Symmetric Determinants and the Cayley and Capelli Operators

By H. W. TURNBULL

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§ 1. *Historical Note*

The result[1] obtained by Lars Gårding, who uses the Cayley operator upon a *symmetric* matrix, is of considerable interest. The operator $\Omega = | \partial/\partial x_{ij} |$, which is obtained on replacing the n^2 elements of a determinant $| x_{ij} |$ by their corresponding differential operators and forming the corresponding n-rowed determinant, is fundamental in the classical invariant theory. After the initial discovery in 1845 by Cayley further progress was made forty years later by Capelli[2] who considered the minors and linear combinations (polarized forms) of minors of the same order belonging to the whole determinant Ω: but in all this investigation the n^2 elements x_{ij} were regarded as independent variables. The apparently special case, undertaken by Gårding when $x_{ij} = x_{ji}$ and the matrix $[x_{ij}]$ is symmetric, is essentially a new departure: and it is significant to have learnt from Professor A. C. Aitken in March this year 1946, that he too was finding the symmetrical matrix operator $[\partial/\partial x_{ij}]$ of importance and has already written on the matter.

From the point of view of invariant theory it was natural to take all the x_{ij} as independent, since the matrix belongs to the coefficients of a linear transformation possessing group properties which must survive matrix multiplication, whereas the product of two symmetric matrices is not necessarily symmetric. There was not therefore the same *a priori* reason for supposing that the symmetric operator would have significance, as there was in the general case. By these latest discoveries fresh light has been thrown upon the theory, including that of the determinantal identities on which the proofs depend. While the Cayley-Capelli results follow essentially upon the generalized identities of Sylvester, which in their simplest forms were known to Bézout, and which involve sums of products of pairs of determinants,

[1] Gårding, these *Proceedings* (2), **8**, p. 73.

[2] Capelli (1882, 1886), *Math. Annalen* **29**, 331-338.

the present results depend upon the Kronecker identities which involve sums of single determinants, namely minors of the same order belonging to a symmetric determinant. There is much information to be found in Muir's *History* about these identies of Kronecker, which first appeared almost contemporaneously with those of Capelli in 1882, and which have frequently been studied since, but ingenious as the treatment has been it seems to have lacked a unifying principle. The formula found by Lars Gårding has however suggested an extension of a procedure, which has been useful in the case of the Sylvester identities, to cover the less obvious case of the Kronecker identities.

§ 2. *Gårding's Theorem*

This theorem may be illustrated by taking it in a more general and polarized form. What follows applies at once to n-rowed determinants, but is expounded for brevity by the quaternary case. Thus let

$$\Delta = (xyzt)_{1234} = \Sigma \pm x_1 y_2 z_3 t_4 = | x_{ij} |$$

be a four-rowed determinant whose columns are x, y, z, t respectively, the suffixes denoting the rows. Let a, b, c, d denote further such columns of elements which are independent of the elements x_{ij}.

Let
$$\left(a \mid \frac{\partial}{\partial x}\right) = \Sigma\, a_i \frac{\partial}{\partial x_i} = a_x \tag{1}$$

be called a polar operator, where $i = 1, 2, 3, 4$, the notation being a convenient abbreviation. Such an operator at once yields the identity $a_x \Delta = (ayzt)$ which replaces the column x by the column a. Similarly we may utilise further operators a_y, b_x, etc., and their compounds $(ab)_{xy}$, $(ab)_{xyz}$, etc., where, for example,

$$(ab)_{xy} = \left(ab \,\middle|\, \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) = \begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix} \tag{2}$$

which is the second-compound operator. Then if s is any positive integer it follows, from Cayley's results, that

$$\begin{aligned} a_x \Delta^s &= s \Delta^{s-1} (ayzt)_{1234}, \\ (ab)_{xy} \Delta^s &= s(s+1)\, \Delta^{s-2} (abzt)_{1234}, \end{aligned}$$

and in general, if $\Delta = | XZ |$ and $(A \mid \partial/\partial X) = A_X$ is the bi-determinant of the m^{th} order,

$$A_X \Delta^s = s(s+1)\ldots(s+m-1)\, \Delta^{s-m} (AZ),\ 0 < m \leqq n. \tag{3}$$

If A is replaced by any m different columns of the unit $n \times n$

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matrix I this becomes an identity between an m-rowed minor of $\Omega = | \partial/\partial x_{ij} |$ operating upon Δ^s and a product involving a complementary minor belonging to Δ. For example, if $a = \{1, 0, 0, 0\}$, $b = \{0, 0, 1, 0\}$ then $a_x = \partial/\partial x_1$, $b_x = \partial/\partial x_3$, and

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)_{13} \Delta^s = s(s+1)\,\Delta^{s-2}(zt)_{42},$$

where the complementary nature both of xy, zt and of 13, 42 is apparent. (Partition of $xyzt$ into xyz,t etc. would be equally relevant.)

Now let Δ be *symmetric* whereas a, b, c, d are still quite arbitrary. The appropriate operator is obtained by affixing a coefficient $\frac{1}{2}$ to every element except those which stand upon the principal diagonal, thus:

$$\Omega = \begin{vmatrix} \dfrac{\partial}{\partial x_{11}} & \dfrac{\partial}{2\partial x_{12}} & \cdots & \dfrac{\partial}{2\partial x_{1n}} \\ \dfrac{\partial}{2\partial x_{21}} & \dfrac{\partial}{\partial x_{22}} & \cdots & \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial}{2\partial x_{n1}} & \dfrac{\partial}{2\partial x_{n2}} & \cdots & \dfrac{\partial}{\partial x_{nn}} \end{vmatrix} \qquad x_{ij} = x_{ji}, \tag{4}$$

and the corresponding identity is

$$A_X\Delta^s = s(s+\tfrac{1}{2})(s+1)\ldots\left(s+\frac{m-1}{2}\right)\Delta^{s-m}(AZ), \qquad 0 < m \leqq n. \tag{5}$$

If A is replaced by m columns of the unit matrix this becomes Gårding's identity. It is convenient to retain the same notation such as a_x, $(ab)_{xy}$, A_X for the polar operator, but to speak of it as the *adjusted* operator; namely the cœfficient $\frac{1}{2}$ is attached to each $\partial/\partial x_{ij}$ whenever i and j differ, but is omitted when $i = j$.

The following method proves the theorem, and also adapts the polarizing process to symmetric determinants. Let Δ be bordered with the constants a_i and written

$$\Delta = \begin{vmatrix} 1_1 & 2_1 & 3_1 & 4_1 \\ 1_2 & 2_2 & 3_2 & 4_2 \\ 1_3 & 2_3 & 3_3 & 4_3 \\ 1_4 & 2_4 & 3_4 & 4_4 \end{vmatrix} \qquad \Phi = \begin{vmatrix} . & 1_a & 2_a & 3_a & 4_a \\ a_1 & 1_1 & 2_1 & 3_1 & 4_1 \\ a_2 & 1_2 & 2_2 & 3_2 & 4_2 \\ a_3 & 1_3 & 2_3 & 3_3 & 4_3 \\ a_4 & 1_4 & 2_4 & 3_4 & 4_4 \end{vmatrix},$$

where

$$a_a = 0, \qquad i_a = a_i, \qquad \text{and } i_j = x_{ij} = x_{ji} = j_i \tag{6}$$

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by definition, for all values of i, j from 1 to n inclusive.

More briefly

$$\Delta = (1234)_{1234}, \qquad \Phi = (a1234)_{a1234}. \tag{7}$$

Let $x = [x_1, x_2, x_3, x_4]$ denote the top row of Δ, and x' the first column. Then differentiating Φ with regard to x_1 we have

$$\partial\Phi/\partial x_1 = \partial\Phi/\partial 1_1 = (a234)_{a234}$$

but, on the other hand,

$$\partial\Phi/\partial x_2 = \partial\Phi/\partial 1_2 = -(a134)_{a234} - (a234)_{a134}$$

since x_2 occurs twice, as 1_2 and 2_1, whereas x_1 only occurs once in Φ. On interchanging the whole lower with the whole upper set of indices, since $i_j = j_i$ always, we have

$$(a134)_{a234} = (a234)_{a134}.$$

Thus $\frac{1}{2}\partial\Phi/\partial x_2 = -(a134)_{a234}$.

Next differentiate Δ in the same way, and we find that

$$\partial\Delta/\partial x_1 = (234)_{234}, \qquad \tfrac{1}{2}\partial\Delta/\partial x_2 = -(134)_{234};$$

whence

$$(a \mid \partial/\partial x)\Delta \equiv \left(a_1\frac{\partial}{\partial x_1} + \tfrac{1}{2}a_2\frac{\partial}{\partial x_2} + \tfrac{1}{2}a_3\frac{\partial}{\partial x_3} + \tfrac{1}{2}a_4\frac{\partial}{\partial x_4}\right)\Delta, \tag{8}$$

that is, $\qquad a_x\Delta = (a234)_{1234} = \Delta_a, \qquad (9)$

which is Gårding's theorem with $s = 1$. Since the operator a_x is linear in the $\partial/\partial x_i$, therefore

$$a_x\Delta^s = s\Delta^{s-1}\Delta_a, \tag{10}$$

which is the theorem with $m = 1$ and a general s.

To polarize a step further we need the first minors of the polarized bordered determinant

$$\Phi(a, b) = (a1234)_{b1234}$$

where the set b has replaced one set (row or column) of a. We find that

$$b_y\,\Delta = (1b34)_{1234} = (1234)_{1b34} \tag{11}$$

and
$$2\,b_y\,\Delta_a = (ab34)_{1234} + (a234)_{1b34} \tag{12}$$

where b replaces the index 2 (belonging to y) within the operand in every possible way. Hence, operating on (10) with b_y, we have

$$b_y a_x\,\Delta^s = s(s-1)\Delta^{s-2}\Delta_b\Delta_a + s\Delta^{s-1}b_y\Delta_a. \tag{13}$$

Now, by a Sylvester identity,

$$\Delta_b\Delta_a - \Delta_a\Delta_b = (1b34)_{1234}(a234)_{1234} - (1a34)_{1234}(b234)_{1234} = (ab34)_{1234},$$

and by a Kronecker identity

$$(a234)_{1b34} - (b234)_{1a34} = (ab34)_{1234} \tag{14}$$

Hence, by (12), $2(b_y\Delta_a - a_y\Delta_b) = 3(ab34)_{1234}$, so that, if a, b are permuted determinantally, (13) becomes

$$(a_x b_y - a_y b_x)\Delta^s = s(s-1)\Delta^{s-1}\Delta_{ab} + \tfrac{3}{2}s\Delta^{s-1}\Delta_{ab},$$

that is,
$$(ab)_{xy} \quad \Delta^s = s(s+\tfrac{1}{2})\Delta^{s-1}\Delta_{ab}, \tag{15}$$

which is the required identity for the second-compound operator.

Similarly by operating with c_z on this and then permuting ab, c determinantally we have

$$(abc)_{xyz}\Delta^s = s(s+\tfrac{1}{2})\,[(s-1)\Delta^{s-2}\Delta_c\Delta_{\ddot{a}\ddot{b}} + \Delta^{s-1}\Delta']$$

where
$$\Delta' = \dot{c}_z\Delta_{\ddot{a}\ddot{b}} = \tfrac{1}{2}\{c,\, ab\}'[(abc4)_{1234} + (ab34)_{12c4}]$$

(using a convenient confluent modification $\{c,\, ab\}'$ of three terms instead of Young's substitutional expression $\{abc\}'$ of six terms permuted determinantally). By a Kronecker identity

$$\{c,\, ab\}'(ab34)_{12c4} = (abc4)_{1234} = \Delta_{abc}:$$

whence
$$(abc)_{xyz}\Delta^s = s(s+\tfrac{1}{2})\,[s-1+\tfrac{1}{2}(3+1)]\Delta^{s-1}\Delta_{abc}$$
$$= s(s+\tfrac{1}{2})\,(s+1)\Delta^{s-1}(abc4).$$

At the next stage $(abcd)_{xyzt}$ factorizes into $(abcd)\,\Omega$ yielding $\Omega\,\Delta^s = s(s+\tfrac{1}{2})\,(s+1)\,(s+\tfrac{3}{2})\Delta^{s-1}$ after cancelling the determinant $(abcd)$. The method is general and proves the theorem.

§3. *The Kronecker Identities*

The above theorem therefore depends on a particular type of Kronecker's identity which is exemplified by

$$\{c,\, ab\}'(ab345)_{12c45} = (abc45)_{12345} \tag{1}$$

where the right-hand expression is a five (n) rowed symmetric determinant with three (m) quite arbitrary columns abc substituted for its first three (m) columns, while the left-hand expression is the sum of three (m) bordered determinants obtained by permuting one column c with two ($m-1$) others. Here

$$(ab345)_{12c45} = \begin{vmatrix} a_1 & b_1 & 3_1 & 4_1 & 5_1 \\ a_2 & b_2 & 3_2 & 4_2 & 5_2 \\ 0 & 0 & 3_c & 4_c & 5_c \\ a_4 & b_4 & 3_4 & 4_4 & 5_4 \\ a_5 & b_5 & 3_5 & 4_5 & 5_5 \end{vmatrix} \tag{2}$$

where the elements a_c, b_c both vanish, and $i_c = c_i$ for each of the numerical indices i. The sum of three such determinants obtained by permuting ab, c cyclically is therefore the determinant

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$$\begin{vmatrix} a_1 & b_1 & c_1 & 4_1 & 5_1 \\ a_2 & b_2 & c_2 & 4_2 & 5_2 \\ a_3 & b_3 & c_3 & 4_3 & 5_3 \\ a_4 & b_4 & c_4 & 4_4 & 5_4 \\ a_5 & b_5 & c_5 & 4_5 & 5_5 \end{vmatrix}.$$

The determinant (2) is a five (n) rowed minor of an eight $(m+n)$ rowed symmetrically bordered determinant

$$\Phi(a,b,c) \equiv (abc12345)_{abc12345} \tag{3}$$

consisting of a leading square array of nine (m^2) zeros at the meeting places of the rows a, b, c with the columns a, b, c. The minor in question is obtained by deleting rows a, b, 3 and columns 1,2, c. A sum of such minors, from a symmetric determinant, obtained by derangement of several upper indices with one single lower index after the manner of (1) above is the Kronecker identity of 1882.

Now there is a parallelism [1] running through the Sylvester and the Kronecker identities which is remarkable and complete, and it leads to generalized forms of the latter identities corresponding to those of the former. It may be illustrated very simply as follows. Take a pair of three (n) rowed determinants and form a Sylvester identity [2]

$$(12\dot{3})\,(\dot{4}\dot{5}6) = (345)\,(126). \tag{4}$$

Now lower either row of indices in each expression so written down, and the result is a Kronecker identity

$$(12\dot{3})_{\dot{4}\dot{5}6} = (345)_{126}. \tag{5}$$

In formula (4) each factor (ijk) is understood to mean a determinant whose columns are i, j, k but whose rows have indices p, q, r, *the same for each determinant.* The determinants are general, whereas in (5) they are minors of the symmetric matrix $[i_j]$. A proof of this parallelism implies that just as there are three types of Sylvester identity, so there are for the Kronecker identities, which are as follows:

$$\begin{array}{llll} \text{I.} & (123\dot{4}\dot{5})_{\dot{6}\dot{7}890} = (\dot{1}4567)_{\dot{2}\dot{3}890}, & \mu+\nu<n, & \\ \text{II.} & (123\dot{4}\dot{5})_{\dot{6}\dot{7}\dot{8}90} = (45678)_{12390}, & \mu+\nu=n, & (6) \\ \text{III.} & (12\dot{3}\dot{4}\dot{5})_{\dot{6}\dot{7}\dot{8}90} = 0, & \mu+\nu>n. & \end{array}$$

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[1] Muir: *History* 5 138: First noticed by H. S. White, *Bulletin American Math. Soc.* 2 (1895) 136-138: Cf. E. J. Nanson *Messenger of Mathematics* 31 (1902) 140-143.

[2] Turnbull: *Determinants Matrices and Invariants* p. 45.

Here μ denotes the number of upper and ν the number of lower indices in the left-hand expression, which undergo determinantal permutation as indicated by the accompanying dots, while n is the order of the determinant. Each identity is an aggregate of single five (n) rowed minors chosen from a symmetric ten ($2n$) rowed matrix. The series on the left have six, ten and twenty terms respectively: and there is nothing to exclude the cases where some of the upper and the lower indices are identical, so that extensionals of the identities are included. Gårding's theorem depends upon such an extensional, as at § 2 (14), a type of Kronecker identity first considered by Muir.

To prove these identities consider the corresponding Sylvester identities, which are obtained by raising the lower rank of indices to the upper level and treating the two ranks as a product of two n-rowed determinants. For example

$$(1\dot{2}3)_{pqr}\,(\ddot{4}56)_{pqr} = (345)_{pqr}\,(126)_{pqr}. \tag{7}$$

Here the six columns are arbitrary while the row indices p, q, r are the same in each determinant. Now let the symmetric matrix $[x_{ij}]$ be regarded as belonging to a quadratic in n variables u_i, which is cast into symbolic form

$$u[x_{ij}]u' = u_p{}^2 = u_q{}^2 = u_r{}^2$$

in the familiar way by means of equivalent symbols p, q, r. It follows at once that the typical coefficient is symbolized by

$$i_j = j_i = x_{ij} = i_p j_p = i_q j_q = i_r j_r.$$

Furthermore, any identity such as (7), which possesses a pair of indices p (or q or r) in each term, can be interpreted as belonging to this quadratic. The process of putting this into nonsymbolic form yields the parallel expression

$$3!\,(1\dot{2}3)_{\dot{4}56} = 3!\,(345)_{126} \tag{8}$$

and on dividing by $3!$ ($n!$ in general) we at once have the corresponding Kronecker identity.

For example, $(345)_{126} = \Sigma \pm 3_1.4_2.5_6 = \Sigma \pm 3_p 1_p 4_q 2_q 5_r 6_r$

$$= \dot{3}_p \dot{4}_q \dot{5}_r 1_p 2_q 6_r = (345)_{pqr} 1_p 2_q 6_r$$

which gives the desired result (8) after interchanging the equivalent symbols in every possible way and adding the results.[1]

This proves the generalized Kronecker identities; and it suggests their further generalizations corresponding to the Sylvester identities which involve three or more [2] determinantal factors in each term.

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[1] *Determinants* p. 194. [2] p. 48.

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If the number of factors is m then the suffixes $p, \ldots$ will occur repeated m times each and may be regarded as belonging to an n-ary m-ic form, so that the identity will furnish a property of a vanishing polynomial in the coefficients of this form.

Still another generalization is possible, in the case when m is even and the dual form of the Sylvester identities is taken.[1] This leads in its simplest examples to sums of products of minors belonging to a symmetric determinant, but the minors are not necessarily all of the same order.

For example

$$3\ (\dot{1}\dot{2})_{34}\ (\dot{5}6)_{78} = 2\ (125)_{34\dot{7}}\ 6_{\dot{8}}$$

is an identity of the Kronecker type which arises from the Sylvester identity [2]

$$(\dot{1}\dot{2})_{\dot{p}q}\ (34)_{pq}\ (\dot{5}6)_{rs}\ (78)_{rs} = (125)_{pq\dot{r}}\ (34)_{pq}\ 6_{\dot{s}}\ (78)_{rs}$$

on treating p, q, r, s as equivalent symbols. The numerical coefficients are due to convolving pqr twice on the right-hand side. This matter could be pursued considerably further.

As far as I know, the cases which have been discussed hitherto are all to be found [3] in Muir's *History*. Kronecker (1882) initiated the study with the identity (6) II in which $\nu = 1$: Muir (1897) gave its extensional with $\nu = 1$ and $n - \mu$ upper and lower indices identical, while Metzler discussed the case when $\nu = n$ and $0 < \mu \leqq n$. Muir also gave many remarkable results connecting this work with Pfaffians and skew symmetric determinants. Muir, in **5** p. 138, speaks of the method of parallelism between Sylvester and Kronecker identities as one "whose own logical soundness is not too patent." The present treatment gives, in my view, the necessary basis; and since writing the above I find that Nanson (*loc. cit.*) definitely invoked the symbolic methods of invariants to establish the parallelism which resolves the difficulty.

§ 4. *The Capelli Operator for the symmetric matrix*

The formula of § 2 (2) is an example of the Binet-Cauchy theorem which continues to hold for differential operators only if the polarizing columns, a, b are independent of the variables x_{ij}. But Capelli found the necessary modification in his celebrated theorem which

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[1] *loc. cit.* pp. 93-97 and 342-346 (2nd edition (1946)).

[2] Cf. *loc. cit.* p. 97 and p. 194.

[3] *History* **4** p. 113, 121, 137 ; **5** p. 132, 134, 135, 143.

covers the case when a is replaced by the original column x, and b by y and so on. The formula becomes

$$(xy)_{xy} = \left(xy \,\middle|\, \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right) = \begin{vmatrix} x_x & y_x \\ x_y & y_y + 1 \end{vmatrix}, \tag{1}$$

where the integers 0, 1, 2,..., $(m-1)$ are added to the respective elements upon the principal diagonal, m being the order of the determinant (it is second order in this illustration). It is understood that such a determinant is to be expanded by columns and from left to right, otherwise an ambiguity arises. If we denote this determinant on the right of the identity by $\Delta(0, 1, 2, \ldots, m-1)$ and that on the left by $H = \left(x \, . \, . \, t \,\middle|\, \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial t}\right) = \sum_I (x \ldots t)_I \left(\frac{\partial}{\partial x} \cdots \frac{\partial}{\partial t}\right)_I$ where I denotes a rank of m different indices taken from the first n positive integers, then Capelli's theorem states that if all the x_{ij} are independent then

$$H = \Delta(0, 1, 2, \ldots, m-1). \tag{2}$$

This can be adapted to the case where the x_{ij} are symmetric, and the operators x_x, x_y, etc. have the significance already explained as in § 2 (8); and it is interesting to observe that the same characteristic rise by half steps is apparent as it is in Gårding's theorem.

THEOREM *If $[x_{ij}]$ is a symmetric matrix then*

$$H = \Delta\left(0, \tfrac{1}{2}, 1, \ldots, \frac{m-1}{2}\right), \; 0 < m \leqq n. \tag{3}$$

For example

$$\left(xyz \,\middle|\, \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right) = \sum_{i,j,k}(xyz)_{ijk}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right)_{ijk} = \begin{vmatrix} x_x & y_x & z_x \\ x_y & y_y + \frac{1}{2} & z_y \\ x_z & y_z & z_z + 1 \end{vmatrix}. \tag{4}$$

Proof. This depends on separating the terms of the expanded forms into intrinsic and extrinsic terms, just as in the proof of Capelli's theorem.[1] The extrinsic terms are those due to treating all x_{ij} which occur in the operators as constants, in which case the additions, of $\frac{1}{2}$, 1, etc., upon the principal diagonal, are unnecessary and the identity is true by the Binet-Cauchy theorem. It remains to be proved that all the intrinsic terms cancel themselves out. Now such

[1] *Determinants* p. 116.

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terms occur only when a variable x_{ij} stands on the right of a differential operator within the same term, so that, for example, no intrinsic terms occur in the expression H on the left of the identity.

Let θ denote the j^{th} of the m columns $x, y, \ldots, z, t$ taken in this order, and let θ_i denote its i^{th} component. Similarly let ϕ denote the k^{th} such column. Then owing to symmetry we shall have

$$\theta_k = x_{kj} = x_{jk} = \phi_j. \tag{5}$$

Now consider the intrinsic terms within the minors of the first two columns of $\Delta\,(0, \tfrac{1}{2}, \ldots)$. From the operator $x_y y_\phi$, that is

$$\tfrac{1}{4}\left(x_1 \frac{\partial}{\partial y_1} + 2x_2 \frac{\partial}{\partial y_2} + x_3 \frac{\partial}{\partial y_3} + \ldots + x_n \frac{\partial}{\partial y_n}\right)\left(\sum_{i=1}^{n} y_i \frac{\partial}{\partial \phi_i} + y_k \frac{\partial}{\partial \phi_k}\right)$$

we get one such term for each y_i except y_k which supplies two as here written. We write the sum of these intrinsic terms thus:

$$\operatorname{int}\,(x_y y_\phi) = \tfrac{1}{4}\left(x_1 \frac{\partial}{\partial \phi_1} + 2x_2 \frac{\partial}{\partial \phi_2} + x_3 \frac{\partial}{\partial \phi_3} + \ldots\right) + \tfrac{1}{4} x_k \frac{\partial}{\partial \phi_k}.$$

But, by definition, since the element ϕ_k of ϕ alone is on the principal diagonal of $[x_{ij}]$,

$$x_\phi = \tfrac{1}{2} \sum_{r=1}^{n} x_r \frac{\partial}{\partial \phi_r} + \tfrac{1}{2} x_k \frac{\partial}{\partial \phi_k}.$$

Hence

$$\operatorname{int}\,(x_y y_\phi) = \tfrac{1}{2} x_\phi + \tfrac{1}{4} x_2 \frac{\partial}{\partial \phi_2}.$$

On the other hand if θ is any column but the second, which is y, then

$$\operatorname{int}\,(x_\theta y_\phi) = \tfrac{1}{4} x_2 \frac{\partial}{\partial \phi_j},$$

for in this case θ_2 alone supplies an operator which belongs to y, namely $\theta_2 = y_j$. Similarly

$$\left.\begin{aligned} \operatorname{int}\,(\theta_\phi \phi_\psi) &= \tfrac{1}{2}\theta_\psi + \tfrac{1}{4}\theta_k \frac{\partial}{\partial \psi_k}, \\ \text{and} \qquad \operatorname{int}\,(\theta_\phi \psi_\chi) &= \qquad \tfrac{1}{4}\theta_l \frac{\partial}{\partial \chi_k} \end{aligned}\right\} \tag{6}$$

where ϕ and ψ are the k^{th} and l^{th} columns of $[x_{ij}]$. This term on the right, $\tfrac{1}{2}\theta_\psi$, may be called the *block* term, and the other the *isolated* term. Of these the block term is identical with that of the general unsymmetrical case, save for its cœfficient $\tfrac{1}{2}$ and indeed it arises in the same way.[1] It therefore leads to the same result as before, that is a

[1] Cf. *loc. cit.* pp. 116-118.

sequence of integer additions to the principal diagonal, only each is now to be multiplied by $\frac{1}{2}$. This accounts for the structure $\Delta(0, \frac{1}{2}, 1, \ldots.)$, if we can prove that the isolated terms cancel themselves out.

Now this is true for each two-rowed minor chosen from the first two columns since, by (6), the *isolated* intrinsic terms of

$$\begin{vmatrix} x_\theta & y_\theta \\ x_\phi & y_\phi \end{vmatrix}$$

are $\frac{1}{4}x_2\left(\frac{\partial}{\partial\phi_j} - \frac{\partial}{\partial\theta_k}\right)$ which cancel out because $\phi_j = x_{jk} = \theta_k$. The same induction by columns as is used to prove the general case now applies for the isolated terms. For assuming the result true up to $m-1$ columns and expanding by the m^{th} and final column, we obtain for the first two of m terms

$$(-)^m\left(xy\ldots z\,\Big|\,\frac{\partial}{\partial y}\cdots\frac{\partial}{\partial z}\,\frac{\partial}{\partial t}\right)t_x + (-)^{m-1}\left(xy\ldots z\,\Big|\,\frac{\partial}{\partial x}\cdots\frac{\partial}{\partial z}\,\frac{\partial}{\partial t}\right)t_y \qquad (7)$$

which can be written with the differential operators in the same order but with $\partial/\partial y$ and $\partial/\partial x$ *last* in their respective terms. The isolated terms arising from $\left(\theta\frac{\partial}{\partial y}\right)t_x$ and $\left(\theta\frac{\partial}{\partial x}\right)t_y$ are then $\frac{1}{4}\theta_m\frac{\partial}{\partial x_2}$ and $\frac{1}{4}\theta_m\frac{\partial}{\partial y}$ which are equal since $x_2 = y_1$ by symmetry. Since the left-hand wings$_1$ are the same $xy\ldots z$ in the two terms we deduce at once that the isolated terms due to the x and y, occurring in the above expressions and to the right of the vertical lines, cancel since $(-)^m$ and $(-)^{m-1}$ are opposites. Similarly for each pair of the m different symbols $x, y, \ldots, t$. This accounts for each isolated term arising from the m terms, namely $m(m-1)$ isolated terms which cancel in pairs, there being $(m-1)$ such terms arising from the x of t_x and the remaining $y, \ldots., t$ in the first term of (7), and $(m-1)$ in the second, and so on. This proves the theorem.

§ 5. *The skew symmetric case*

It may be shewn that a corresponding adaptation of Capelli's operator can be made to cover the case when $x_{ij} = -x_{ji}$ for all indices i, j, but that it rests upon a *permanent*

$$P(0, -1, -2, \ldots, 1-m)$$

analogous to the above Δ, where, for example, $a_x b_y + a_y b_x$ is a typical permanent of order two.

THE UNIVERSITY,
ST ANDREWS.