{d}s,$$
$$N(D,z,f)=\sum_{b_n\in D}G_D(b_n,z),$$

G_D is the Green function for D and all a_m and b_n are respectively zeros and poles of f in D counted according to their multiplicities. Then for z with $|z|=r'$, we have

$$\begin{aligned}m(D,z,f)&=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(R'\mathrm{e}^{i\theta})|\frac{R'^2-|z|^2}{|R'^2-z|^2}\mathrm{d}\theta\\&\leqslant\frac{R'+|z|}{R'-|z|}m(R',f)\\&\leqslant\frac{R'+|z|}{R'-|z|}T(R',f)\end{aligned}$$

and, by denoting the number of poles of f in D by $n(D,\,f)$, we have

$$\begin{aligned}N(D,z,f)&\leqslant\sum_{b_n\in D}\log\frac{R'}{|b_n-z|}\\&\leqslant n(D,f)\log\frac{R'}{r'-r}\\&=n(r,f)\log\frac{R'}{r'-r}\\&=\left(\log\frac{R}{r}\right)^{-1}\int_r^R\frac{n(t,f)}{t}\mathrm{d}t\log\frac{R'}{r'-r}\\&\leqslant\left(\log\frac{R}{r}\right)^{-1}\left(\log\frac{R'}{r'-r}\right)T(R,f).\end{aligned}$$

Therefore we immediately have (3.4). □

The following is extracted from the proof of theorem 1.1 in [**28**], but condition (1.4) in that paper is replaced by the more general condition (1.4) in this paper. Condition (1.4) in [**28**] is that for arbitrarily large C there exists an $s(C)>0$ such that for $r\geqslant s(C)$ we have

$$T(Cr,f)-N(Cr,f)\geqslant T(2r,f)+7\pi\log C.$$

Basically, in order to be able to use lemma 2.1 to obtain the large annulus in the proof of theorem 1.1 in [**28**], we only deal with the following

$$T(2r,f)+7\pi\log C\leqslant m(Cr,f).$$

Thus we can take a point z_1 such that

$$\log|f(z_1)|\geqslant T(2r,f)+7\pi\log C.$$

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Of course, we can also obtain z_1 required from the following inequality

$$T(2r, f) + 7\pi \log C \leqslant \log M(Cr, f).$$

The inequality follows naturally from (1.4) in this paper. In fact, under (1.4), we have

$$\begin{aligned} \log M(2\mathrm{e}^{c+2}Cr, f) &\geqslant \left(1 - \frac{c}{\log(2\mathrm{e}^{c+2}Cr)}\right) T(2\mathrm{e}^{c+2}r, f) \\ &\geqslant \left(1 - \frac{c}{\log(2r)}\right)\left(1 + \frac{c+2}{\log(2r)}\right) T(2r, f) \\ &> T(2r, f) + \frac{T(2r, f)}{\log 2r} \\ &> T(2r, f) + 7\pi \log(2\mathrm{e}^{c+2}C), \end{aligned}$$

for large r. We can establish the following by replacing C in the proof of theorem 1.1 in [**28**] with $D = 2\mathrm{e}^{c+2}C$.

LEMMA 3.2. *Let f be a meromorphic function such that inequality (1.4) holds. Let U be a Fatou component of f. Then there exist constants $D > 1$, $d > 1$ and $R_0 > 0$ such that if for some $m > 0$ and $r \geqslant R_0$,*

$$A(D^{-d}r, D^d r) \subseteq f^m(U), \tag{3.6}$$

then for all sufficiently large n, $f^n(U) \supset A_n$ and $A_{n+1} \subset f(A_n)$ with $A_n = A(r_n, R_n)$, $R_n \geqslant r_n^\sigma$, $r_{n+1} > R_n$, $r_n \to \infty$ $(n \to \infty)$ and $\sigma > 1$.

We remark that if the conclusion of lemma 3.2 holds, that is to say, for large n, $f^n(U)$ contains the large annulus given by lemma 3.2, then condition (1.4) can be replaced by the following weaker inequality

$$\log M(Cr, f) \geqslant \left(1 - \frac{\log\log r}{\log r}\right) T(r, f), \quad \forall\, r > r_0. \tag{3.7}$$

We will complete the proof of theorem 1.2 under condition (3.7) instead of (1.4) after we obtain the large annulus sequence given in lemma 3.2. Now we are in a position to prove theorem 1.2.

Under the conditions of theorem 1.2, in view of lemma 3.2, for all sufficiently large n, we have

$$A(r_n, R_n) \subset f^n(U) \tag{3.8}$$

with $R_n \geqslant r_n^\sigma$, $r_{n+1} > R_n$, $r_n \to \infty$ $(n \to \infty)$ and $\sigma > 1$. Therefore, for a sufficiently large m and $n \geqslant m$ we have $r_{n+1} > R_n \geqslant R_m^{\sigma^{n-m}}$. Without loss of generality, for a sufficiently large m, we rewrite r_m, R_m and $\sigma > 1$ such that

$$f^m(U) \supset A(r_m^\alpha, R_m^\beta) \supset A(r_m, R_m)$$

with $R_m = r_m^\sigma$, $(\sigma - 1)\log r_m > 2m^2$, $0 < \alpha < 1$ and $\beta > 1$ (e.g. with $n = m$ in (3.8), we replace r_m, R_m by $r_m^{1/\alpha}$, $R_m^{1/\beta}$ respectively and choose $\beta = \alpha\sigma$). Take

$R'_m = \mathrm{e}^{-m^2} R_m$ and $r'_m = er_m$. Applying lemma 2.3 to f on $A(r_m, R_m)$, we have

$$\hat{m}(R'_m, f) \geqslant M(R'_m, f)^{s_m}, \ s_m = \exp\left(-\frac{\pi^2}{2} \max\left\{\frac{1}{\log(R_m/R'_m)}, \frac{1}{\log(R'_m/r_m)}\right\}\right). \tag{3.9}$$

Next we estimate s_m. Since

$$\log \frac{R_m}{R'_m} = m^2 \text{ and } \log \frac{R'_m}{r_m} = (\sigma - 1)\log r_m - m^2 > m^2,$$

we have

$$s_m = \exp\left(-\frac{\pi^2}{2m^2}\right).$$

Select a $\tau \in (1, \sigma)$ such that $s_m\sigma > \tau$ and take two points a and b with $|a| = r'_m$ and $|b| = R'_m$. Set $\hat{R}_m = r'_m \mathrm{e}^{m^3}$. Finding σ_m such that $R'_m/C = \hat{R}_m^{\sigma_m}$, we have

$$\sigma_m = \frac{\log R'_m/C}{\log \hat{R}_m} = \sigma - \frac{\sigma m^3 + m^2 + \sigma + \log C}{\log r_m + m^3 + 1}.$$

Therefore, we have $|f(a)| \leqslant M(r'_m, f)$ and in view of (3.7), (3.9) and lemma 2.4, we have

$$\begin{aligned} |f(b)| \geqslant M(R'_m, f)^{s_m} &\geqslant \exp(s_m\tau_m T(R'_m/C, f)) \\ &= \exp(s_m\tau_m T(\hat{R}_m^{\sigma_m}, f)) \geqslant \exp(s_m\sigma_m\tau_m T(\hat{R}_m, f)), \end{aligned} \tag{3.10}$$

where $\tau_m = 1 - (\log\log(R'_m/C)/\log(R'_m/C))$. Since $r'_m > r_m > r_0^{\sigma^m}$, where $r_0 > e$, we have

$$\sigma_m > \sigma - \frac{(C+\sigma)m^3}{\log r_m} > \sigma - \frac{(C+\sigma)m^3}{\sigma^m \log r_0}$$

and $s_m\sigma_m\tau_m \to \sigma > 1 (m \to \infty)$.

In view of lemma 3.1 with $R' = r'_m(m^2+1)$, $r' = r'_m$, $r = r_m$ and $R = \hat{R}_m$, we have

$$\begin{aligned} \log M(r'_m, f) &\leqslant \left(\frac{m^2+2}{m^2} + \frac{\log((r'_m(m^2+1))/(r'_m - r_m))}{\log(\hat{R}_m/r_m)}\right) T(\hat{R}_m, f) \\ &= \left(\frac{m^2+2}{m^2} + \frac{\log(e/(e-1))(m^2+1)}{m^3+1}\right) T(\hat{R}_m, f) \\ &\leqslant \left(1 + \frac{t}{m^2}\right) T(\hat{R}_m, f), \end{aligned}$$

where t is an absolute constant. Set $t_m = (1 + t/m^2)^{-1}$. Thus combining (3.10) together with the above inequality yields that

$$|f(b)| \geqslant \exp(s_m\sigma_m\tau_m T(\hat{R}_m, f)) \geqslant M(r'_m, f)^{t_m s_m\sigma_m\tau_m} \geqslant |f(a)|^{t_m s_m\sigma_m\tau_m}. \tag{3.11}$$

Set $r'_{m+1} = |f(a)|$ and $R'_{m+1} = |f(b)|$. Then we have $R'_{m+1} \geqslant (r'_{m+1})^{t_m s_m\sigma_m\tau_m}$. Set $R_{m+1} = \mathrm{e}^{(m+1)^2} R'_{m+1}$ and $r_{m+1} = r'_{m+1}/e$.

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Now we estimate $d_{U_m}(a,\, b)$ with $U_m = f^m(U)$ in terms of lemma 2.2. Set $A'_m = A(r_m^\alpha,\, R_m^\beta)$. Since

$$\sin\left(\pi\frac{\log(R_m^\beta/r'_m)}{\operatorname{mod}(A'_m)}\right) = \sin\left(\pi\frac{(\beta-1/\sigma)\log R_m - 1}{(\beta-\alpha/\sigma)\log R_m}\right) \to \sin\left(\pi\frac{\sigma\beta-1}{\sigma\beta-\alpha}\right) \quad (m\to\infty)$$

and

$$\sin\left(\pi\frac{\log(R_m^\beta/R'_m)}{\operatorname{mod}(A'_m)}\right) = \sin\left(\pi\frac{(\beta-1)\log R_m + m^2}{(\beta-\alpha/\sigma)\log R_m}\right) \to \sin\left(\pi\frac{\sigma(\beta-1)}{\sigma\beta-\alpha}\right) \quad (m\to\infty),$$

we assume that $0 < \hat{K}_1 \leqslant \hat{K}_0 < \min\{\sin(\pi((\sigma\beta-1)/(\sigma\beta-\alpha))),\, \sin(\pi((\sigma(\beta-1))/(\sigma\beta-\alpha)))\}$. In terms of lemma 2.2, we have, for $|a| = r'_m$ and $|b| = R'_m$,

$$\begin{aligned}\delta &= d_{U_m}(a,b)\\ &\leqslant d_{A'_m}(a,b)\\ &\leqslant \frac{\pi^2}{\operatorname{mod}(A'_m)\hat{K}_0} + \frac{\pi}{\operatorname{mod}(A'_m)\hat{K}_1}\log\frac{R'_m}{r'_m}\\ &= \frac{\pi^2}{\operatorname{mod}(A'_m)\hat{K}_0} + \frac{\pi}{(\beta-\alpha/\sigma)\hat{K}_1\log R_m}((1-1/\sigma)\log R_m - m^2 - 1)\\ &\to \frac{\pi(\sigma-1)}{(\sigma\beta-\alpha)\hat{K}_1} \quad (m\to\infty).\end{aligned}$$

Therefore, for sufficiently large m and $\sigma > 1$ close to 1, we have $\delta = d_{U_m}(a,\, b) < 1/6$. In view of the inequality, (3.11) and (3.10), we have

$$\begin{aligned}\mathrm{e}^{-\kappa}\left(\frac{|f(b)|}{|f(a)|}\right)^{1/\delta-1} &\geqslant \mathrm{e}^{-\kappa}\left(|f(b)|^{1-1/(t_m s_m \sigma_m \tau_m)}\right)^5\\ &\geqslant \exp\left[5\left(1-\frac{1}{t_m s_m \sigma_m \tau_m}\right)s_m\sigma_m\tau_m T(\hat{R}_m, f) - \kappa\right]\\ &> \exp[3(\sigma-1)T(\hat{R}_m, f)]\\ &> \mathrm{e}^{(m+1)^2}.\end{aligned}$$

Thus, for $|f(a)| \leqslant |f(\hat{z})| \leqslant |f(b)|$ with $\hat{z} \in U_m$ we have

$$\mathrm{e}^{\kappa}\left(\frac{|f(a)|}{|f(b)|}\right)^{1/\delta}|f(\hat{z})| \leqslant \mathrm{e}^{\kappa}\left(\frac{|f(a)|}{|f(b)|}\right)^{1/\delta-1} r'_{m+1} < r'_{m+1}/e = r_{m+1};$$

and

$$\mathrm{e}^{-\kappa}\left(\frac{|f(b)|}{|f(a)|}\right)^{1/\delta}|f(\hat{z})| \geqslant \mathrm{e}^{-\kappa}\left(\frac{|f(b)|}{|f(a)|}\right)^{1/\delta-1} R'_{m+1} > \mathrm{e}^{(m+1)^2}R'_{m+1} = R_{m+1}.$$

Since $d_{U_m}(a,\, b) < 1/6$, according to lemma 2.1, we have

$$U_{m+1} = f(U_m) \supset A(r_{m+1}, R_{m+1}) \supset \overline{A}(r'_{m+1}, R'_{m+1}).$$

Thus, by noting that $d_{U_{m+1}}(f(a),\, f(b)) \leqslant d_{U_m}(a,\, b) < 1/6$ (β and α are just used to imply the inequality), $|f(a)| = r'_{m+1}$ and $|f(b)| = R'_{m+1}$, we can repeat the above step and obtain that

$$|f^n(b)| \geqslant |f^n(a)|^{\prod_{k=0}^{n-1} s_{k+m} t_{k+m} \sigma_{k+m} \tau_{k+m}}. \tag{3.12}$$

For sufficiently large m, we can require, for $n \geqslant 1$,

$$\prod_{k=0}^{n-1} s_{k+m} t_{k+m} \sigma_{k+m} \tau_{k+m} > \frac{\sigma+1}{2} > 1.$$

Take a point $c \in U$. Define

$$h_n(z) = \frac{\log|f^n(z)|}{\log|f^n(c)|}, \quad \forall\ z \in U.$$

Since U is wandering and escaping, we can assume that $|f^n(z)| > 1$ on U and hence with (1.1) we conclude that $h_n(z)$ is harmonic and positive on U. In view of (1.1) or by Harnack's inequality, the family $\{h_n\}$ of harmonic functions is locally normal on U and hence

$$\overline{h}(z) := \limsup_{n\to\infty} h_n(z), \quad \forall\ z \in U$$

exists and $\overline{h}$ is harmonic and positive on U. Then it follows from (3.12) that

$$h_n(b) \geqslant \frac{\sigma+1}{2} h_n(a), \quad \overline{h}(b) \geqslant \frac{\sigma+1}{2}\overline{h}(a) > \overline{h}(a).$$

Therefore, $\overline{h}(z)$ is not a constant on U. The same argument implies that

$$\underline{h}(z) := \liminf_{n\to\infty} h_n(z), \quad \forall\ z \in U$$

exists and $\underline{h}$ is a non-constant, harmonic and positive function on U.

Suppose that $\overline{h}(z_0) > \underline{h}(z_0)$ for some $z_0 \in U$. Without any loss of generality, suppose that $\overline{h}(z_0) > \overline{h}(c) = 1$. Take a real number η with $\max\{1,\, \underline{h}(z_0)\} < \eta < \overline{h}(z_0)$. Since $\overline{h}(z)$ is not a constant, we can find z_1 and z_2 such that $\overline{h}(z_1) < 1 = \overline{h}(c) < \overline{h}(z_0) < \overline{h}(z_2)$. In view of lemma 2.1, for some sufficiently large m, we have

$$f^m(U) \supset A\left(|f^m(c)|^{\alpha}, |f^m(c)|^{\beta}\right),$$

where $\overline{h}(z_1) < \alpha < 1$ and $1 < \overline{h}(z_0) < \beta < \overline{h}(z_2)$.

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From the argument of the above paragraph we can take $a = f^m(c)$ and $b = f^m(z_0)$ such that

$$|f^{n+m}(z_0)| > |f^{n+m}(c)|^\eta, \quad \forall\, n \in \mathbb{N}.$$

This implies that $\underline{h}(z_0) \geqslant \eta$. A contradiction is derived and so we have proved that $\overline{h}(z) = \underline{h}(z)$ on U, that is to say,

$$h(z) = \lim_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(c)|}$$

exists on U.

Set

$$H(z) := \lim_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0)|} = \frac{h(z)}{h(z_0)}.$$

Then H is a non-constant, harmonic and positive function on U. Thus we can find two points z_1 and z_2 in V and $0 < \alpha < 1$ such that $H(z_1) < 1 - \alpha < 1 < 1 + \alpha < H(z_2)$. For a sufficiently large m, we have $d_{f^m(V)}(f^m(z_1),\, f^m(z_2)) < 1/6$ and hence, in view of lemma 2.1, for $n \geqslant m$

$$f^n(V) \supset A(|f^n(z_1)|, |f^n(z_2)|) \supset A(|f^n(z_0)|^{1-\alpha}, |f^n(z_0)|^{1+\alpha}).$$

Thus we have (3.6) for sufficiently large m and $f^m(V)$ instead. In view of lemma 3.2, for all sufficiently large n and some $\tau > 1$, we obtain $A_n = A(r_n,\, R_n) \subseteq f^n(V)$ with $R_n \geqslant r_n^\tau \to \infty (n \to \infty)$ such that $A_{n+1} \subset f(A_n)$.

This completes the proof of theorem 1.2.

Proof of theorem 1.3. In view of theorem 1.2, there exist a sufficiently large m and a sufficiently large r_m such that for $n \geqslant m$ and $r_{n+1} > R_n \geqslant r_n^\sigma$ with $\sigma > 1$ we have

$$f^n(U) \supset A(r_n^\alpha, R_n^\beta), \quad 0 < \alpha < 1,\ 1 < \beta.$$

From (1.9) and (1.4), by using lemma 2.3, the version of lemma 2.4 for $M(r,\, f)$ from [**9**] and lemma 3.1, we can show that

$$M(R_m, g) \geqslant \hat{m}(R_m, g) > M(r_m, g) \geqslant \hat{m}(r_m, g)$$

and in view of the maximum principle, we have

$$g(A(r_m, R_m)) \subseteq A(\hat{m}(r_m, g), M(R_m, g)). \tag{3.13}$$

Set $u(z) = f(z) - g(z)$ and $s_m = \exp(-(\pi^2/2m^2))$. For sufficiently large m, we have $s_m > \max\{1/2,\, \delta\}$. Using lemma 2.3 with $r = \mathrm{e}^{-m^2} r_m$, $\rho = r_m$ and $R = R_m$ and

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noting that $\log(1-x) > -2x$ for $0 < x < 1/2$, we have

$$\begin{aligned}\hat{m}(r_m, g) &\geqslant \hat{m}(r_m, f) - M(r_m, u) \geqslant M(r_m, f)^{s_m} - M(r_m, u)\\ &= \left(1 - \frac{M(r_m, u)}{M(r_m, f)^{s_m}}\right) M(r_m, f)^{s_m} \geqslant M(r_m, f)^{s_m\gamma_m},\end{aligned}$$

where $\gamma_m = 1 - (4M(r_m, f)^{\delta - s_m})/(\log M(r_m, f)) > 1 - 1/m^2$, and from the inequality $\log(1+x) < x$ for $x > 0$, we have

$$\begin{aligned}M(R_m, g) &\leqslant M(R_m, f) + M(R_m, u)\\ &= M(R_m, f)\left(1 + \frac{M(R_m, u)}{M(R_m, f)}\right) \leqslant M(R_m, f)^{\tau_m},\end{aligned}$$

where $\tau_m = 1 + 1/(M(R_m, f)^{1-\delta}\log M(R_m, f)) < 1 + 1/m^2$ for sufficiently large m. Thus it follows from (3.13) that

$$g(A(r_m, R_m)) \subseteq A(M(r_m, f)^{s_m\gamma_m}, M(R_m, f)^{\tau_m}). \tag{3.14}$$

On the other hand, we have

$$f^{m+1}(U) \supset f(A(r_m^{\alpha}, R_m^{\beta})) \supset A(M(r_m^{\alpha}, f), \hat{m}(R_m^{\beta}\mathrm{e}^{-m^2}, f)). \tag{3.15}$$

By lemma 2.3 and (1.4), we have

$$\hat{m}(R_m^{\beta}\mathrm{e}^{-m^2}, f) \geqslant M(R_m^{\beta}\mathrm{e}^{-m^2}, f)^{s_m} \geqslant \exp(\sigma_m t_m T(R_m^{\beta}\mathrm{e}^{-m^2}/C, f)),$$

where $t_m = 1 - (\log\log(R_m^{\beta}\mathrm{e}^{-m^2}/C))/(\log(R_m^{\beta}\mathrm{e}^{-m^2}/C)) > 1 - 1/m^2$ for sufficiently large m, and in view of lemma 3.1, we can show that

$$\tau_m \log M(R_m, f) < s_m t_m T(R_m^{\beta}\mathrm{e}^{-m^2}/C, f)$$

and by (1.4) and the version of lemma 2.4 for $M(r, f)$ from [**9**], we have

$$s_m\gamma_m \log M(r_m, f) > \log M(r_m^{\alpha}, f).$$

Thus $M(R_m^{\beta}\mathrm{e}^{-m^2}, f)^{s_m} > M(r_m^{\alpha}, f)$ and from (3.15) we can deduce

$$f^{m+1}(U) \supset A(M(r_m^{\alpha}, f), M(R_m^{\beta}\mathrm{e}^{-m^2}, f)^{s_m}). \tag{3.16}$$

Now we set $\hat{r}_1 = M(r_m^{\alpha}, f)^{s_m\gamma_m/\alpha}$ and $\hat{R}_1 = M(R_m, f)^{\tau_m}$ so that we have $M(r_m, f)^{s_m\gamma_m} \geqslant \hat{r}_1$. Combining (3.14) and (3.16) yields

$$f^{m+1}(U) \supset A(\hat{r}_1^{\alpha\eta_m}, \hat{R}_1^{\beta_m}) \supset A(\hat{r}_1, \hat{R}_1) \supset g(A(r_m, R_m))$$

with $\eta_m = 1/s_m\gamma_m > 1$ and $\beta_m = (s_m/\tau_m)((\log M(R_m^{\beta}\mathrm{e}^{-m^2}, f))/(\log M(R_m, f))) \to \beta$ $(m \to \infty)$. For sufficiently large m, we can require that $0 < \alpha\eta_m < (\alpha+1)/2 < 1$ and $1 < (\beta+1)/2 < \beta_m$.

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By the same argument as above, we have

$$f^{m+2}(U) \supset A(\hat{r}_2^{\alpha\eta_m\eta_{m+1}}, \hat{R}_2^{\beta_{m+1}}) \supset A(\hat{r}_2, \hat{R}_2) \supset g^2(A(r_m, R_m))$$

with $\hat{r}_2 = M(\hat{r}_1^{\alpha\eta_m}, f)^{1/(\eta_{m+1}\eta_m\alpha)}$, $\hat{R}_2 = M(\hat{R}_1, f)^{\tau_{m+1}}$, $0 < \alpha\eta_m\eta_{m+1} < (\alpha+1)/2 < 1$ and $1 < (\beta+1)/2 < \beta_{m+1}$.

Inductively, we have

$$f^{m+n}(U) \supset g^n(A(r_m, R_m)).$$

Thus $A(r_m, R_m)$ is contained in an escaping Fatou component W of g.

We note that in $A(r_m, R_m)$ we have

$$\begin{aligned}\log M(Cr, g) &\geqslant \log(M(Cr, f) - M(Cr, u)) \\ &= \log M(Cr, f) + \log\left(1 - \frac{M(Cr, u)}{M(Cr, f)}\right) \\ &\geqslant \left(1 - \frac{2\log\log r}{\log r}\right) T(r, f)\end{aligned}$$

and for $r_m < r + 1 < r' < R' < R \leqslant R_m$ we have

$$\begin{aligned}\log M(r', g) &\leqslant \log M(r', f) + \log\left(1 + \frac{M(r'u)}{M(r', f)}\right) \\ &\leqslant \left(\frac{R'+r'}{R'-r'} + \left(\log\frac{R}{r}\right)^{-1}\log\frac{R'}{r'-r}\right) T(R, f) + 1.\end{aligned}$$

Thus we can repeat the arguments in the proof of theorem 1.2 and prove that the results of theorem 1.2 hold for g in W. □

We make a remark on (1.9). From the proof of theorem 1.3 we see that in order that theorem 1.3 holds for g, in fact, we only need to require (1.9) holds in a sequence of annuli $A(r_n, r_n^{\sigma_n})$ instead of $f^n(U)$ as long as $A(r_n, r_n^{\sigma_n}) \subset f^n(U)$ for $\forall\ n \geqslant m$ with m being large enough and $f(A(r_n, r_n^{\sigma_n})) \subset A(r_{n+1}, r_{n+1}^{\sigma_{n+1}})$, where $1 < \sigma \leqslant \sigma_n$.

Proof of theorem 1.4. With the help of theorem 1.3 we will find a meromorphic function f with $\delta(\infty, f) = 0$ which has an escaping Fatou wandering domain U such that the results of theorem 1.2 hold for f in U.

In [**9**], Bergweiler *et al.* proved the results of theorem 1.2 for entire functions in their multiply connected Fatou components. The first entire function with multiply connected Fatou component is due to Baker [**2**]. The multiply connected wandering domains which have uniformly perfect boundary or non-uniformly perfect boundary were found in [**10**]. For example, in theorem 1.3 of [**10**], for a sequence $\{r_k\}$ of positive numbers with $r_{k+1} > 2r_k^2$ and a sequence $\{\varepsilon_k\}$ of positive numbers tending to 0 as $k \to \infty$, an entire function f of zero order is constructed such that

$$f(A((1+\varepsilon_k)r_k, (1-\varepsilon_k)r_{k+1})) \subset A((1+\varepsilon_{k+1})r_{k+1}, (1-\varepsilon_{k+1})r_{k+2}).$$

Moreover, the Fatou components U containing $A((1+\varepsilon_k)r_k, (1-\varepsilon_k)r_{k+1})$ may have uniformly perfect boundaries by choosing $\{r_k\}$ suitably.

Now we put

$$g(z) = f(z) + \sum_{n=1}^{\infty} \frac{1}{2^n m_n} \sum_{k=1}^{m_n} \frac{\varepsilon_n}{z - z_{k,n}},$$

where $m_n = [r_{n+1}] + 1$, $[r_{n+1}]$ is the maximal integer not exceeding r_{n+1}, and $z_{k,n} = r_n \mathrm{e}^{i(2k\pi/m_n)}$. It is clear that g is a meromorphic function. For $|z| = r$ with $(1+\varepsilon_k)r_k < r < (1-\varepsilon_k)r_{k+1}$, we have

$$\begin{aligned}
|f(z) - g(z)| &\leqslant \sum_{n=1}^{\infty} \frac{1}{2^n m_n} \sum_{j=1}^{m_n} \frac{\varepsilon_n}{|z - z_{j,n}|} \\
&\leqslant \sum_{\substack{n=1 \\ n \neq k, k+1}}^{\infty} \frac{1}{2^n} \frac{\varepsilon_n}{|r - r_n|} \\
&\quad + \frac{1}{2^k} \frac{\varepsilon_k}{r - r_k} + \frac{1}{2^{k+1}} \frac{\varepsilon_{k+1}}{r_{k+1} - r} \\
&< \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,
\end{aligned}$$

and this implies (1.9) for $|z| = r$ with $(1+\varepsilon_k)r_k < r < (1-\varepsilon_k)r_{k+1}$. In view of theorem 1.3 and from the remark after the proof of theorem 1.3, there exists an escaping wandering domain W of g such that the results of theorem 1.2 hold for g and W.

Next we calculate the deficiency $\delta(\infty, g)$. For $er_k < r < (1-\varepsilon_k)r_{k+1}$, we have $m(r, g) \leqslant m(r, f) + \log 2$ and

$$N(r, g) = \int_0^r \frac{n(t, g)}{t} \mathrm{d}t \geqslant \int_{r_k}^r \frac{m_k}{t} \mathrm{d}t \geqslant r_{k+1} \geqslant r.$$

Then as $r \in \cup_{k=m}^{\infty} A(er_k, (1-\varepsilon_k)r_{k+1}) \to \infty$, we have

$$\frac{m(r, g)}{T(r, g)} \leqslant \frac{m(r, g)}{N(r, g)} \leqslant \frac{m(r, f) + \log 2}{r} \to 0,$$

by using the fact that f is of zero order. This implies that

$$\delta(\infty, g) = \liminf_{r \to \infty} \frac{m(r, g)}{T(r, g)} = 0.$$

Then g is the desired meromorphic function of theorem 1.4. □

Nevertheless we do not know if W has uniformly perfect boundary. Therefore, we ask the following question.

Question $\mathcal{B}$: Is there any meromorphic function with zero Nevanlinna deficiency at poles which has a multiply connected escaping wandering domain with uniformly perfect boundary?

Perhaps the following approach would be possible to solve question $\mathcal{B}$. We try to control the changes of critical values as an entire function, that has a multiply

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connected escaping wandering domain with uniformly perfect boundary, is changed to a meromorphic function with zero Nevanlinna deficiency at poles and then in view of theorem 1.3 we show the meromorphic function also has a multiply connected escaping wandering domain with uniformly perfect boundary.

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