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ARTICLE

On Symmetries and Springs

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Abstract

Imagine that we are on a train, playing with some mechanical systems. Why can't we detect any differences in their behavior when the train is parked versus when it is moving uniformly? The standard answer is that boosts are symmetries of Newtonian systems. In this article, I use the case of a spring to argue that this answer is problematic because symmetries are neither sufficient nor necessary for preserving its behavior. I also develop a new answer according to which boosts preserve the relational properties on which the behavior of a system depends, even when they are not symmetries.

1. Introduction

Why do mechanical systems inside a bigger system behave in the same manner regardless of the (constant) velocity of the bigger system? For example, why is it that a pendulum hanging from the roof of a train's cabin oscillates with the same period when the train is at rest at the station and when it is moving at a speed of 150 km/h in a straight line? The standard answer to these questions appeals to the fact that the laws of mechanical systems are invariant under constant-velocity transformations or, what amounts to the same thing, to the fact that boosts are dynamical symmetries of mechanical systems (one could also say that "Newton's laws are Galilean invariant," which includes being invariant under boosts). In this article, I will argue that the standard answer is wrong. In particular, I will show that some transformations are dynamical symmetries and do not preserve the behavior of a mechanical system, and others are not dynamical symmetries, yet they do preserve the behavior of a mechanical system. Hence, being a dynamical symmetry is neither necessary nor sufficient for a given transformation (like a boost) to preserve the behavior of a Newtonian system. The main takeaway of the article is that we should look beyond dynamical symmetries for an answer to why mechanical systems, such as a pendulum inside a train, behave the same under boosts of the train.

The arguments in this article pose a problem not only for the standard physics explanation of why mechanical systems behave in the same way under boosts (i.e., the explanation that appeals to the fact that boosts are symmetries of Newtonian systems) but also for any philosophical view of symmetries that entails the standard explanation. For example, although they adopt rather different approaches to elucidating the connection between symmetries, observations, and measurements, Healey (2009), Dasgupta (2016), and Wallace (2022) have developed frameworks that seem to agree on this particular point: in order to explain why in Newtonian mechanics experiments confined to the cabin of a train yield the same outcomes regardless of the constant velocity of the train, we must appeal to (among other things) the fact that boosts are dynamical symmetries of the laws that characterize the mechanical systems in the cabin (see Dasgupta 2016, sec. 4.3; Healey 2009, sec. 5; Wallace 2022, sec. 4 and 5). Hence, these different approaches seem to recover the standard physics explanation that aims to derive sameness in behavior for a system from its dynamical symmetries, at least when restricted to the case of boosts in Newtonian mechanics. Hence, these accounts are also vulnerable to the arguments developed in this article.

The structure of the article is as follows. In section 2, I will present a simple mechanical system (a spring inside a spaceship) around which the rest of the article will be structured. The point of using this system is to examine, as concretely as possible, different explanations for why it is that the system remains invariant under the boosts of the spaceship. In section 3, I consider the standard approach to explaining why the system behaves in the same way under boosts; boosts are dynamical symmetries of the laws characterizing the system. Section 4 shows a problem with this kind of approach, having to do with a proliferation of dynamical symmetries. Section 5 shows a second problem: being a dynamical symmetry is not sufficient for explaining sameness in behavior. And section 6 shows a third problem: being a dynamical symmetry is not necessary for explaining the sameness of behavior. Section 7 presents a different approach that is more general and simpler than the one that appeals to dynamical symmetries and that centers on the fact that boosts can preserve the relative quantities on which the behavior of a system depends even when they are not symmetries.

Before moving on, it is worth stressing that other scholars, such as Belot (2013) and Wallace (2022), have already noted that some symmetries do not preserve the physical behavior of systems. However, in contrast to this article, these works do not focus on the explanatory role of symmetries and do not question the claim that symmetries explain why the behavior of simple mechanical systems (such as springs) remains invariant under boosts.

2 The main question

2.1 Terminology

Before considering the various arguments, it is a good moment to fix some basic terminology. Roughly, a dynamical symmetry is a mathematical transformation of the dependent or independent variables appearing in the equation for a law (usually, a differential equation) that leaves it invariant (equivalently, such transformation maps solutions to solutions). For the sake of brevity, "symmetry" and "dynamical symmetry" will be treated as interchangeable unless otherwise noted (this is common in the literature; see, e.g., Dasgupta 2016 or Wallace 2022). Also for brevity, I say "dynamical symmetry of the system" instead of "dynamical symmetry of the system's laws," but it should be clear that the latter is more precise.

2.2 A spring in a spaceship

Suppose that we want to explicitly show that boosts are symmetries of a spaceship moving in (almost) empty space with some constant velocity with respect to a faraway star (which is not accelerating). In order to explicitly show that boosts are symmetries of the spaceship, one can show that the equations representing its dynamics are invariant under boosts. In practice, this involves showing either that the two sides of the differential equation remain unchanged under the transformation *or* that the two sides are changed by the very same factor so that it cancels out at the end. In particular, for the present case, one can do the following: because the spaceship is a mechanical system, and because it is assumed to be isolated (i.e., no external force acts on it), its dynamics can be represented by the equation $0 = mdv/dt$ (Newton's second law for zero force). Then, we note that a boost of the form $v \mapsto v + k$ (with k constant) produces no change in the acceleration because $d(v + k)/dt = dv/dt$ if k is constant, meaning that the right-hand side of equation $0 = mdv/dt$ does not change. And then, we show that $v \mapsto v + k$ produces no changes in the force $F = 0$ because, we assume, the system remains isolated after the boost. Hence, both sides of $0 = mdv/dt$ remain invariant under a boost, indicating that the simple dynamical law characterizing the spaceship is preserved by this transformation. We can then say that the transformation (a boost) is a symmetry of the equation. This entails that such a boost will map solutions of $0 = mdv/dt$ into other solutions with higher or slower velocity (e.g., it will map $v = c$ into $v' = c \pm k$). This is all very familiar: symmetry transformations map solutions of the equations of motion of the system into other (or the very same) solutions.

Now, even though we just showed that the boosts of the spaceship are symmetries (they preserve the law describing the spaceship's behavior), notice that the previous explanation does not really say anything concrete about what is happening inside the spaceship. Arguably, the most important feature of Galileo's famous ship thought experiment is that mechanical systems confined to the ship behave in the same manner no matter what the velocity of the ship with respect to an external system is (see Brown and Sypel (1995) for a historically informed discussion of this thought experiment). So, how do we show in an explicit way that the behavior of mechanical systems inside the spaceship in question (which is a version of Galileo's ship) remains the same under the boosts of the spaceship? To be as concrete as possible, let's consider this question in the context of a particular Newtonian system inside the spaceship's cabin. In particular, let's focus on a block of mass M, which is connected to the back cabin's wall by an ideal spring that is measured by a detector inside the cabin that is located at the equilibrium position (see figure 1). Given this setup, consider the following question:

> **Main question:** Why is it that the behavior of the block, as seen inside the cabin (e.g., as measured by the detector), remains invariant under the boosts of the spaceship?

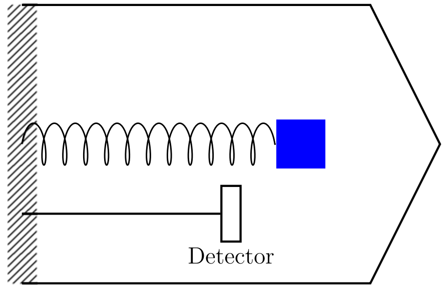

Figure 1. A block attached to an ideal spring that is itself attached to a wall of the spaceship. The detector is also rigidly attached to the same wall, and it is located at the equilibrium position. We want to understand why the behavior of the spring with respect to the detector looks the same after boosts.

This apparently simple question will be the focus of attention of the present article.

3 The dynamical symmetries approach

The standard way to answer why the behavior of a system remains invariant under boosts appeals to the dynamical symmetries of the system (in particular, it appeals to showing that boosts are symmetries of such a system). For example, Steven Weinberg (2021, 90) illustrates that Newtonian gravity is invariant under boosts by showing explicitly that the equation that describes two massive bodies interacting via Newtonian gravitation is invariant under transformations of the form $\mathbf{r} \mapsto \mathbf{r} + \mathbf{v}t$ (with $\mathbf{v}$ constant). The idea, then, is to use the exact same kind of strategy in order to answer the main question (where the relevant system is a spring seen by a detector at the equilibrium position).

It will be convenient to call the view that purports to answer the main question along these lines—that is, by focusing on the relevant dynamical symmetries of the laws that characterize the system—"dynamical symmetries explain" (DSE). Although DSE is widely defended in physics circles, it is no less popular in philosophical accounts of symmetries. For example, according to Dasgupta (2016), symmetries preserve appearances (according to Dasgupta, they do so by definition). Hence, if one can show that boosts are symmetries of a certain law characterizing a given system, then it follows that the measurement outcomes of such a system will be preserved under boosts (see also Roberts 2008). To give another example, Healey (2009, 707–8) explicitly points out that the fact that boosts are dynamical symmetries of Newtonian mechanics *explains* why mechanical systems look the same after a boost.

Let's then follow DSE for the case of the spring. First, it is clear that the relevant law characterizing the block's behavior is Hooke's because the block is attached to an ideal spring. Second, according to DSE, we need to show that Hooke's law is invariant under boosts—showing this would suffice for answering the main question because all the detector does is measure the displacement of the spring with respect to equilibrium. Mathematically, Hooke's law can be expressed as

$$x_b = -\omega^2 \tfrac{d^2}{dt^2} x_b, \tag{1}$$

where x_b is the position of the endpoint of the spring (the point at which the block is attached) measured with respect to the equilibrium position of the spring (we follow physics practice and use a coordinate system where the equilibrium position is the origin), ω depends on the spring's constant k and the block's mass via $\omega = \sqrt{m/k}$, and t represents the time. The general solution of equation 1 can be written as $x_b = A\,\cos(\omega t) + B\,\sin(\omega t)$, where A represents the initial amplitude and B the initial speed. Hence, the actual motion of the spring as a function of time is determined by a specific value of A and B (the initial conditions). For simplicity, I will refer to equation 1 as "Hooke's law," although the more precise terminology would be "the law for a classic harmonic oscillator."

Now, let's try to answer the main question by showing that equation 1 is invariant under a boost, that is, under a transformation of the form $x \mapsto x + vt$. Notice that the right-hand side of equation 1 remains invariant under this transformation because the second derivative of vt vanishes.[1] However, the left-hand side of equation 1 is not invariant, for x_b is obviously different from $x_b + vt$. Hence, the transformation $x \mapsto x + vt$ takes equation 1 into equation $x_b + vt = -\omega^2 \frac{d^2}{dt^2} x_b$, which is *not* the equation of an ideal spring. This seems to be suggesting that, contrary to what we expected, boosts are not dynamical symmetries of ideal springs! And this suggestion seems reinforced by the fact that the transformation in question does not map solutions into solutions. For instance, it takes $x_b = A_1\,\cos(\omega t)$ (a particular solution of equation 1 for $B = 0$ and $A = A_1$) into $x_b = A_1\,\cos(\omega t) - vt$, which is *not* a solution. Does this then mean that, contrary to what DSE says, the reason a spring behaves in the same manner in a spaceship at rest and in a spaceship that is moving is not a consequence of boosts being dynamical symmetries of the spring? Not so quickly!

A defender of DSE might point out that we have misinterpreted the transformation $x \mapsto x + vt$. Recall that x_b represents the position of the block (the endpoint of the spring) with respect to the equilibrium position. Hence, if we take the transformation $x \mapsto x + vt$ as acting only on the endpoint x_b, then we seem to be representing a boost of the block *with respect to the equilibrium position* as if we were stretching the end of the spring in a way proportional to speed and time. And of course, if we stretch the spring in this manner, we no longer expect the spring to behave like an ideal spring. Hence, what we really need to do according to DSE is show that the behavior of the block remains the same under boosts of the spring taken as a whole (instead of boosts of just the spring's endpoint).

Motivated by the previous remarks, let's write Hooke's law (or something that looks like it) in a different form that explicitly relates the endpoint to the equilibrium position of the spring via $x_b = x_f - x_{\text{eq}}$. Here, x_f is the position of the block (the endpoint), and x_{eq} is the position of the equilibrium point with respect to a certain coordinate system (initially, we take the equilibrium position to be at the origin of such a system). With these conventions, we can write the following equation:

$$x_f - x_{\text{eq}} = -\omega^2 \frac{d^2}{dt^2}\left(x_f - x_{\text{eq}}\right). \tag{2}$$

Mathematically speaking, equation 2 is what we get if we make the identification given by $x_b = x_f - x_{\text{eq}}$ in equation 1. The point of this identification is simply that it makes it easier to keep track of the equilibrium position, which will be useful for what

[1] Explicitly, $-\omega^2 \frac{d^2}{dt^2}(x_b + vt) = -\omega^2 \frac{d^2}{dt^2}(x_b) - \omega^2 \frac{d^2}{dt^2}(vt) = -\omega^2 \frac{d^2}{dt^2}(x_b)$ (because v is constant).

comes next. As before, the general solution of this differential equation is $x_b = x_f - x_{eq} = A\,\cos(\omega t) + B\,\sin(\omega t)$.[2]

Consider a boost that acts on both the endpoint x_f and the equilibrium point x_{eq} (and any other point of the spring, for that matter). Explicitly, let's interpret the boost transformation $x \mapsto x + vt$ as a transformation acting on every single point of the spring so that in particular, it acts on x_f like $x_f \mapsto x_f + vt$ and on x_{eq} as $x_{eq} \mapsto x_{eq} + vt$ (this means that the equilibrium point of the spring is no longer at rest). It is trivial to show that equation 2 remains invariant under such transformation, for $x_f - x_{eq}$ is just a difference between two terms, and so the left-hand side of equation 2 is invariant under the transformation. And of course, if this difference is preserved, so is any derivative of it, meaning the right-hand side is invariant too. Hence, both sides of equation 2 remain invariant, meaning that $x \mapsto x + vt$ (when interpreted as explained here) is indeed a dynamical symmetry, just as DSE expected to show!

Similarly, it is easy to show that the transformation in question maps solutions to solutions (this is a different way of establishing the same thing, namely, that boosts are symmetries of equation 2). To see this, notice that the only reference the general solution (i.e.., $x_f - x_{eq} = A\,\cos(\omega t) + B\,\sin(\omega t)$) of equation 2 makes to either x_f or x_{eq} comes from the term $x_f - x_{eq}$, which is preserved by the transformation, as explained earlier. Hence, the transformation maps solutions into themselves: $(x_f - x_{eq}) \mapsto (x_f - x_{eq})$ or, using the variables of equation 1, $x_b \mapsto x_b$ (contrast this with the case of the spaceship at the beginning of the section, where a solution is mapped to a *different* solution, $v = c \mapsto v' = c \pm k$).

It is worth mentioning that the argument just given in no essential way depends on choosing the formulation given in equation 2 as opposed to the one given in equation 1. One only needs to be careful, remembering that if the boost acts on the whole spring, this amounts to $x_b \mapsto x_b$ in equation 1 and to $(x_f - x_{eq}) \mapsto (x_f - x_{eq})$ in the newer formulation. That is, once we recognize that the boost acts on the whole spring, both formulations of the law are left invariant under boosts. In short, it seems that DSE offers the right answer to our original question:

> DSE's answer: The spring (and so the block attached to it) behaves the same way before and after the boost because boosts (understood as applying to all points of the spring) are dynamical symmetries of Hooke's law, which is the law that characterizes the motion of the spring. And because the spring's behavior does not change under the boosts of the spaceship, the detector will not detect any differences in behavior.

4 Problem I: Proliferation

Although DSE's answer seems to match our expectations, a closer inspection reveals an interesting problem. If DSE is right, then, contrary to conventional wisdom, there is nothing special about boosts or spatial translations because *infinitely many*

[2] To be more precise, we are solving for x_f as a function of time, and we take x_{eq} as an external parameter that depends on the situation (if the spaceship is at rest, $x_{eq} = 0$; if the spaceship is moving at a constant velocity, $x_{eq} = vt$, and so on). So the general solution can be written as $x_f(t) = A\,\cos(\omega t) + B\,\sin(\omega t) + x_{eq}(t)$.

transformations in one dimension preserve the difference $x_f - x_{eq}$ and hence leave equation 2 invariant (this includes transformations that are not normally counted as dynamical symmetries of springs). Hence, according to DSE, infinitely many transformations would make the behavior of the block remain invariant. To see this, notice that $x \mapsto x + g(t)$, *if* applied to both x_f and x_{eq}, will leave equation 2 invariant, *no matter what alteration of one-dimensional motion g(t) is taken to represent*![3] For example, $g(t)$ can represent a constant acceleration (in which case we can write the transformation as $x \mapsto x + 0.5at^2$),[4] or it can also represent a variable acceleration, or it can correspond to a bizarre transformation such as this one: $x \mapsto x + \cos t + t^3$ (here, $g(t) = \cos t + t^3$). This last transformation has no natural physical interpretation (it looks like a weird combination of a harmonic motion combined with a jolt term). However, whatever motion $g(t)$ represents here (if any!), *if* the transformation were to act on all the points of the spring simultaneously, then it would leave equation 2 invariant.

The fact that so many transformations, from the very mundane ones like shifts and boosts all the way to rather bizarre ones like $x \mapsto x + \cos t + t^3 + t^6$, would, *if* implemented, leave Hooke's law invariant should puzzle defenders of DSE. At the very least, it seems puzzling that the reasoning in DSE's answer ends up generalizing way beyond the transformations that physicists usually take to be relevant when discussing Newtonian systems (e.g., see Weinberg 2021, ch. 4; Feynman et al. 1963, ch. 52). And it is worth pointing out that the proliferation of symmetries for Hooke's law affects not only springs but also any other systems that obey the equations of the harmonic oscillator, including, for instance, pendulums undergoing small oscillations, the vibrating particles of the medium in a sound wave, or even a ball rolling in a curved dish.

The proliferation of dynamical symmetries is also problematic for formal reasons. For example, it has been shown that ideal springs are characterized by a *finite* group of eight independent symmetry transformations (e.g., see Wulfman and Wybourne 1976), which would be in clear tension with the existence of infinite independent symmetries (and none of the symmetries discussed in the literature include transformations such as $x \mapsto x + \cos t + t^3 + t^6$). Related to this, as Belot (2013) points out, dynamical symmetries are hard to come by. The fact that we just found an infinite number of them ($g(t)$ can be *any* smooth function) with one simple line of elementary math should make us a bit suspicious of the approach!

[3] For $\left(x_f + g(t)\right) - \left(x_{eq} + g(t)\right) = x_f - x_{eq}$, and if $x_f - x_{eq}$ is preserved, then any derivatives of it are preserved too.

[4] One might suggest that constant accelerations of mechanical systems are indeed dynamical symmetries of Newtonian systems (including springs) by appealing to something like Corollary VI to the laws of Newton's *Principia* (e.g., see Saunders 2013). As provocative and interesting as this might sound, however, this kind of approach faces some problems. To name a rather obvious one, notice that equation $-G\frac{m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2) = \frac{d^2\mathbf{r}_1}{dt^2}$ (describing the motion of a body gravitating with respect to another one) *is not* invariant under constant accelerations, and neither is the equation of the spaceship at the beginning of section 2.2. More generally, the fact that, locally, a system looks the same in the presence of a constant acceleration does not imply that the differential equation characterizing such a system is invariant under constant accelerations (e.g., the differential equation might explicitly characterize the system with respect to an inertial frame, in which case the equation will fail to be invariant under constant accelerations). We will come back to this point in section 6.

5 Problem 2: Stretching the spring

Given the nature of the questions discussed here, it seems rather important to look at what physicists and mathematicians working on the dynamical symmetries of the classical harmonic oscillator have said. It turns out that if we do this, a different problem for DSE appears (a problem already identified by Belot (2013), although in a different context). If we read physics articles on the symmetries of the harmonic oscillator (e.g., see Wulfman and Wybourne 1976; Lutzky 1978), we find that there are plenty of dynamical symmetries that *do not preserve the physical behavior of springs* (it is also worth pointing out that in those same articles, boosts are *not* counted among the symmetries of Hooke's law). This highlights a second problem for DSE: to be a dynamical symmetry is not a sufficient condition for preserving the behavior of mechanical systems. Schematically, this suggests that we need to replace DSE with DSE + X, where X refers to some property that distinguishes those dynamical symmetries that preserve behaviors from those that do not.

Consider the following symmetry transformation of the harmonic oscillator equation:[5]

$$x_b \mapsto x_b + \cos(t). \tag{3}$$

Here, we assume that the equilibrium point is initially placed at the origin of the coordinate system, and x_b is understood as the position of the endpoint with respect to such origin (following Wulfman and Wybourne (1976), we also assume units such that $\omega = 1$). Here, the transformation is understood as *only* acting on the endpoint of the spring with respect to the origin (but see sec. 6 for alternative interpretations). Using the notation introduced earlier, this means that $x_f \mapsto x_f + \cos(t)$ and $x_{eq} \mapsto x_{eq}$ (or $x_b \mapsto x_b + \cos(t)$ using the original variables).[6]

For our purposes, the crucial point of this transformation is this: in contrast to the transformations we discussed in prior sections, this one does map a solution of the classic harmonic oscillator equation to a *different* solution. For instance, it takes $x_b = A\cos(t)$ into $x_b = (A-1)\cos t$, which is a solution with a *different* amplitude. Hence, this dynamical symmetry of the spring does not preserve the behavior of the spring (the spring is less stretched or more stretched after the transformation takes place). More generally, one can check that *any* given spring state (even a spring that is not oscillating) can be mapped to *any other* spring state via a symmetry of the form $x_b \mapsto x_b + \alpha\sin(t) + \epsilon\cos(t)$, where α and ϵ are real numbers (see footnote 6). This illustrates how some dynamical symmetries of the spring disrupt the behavior in a rather significant way.

This is a good moment to point out in passing that, contrary to what is often said in the philosophy literature, this is a dynamical symmetry that does not fix how "things

[5] A slightly different version of this symmetry transformation is discussed by Wulfman and Wybourne (1976, 516). The authors do not write the generator of this symmetry explicitly, but given their prior results, one can show that the generator is $S_B = \sin t \frac{\partial}{\partial x}$.

[6] To see that this transformation is a symmetry, note that

$$x_b + \cos(t) = -\frac{d^2}{dt^2}(x_b + \cos(t))$$
$$x_b + \cancel{\cos(t)} = -\frac{d^2}{dt^2}x_b + \cancel{\cos(t)}.$$

This is actually an instance of the much more general symmetry $x_b \mapsto x_b + \alpha\cos(t) + \epsilon\sin(t)$ (where α and ϵ are real numbers) that can map a spring state to any other state given a suitable choice of α and ε.

look" (e.g., see Dasgupta 2016), and it does not seem to map a solution to another one with the same representational capacities (see, e.g., Fletcher (2020) for a recent defense of a nuanced version of this thesis). The amplitude, the period, and the elastic energy of the spring (all things that we can measure inside the spaceship) are all changed by this symmetry transformation. Hence, equation 3 provides a surprisingly simple but effective illustration of a point already suggested by Belot (2013) (see also Belot 2018 for a similar point, and Luc 2022 and Wallace 2022 for recent responses), namely, that many of the allegedly important features about symmetries so often discussed in philosophy fail to hold for some of the symmetries that physicists discuss in their research (and this is true of other cases as well, including physicists' discussions of the symmetries of the Kepler problem; e.g., see Prince and Eliezer 1981).[7]

Going back to the discussion of DSE, the fact that $x \mapsto x + \cos(t)$ is a symmetry of the spring shows that defenders of DSE need to say more when presenting their accounts. In particular, at the very least, they need to exclude those dynamical symmetries that change the behavior of the spring with respect to equilibrium. Hence, the defenders of DSE need to add the requirement that the symmetries must preserve the values of A and B (changes in these variables would represent changes in the initial amplitude or the initial speed, respectively). This requirement amounts to demanding that the symmetry acts like the identity in the space of solutions of the harmonic oscillator—that is, it maps every solution of the harmonic oscillator to itself: $x_b \mapsto x_b$ (or $x_f - x_{eq} \mapsto x_f - x_{eq}$). Let's follow mathematical jargon and call "trivial" those symmetry transformations that act like the identity. DSE's answer to the main question might then be modified as follows:

> DSE*'s answer: The spring (and so the block attached to it) behaves the same way before and after the boost because boosts, when understood as applying to all points of the spring, are *trivial* dynamical symmetries of Hooke's law.

Schematically, X in DSE + X corresponds to something like "the transformation is trivial." Notice that $x \mapsto x + \cos(t)$ is a nontrivial symmetry, so it does not pose a problem for DSE*. Also, notice that boosts acting on the whole spring are trivial, but boosts acting on the endpoint are not (in that second case, they are not even symmetries).

How compelling is DSE*? It is better than DSE in that it avoids, by fiat, the case of dynamical symmetries that do not preserve the behavior of the spring. But unfortunately for the defenders of DSE*, it still suffers from the other problems affecting DSE. For instance, it is still vulnerable to the proliferation of symmetries because the bizarre transformations that we considered then are also trivial (all these transformations act trivially in the solution space if they are applied to both x_f and x_{eq}).

In addition to not solving the proliferation issue, DSE* seems to be *ad hoc*; the justification for adding the *triviality* condition seems to be that not adding it leads to counterexamples. Third, DSE* seems so restrictive that it leaves out other cases that

[7] For example, symmetries like the one discussed in this section suggest that Dasgupta's definition of symmetries is not extensionally adequate because it leaves out symmetries that relate empirically inequivalent states.

one might have wanted to explain by appealing to symmetries. For instance, go back to the case of the spaceship governed by $0 = m\frac{d}{dt}v$. If someone asks why it is that the spaceship still behaves like an object in free fall even after being boosted, one would imagine that part of the answer (at least for a sympathizer of DSE) is that $0 = m\frac{d}{dt}v$ is invariant under boosts. And yet, boosts map solutions of that equation to *different solutions* (boosts do not act trivially over the space of solutions). And fourth, there just seems to be something highly suspicious in trying to explain a deep fact about mechanical systems of our world (i.e., the fact that their behavior is preserved under boosts) in terms of mathematically trivial transformations such as $x_b \mapsto x_b$. The fact that mapping x_b to itself does not change equations that contain x_b seems too trivial of a fact to offer a substantive resolution to the main question.

6 Problem 3: Active and passive

6.1 Where did the inertial effects go?

In this section, we will consider another problem that a defender of DSE (or DSE*) faces. We will see how the resolution to this problem provides some insights into both the issue of proliferation of symmetries and the distinction between trivial and nontrivial symmetries introduced in the previous section.

It is a basic fact of Newtonian mechanics that if we are using an equation to describe a system from the perspective of an inertial frame, and we "switch" to an accelerating frame, then we need to use a modified equation to take into account the emergence of some inertial or "fictitious" forces. These fictitious forces appearing in the modified equation for the system signal that we are no longer describing the target system from the perspective of an inertial frame. To illustrate this, consider again the case of a spaceship in outer space whose motion is described by $0 = mdv/dt$. Take a case where we transform the frame via $v \mapsto v + at$ (this is not a symmetry of this differential equation). Notice that if we do that, we can no longer use $0 = mdv/dt$ in order to describe the spaceship's motion because, from the perspective of the new frame, it looks as if the spaceship had started accelerating. Instead, we need to consider a fictitious force F_f to account for the apparent acceleration of the object, and so the equation for the spaceship in this new frame will be $F_f = mdv/dt$ (Newton's second law, now with a nonzero fictitious force).

Problems seem to arise, however, when we go back to the block attached to the spring. Take, once again, Hooke's law as written in equation 2. And now take the transformation $x \mapsto x + 0.5at^2$ interpreted *passively*, indicating a shift from an inertial frame to a non-inertial frame that moves with constant acceleration. As with the case of the spaceship, we expect that going to such a frame brings about fictional forces in the (new) equation describing the system. And yet, none of that seems to happen because, as we already showed, $x_f - x_{\mathrm{eq}} = -\omega^2 \frac{d^2}{dt^2}\left(x_f - x_{\mathrm{eq}}\right)$ (or $x_b = -\omega^2 \frac{d^2}{dt^2} x_b$ under the understanding that $x_b = x_f - x_{\mathrm{eq}}$) is invariant under a transformation that affects both x_f and x_{eq} in the same manner, and in particular, it is invariant under $x \mapsto x + 0.5at^2$ if applied to both x_f and x_{eq}. Mathematically speaking, this is not surprising: if $x \mapsto x + 0.5at^2$ is a symmetry of Hooke's law (assuming we act on all the points of the spring in the same manner), then this remains true regardless of whether we interpret it actively or passively. But physically speaking, this is concerning; Hooke's law seems to be completely blind to the non-inertial effects that should arise when the observer jumps into an accelerating frame.

6.2 Recovering the inertial effects

It turns out that the reason inertial effects seem to be missing is connected to an interesting ambiguity in how we characterize the displacement of a spring. In particular, let's go back to the standard formulation of Hooke's law, $x_b = -\omega^2 \frac{d^2}{dt^2} x_b$ (the subscript b in x_b refers to the block). There are at least two ways of interpreting this equation when modeling a spring. First, this equation can be taken "internally," as representing the displacement of the endpoint with respect to the equilibrium point of the spring (I call it "internal" because it focuses on the displacement between two points in the spring). In this interpretation, it is natural to define x_b via $x_b = x_f - x_{\text{eq}}$ because the latter is the displacement between the endpoint and equilibrium, and so it is natural to use the equation $x_f - x_{\text{eq}} = -\omega^2 \frac{d^2}{dt^2}\left(x_f - x_{\text{eq}}\right)$. Second, the equation can be interpreted "externally," as representing the displacement of the endpoint with respect to the origin of an external reference frame (henceforth "external interpretation"). Notice that in this second interpretation, it is misleading to define x_b in terms of $x_f - x_{\text{eq}}$ because in this interpretation, x_b in equation $x_b = -\omega^2 \frac{d^2}{dt^2} x_b$ is *not* modeling the displacement from the endpoint to equilibrium. Rather, in this interpretation, x_b models the same thing x_f does, namely, the displacement of the endpoint with respect to the origin of the frame. Hence, instead of $x_b = x_f - x_{\text{eq}}$, we should write $x_b = x_f$. That is, in this interpretation, the differential equation for the spring can be written as $x_f = -\omega^2 \frac{d^2}{dt^2} x_f$, where x_f stands for the endpoint's position and where the equilibrium point is not modeled (in the external interpretation, the quantity $x_f - x_{\text{eq}}$ still represents the internal displacement of the spring; it is just that this quantity does not appear in the differential equation). The difference between these two interpretations of equation $x_b = -\omega^2 \frac{d^2}{dt^2} x_b$ is reflected in the number of dependent variables of the differential equation; in the internal case, this is a differential equation consisting of two dependent variables, one representing the endpoint and the other one representing the equilibrium point (this becomes apparent once we insert $x_b = x_f - x_{\text{eq}}$ in the equation). In the external case, $x_b = -\omega^2 \frac{d^2}{dt^2} x_b$ is a differential equation consisting of one dependent variable, the one representing the endpoint. Note, by the way, that in the internal case, there are some "external elements" because both x_f and x_{eq} represent the positions of parts of the system with respect to the origin of an external frame. But only in the internal case does $x_b = -\omega^2 \frac{d^2}{dt^2} x_b$ model the behavior of the "internal quantity" $x_b = x_f - x_{\text{eq}}$ (in the external case, that equation models the behavior of the endpoint alone).

Usually, from a practical point of view, this ambiguity in how to read $x_b = -\omega^2 \frac{d^2}{dt^2} x_b$ has little to no importance; in both cases, the displacement is exactly the same because the standard convention for the frame places the equilibrium position at the origin (i.e., $x_b = x_f - x_{\text{eq}}$ and $x_b = x_f$ coincide when $x_{\text{eq}} = 0$). But the external and internal interpretations entail different results under certain specific circumstances: a change in the frame (a shift in the origin, or in the velocity of the frame, or in the acceleration) will bring no changes to the equation of a spring according to the internal interpretation because changes in the frame will preserve relational quantities such as

$x_f - x_{eq}$. This is precisely why $x_f - x_{eq} = -\omega^2 \frac{d^2}{dt^2}(x_f - x_{eq})$ is invariant under accelerations of the frame such as $x \mapsto x + 0.5at^2$, and so this is why no inertial effects appear in such a case. Things are very different, however, if we adopt the external interpretation.

Passive-external acceleration: Consider the constant-acceleration transformation given by $x \mapsto x + 0.5at^2$, and let's interpret it passively. From the perspective of the new accelerating frame, the endpoint of the spring no longer behaves like a spring because it acquires some constant acceleration with respect to the frame. Mathematically, this is reflected in the fact that $x_b = -\omega^2 \frac{d^2}{dt^2} x_b$ (interpreted externally) is not invariant under $x_b \mapsto x_b + 0.5at^2$. Hence, the transformation is not a dynamical symmetry of Hooke's law so interpreted. Instead of Hooke's law, we must now describe the endpoint by using an equation such as $-kx_b + F_f = m \frac{d^2}{dt^2} x_b$, where F_f is a fictitious force (coming from the choice of a non-inertial frame). Thus, the external interpretation recovers the inertial effects coming from an accelerating frame. Notice that an acceleration of the frame will not affect the internal displacement of the spring, which is given by $x_f - x_{eq}$ (recall that x_{eq} does not appear in the external interpretation of $x_b = -\omega^2 \frac{d^2}{dt^2} x_b$). In general, a change in the frame modifies the coordinates of both the equilibrium point and the endpoint by the very same factor so that the relative displacement of those two points is preserved (clearly, a change in the frame will not make the spring get more stretched or less stretched). This is important because it shows a case in which a transformation fails to be a dynamical symmetry and still preserves the "internal behavior" of a system.

Passive-external oscillation: To give another example, consider the transformation $x \mapsto x + \cos(t)$, which we showed to be a dynamical symmetry of $x_b = -\frac{d^2}{dt^2} x_b$ in section 5 (remember that we are using units so that $\omega = 1$). If we adopt the external interpretation, then this means that the relative displacement of the endpoint $x_b = x_f$ with respect to an external frame (originally at rest) now exhibits some sort of oscillatory behavior. Read passively, this means that the frame used to describe the endpoint is starting to oscillate, so it is no longer an inertial frame. Hence, we should expect some inertial effects! However, as we saw in section 5, the transformation $x_b \mapsto x_b + \cos(t)$ is a symmetry of $x_b = -\frac{d^2}{dt^2} x_b$. Back then, this transformation was understood actively (as deforming the spring), but if T is a symmetry of an equation when interpreted actively, it remains one when interpreted passively (either an equation remains invariant under T, or it does not). Hence, $x_b = -\frac{d^2}{dt^2} x_b$ remains invariant if the frame starts oscillating according to $x \mapsto x + \cos(t)$. This seems puzzling; we are jumping to a non-inertial frame, and yet the equation describing how the endpoint behaves with respect to the new oscillatory frame does not change at all. So where are the inertial effects? It turns out that the fictional force that the non-inertial frame "produces" is an oscillatory force that can be modeled as the force of an ideal spring. This means that the original spring still looks like a spring from the perspective of the new oscillatory frame in the sense that it still obeys $x_b = -\frac{d^2}{dt^2} x_b$. However, as seen from the oscillatory frame, the total force acting on the endpoint is different from the original force: if $F = kx_b$ was the original force acting on the block in the stationary frame, then $F' = k'x_b$ ($k' \neq k$) is the new (partially fictional) force "acting" on the block according to the oscillatory frame. Given that the magnitude of the force is different, the amplitude and acceleration as measured with respect to the new frame will be different too (this is why the transformation is a nontrivial symmetry, because it takes us to a different solution). Those differences in force and acceleration are precisely the inertial effects that we were looking for. Also, notice that the oscillatory frame preserves the internal displacement, given by $x_f - x_{\mathrm{eq}}$ (again, these two variables are transformed equally). So *in the external interpretation, both constant accelerations of the frame and oscillatory accelerations give rise to inertial effects (as expected), and both preserve the internal behavior of the spring. But only oscillatory accelerations preserve the form of the equations, and hence only these count as a dynamical symmetry of the system.*

Passive-external boosts: A similar analysis can be done for the case of boosts. We know that in the internal interpretation, boosts are symmetries (see section 4). But in the external interpretation, a boost (either active or passive) is not a symmetry of the equation because it distorts how the oscillation of the endpoint looks with respect to the origin of the frame.[8] Importantly, no inertial effects appear here; from the perspective of a boosted frame, the acceleration of the endpoint is just the same as it was before. Also, a boost of the frame will not change how the endpoint moves with respect to the equilibrium point (i.e., $x_f - x_{\mathrm{eq}}$ is invariant). This is important because it shows once again (as constant accelerations did) that according to the external interpretation, *it is not necessary for a transformation to be a symmetry in order for it to preserve the (internal) behavior of the spring.*

Active-external boosts: Just as there are different ways of passively reading a given transformation of the harmonic oscillator, there are also different ways of reading a transformation actively. For example, according to the external interpretation, we can take a boost as acting only on the endpoint $x_b = x_f$ of the spring ("active-external-endpoint interpretation") or as acting on the whole spring ("active-external-whole interpretation") so that it affects both $x_b = x_f$ and x_{eq}. In both cases, the boost is not a dynamical symmetry from the external perspective because the endpoint $x_b = x_f$ is not oscillating around a fixed point (mathematically, $x_b = -\frac{d^2}{dt^2} x_b$ is not invariant under $x_b \mapsto x_b + vt$). However, in the case of the active-external-endpoint interpretation, the boost does not preserve the internal behavior of the spring (because x_{eq} is not boosted, $x_f - x_{\mathrm{eq}}$ changes into $x_f + vt - x_{\mathrm{eq}}$). In contrast, in the active-external-whole interpretation, the boost does preserve the internal behavior because the relative motion of the endpoint with respect to equilibrium is preserved in this case ($(x_f + vt) - (x_{\mathrm{eq}} + vt) = x_f - x_{\mathrm{eq}}$). This shows, once again, that it is not necessary for a boost to be a symmetry of the equations characterizing a system for it to preserve the (internal) behavior of the system.

Active-external oscillations: Similarly, we can read $x \mapsto x + \cos(t)$ according to the active-external-endpoint interpretation (as we actually did in section 5, where the endpoint alone is made to oscillate in a new manner) or according to the active-external-whole one (where all the points of the spring are made to oscillate with respect to an external frame). In both cases, the transformation is a nontrivial dynamical symmetry of $x_b = -\frac{d^2}{dt^2} x_b$ (read externally), but only in the

[8] Mathematically, this is shown by the fact that $x_b = -\frac{d^2}{dt^2} x_b$ is not invariant under $x_b \mapsto x_b + vt$ (recall that $x_b = x_f$ in this case).

active-external-endpoint case does it change the internal displacement of the spring.[9] Physically, these transformations represent very different states of affairs, even if both are dynamical symmetries.

Active-internal transformations: A similar analysis shows that for a given transformation of the harmonic oscillator equation, we can have an active-internal-whole interpretation or an active-internal-endpoint interpretation. Indeed, the proliferation of symmetries discussed in section 4 is a direct consequence of the adoption of an active-internal-whole interpretation; the displacement between the endpoint and the equilibrium point is not altered if one acts on the endpoint and equilibrium in the same way. This is why boosts, oscillations, constant accelerations, and general one-dimensional transformations are trivial symmetries of $x_b = -\frac{d^2}{dt^2}x_b$ in the active-internal-whole case.

6.3 Is one interpretation more natural in the current context?

Given the fact that, as argued in this section, one can understand a differential equation such as $x_b = -\frac{d^2}{dt^2}x_b$ both internally and externally, it is natural to ask if, in the present context, one of the two interpretations is more natural. The only interpretation that allows a defender of DSE to say that boosts are dynamical symmetries of a spring (as they want to say in order to answer the main question) is an internal-whole interpretation, that is, an interpretation for which the boost acts on the whole spring *and* x_b models the displacement with respect to equilibrium (not with respect to an external coordinate system). Are there any independent reasons to believe that the internal-whole interpretation is a natural or well-motivated interpretation?

I think that there are at least three reasons why the internal-whole interpretation is not a natural one in the context of physical symmetries. First, physicists seem to treat $x_b = -\frac{d^2}{dt^2}x_b$ externally. As evidence for this claim, one can point out at least four (related) things: (i) physicists usually take x_b to represent the position with respect to the origin of an external frame (but they choose the frame so that the equilibrium point is at the origin); (ii) they treat $x_b = -\frac{d^2}{dt^2}x_b$ as having only one dependent variable and the solutions as requiring two initial conditions (the initial speed and initial position); (iii) when physicists investigate the symmetries of $x_b = -\frac{d^2}{dt^2}x_b$, they calculate how x_b *changes* after applying the various symmetry transformations on the spring (in the internal-whole interpretation, x_b does not change); (iv) physicists count $x_b \mapsto x_b \cos t$, but not boosts, among the symmetries of $x_b = -\frac{d^2}{dt^2}x_b$ (see both Lutzky (1978) and Wulfman and Wybourne (1976) for examples of the last two points). Only the external interpretation is consistent with these four points.

Second, an internal-whole interpretation renders trivial paradigmatic cases of physical symmetries such as the ones exhibited in Galileo's ship thought experiment. In these cases, we assume that the relevant equations give us the trajectory of the system with respect to an external frame (e.g., the shore) *and* that the transformation changes those trajectories in certain ways. What is interesting about these cases is that despite the fact that such trajectories do change, they do so in a way that they

[9] In both cases, the new solution is related to the original through $x_b \mapsto x_b + \cos t$. However, in the active-external-whole case, the equilibrium point is also transformed via $x_{eq} \mapsto x_{eq} + \cos t$, and so the internal behavior $x_b - x_{eq}$ is preserved.

still satisfy the same differential equations.[10] For example, take a rock falling in the ship's cabin at rest, which we can model with $\mathbf{g} = \frac{d^2}{dt^2}\mathbf{r}$ (the only acceleration is the gravitational one). If we boost the ship in any direction, this equation remains invariant, and so we infer that the rock still behaves as an object in free fall even though the new trajectory is different (i.e., even if $\mathbf{r}(t) \mapsto \mathbf{r}(t) + \mathbf{v}t$). If one were to understand the trajectories $\mathbf{r}(t)$ for the rock "internally," for example, as relating the rock's motion to the ship's floor (with the understanding that both the floor and the rock are transformed jointly), then it would no longer be interesting to note that $\mathbf{g} = \frac{d^2}{dt^2}\mathbf{r}$ is preserved under boosts (in such a case, $\mathbf{r} \mapsto \mathbf{r}$ simply because the floor and the rock are boosted together). But this last scenario is precisely an instance of the internal-whole interpretation.

Third, one of the most physically significant features of symmetries is that they are tightly connected to the conserved quantities of a system (via Noether's first theorem). But if one adopts the internal-whole interpretation, then there are no conserved quantities (they are all zero) associated with the various symmetries simply because in this interpretation, the apparently different transformations are all just different ways of writing the identity transformation in the solution space (e.g., according to this interpretation, $x \mapsto x + \cos t$ amounts to the identity $x_b \mapsto x_b$ in the space of solutions of the equation). Adopting an interpretation for which there are no interesting conserved charges is problematic because conserved or invariant quantities are usually thought to be paradigmatic examples of quantities that are observable or measurable. The physically interesting link between conserved quantities, measurements, and dynamical symmetries seems to collapse under an internal-whole interpretation. In contrast, the external interpretation does treat the symmetries (e.g., $x \mapsto x + \cos t$) as giving rise to nonzero conserved quantities (see Lutzky (1978, 274) for a discussion).

In sum, the defender of DSE (or DSE*) faces a dilemma. On the one hand, if they want to answer the main question by showing that boosts are symmetries of the spring, then they must adopt the internal-whole interpretation of boosts. On the other hand, this interpretation is problematic for various reasons discussed so far: it is not the one adopted in physics and mathematical circles, it leads to a proliferation of symmetries, it does not recover inertial effects for non-inertial frames, it is not connected to (nonzero) conserved quantities, and it seems *ad hoc*.

7 A relational answer

The previous considerations strongly suggest that answering the main question by adopting an internal-whole interpretation is problematic on various accounts. However, if one does not adopt that kind of interpretation, then one is forced to the conclusion that boosts are not symmetries of springs. If boosts are not symmetries, what, then, explains the fact that springs in trains behave in the same way regardless of the velocity of the train? If boosts are not symmetries, how do we answer the main question?

A clue toward solving these questions comes from the observation, noted in section 6, that a transformation might fail to be a dynamical symmetry and still

[10] As I have argued here, this is not true of all mechanical systems (e.g., springs).

preserve the internal behavior of a system. In particular, recall that according to the external-whole interpretation, boosts are not dynamical symmetries of the harmonic oscillator, and yet they still preserve how the endpoint oscillates with respect to the equilibrium point (they still preserve $x_f - x_{eq}$). If boosts are not symmetries in this interpretation, why do they preserve how the endpoint oscillates with respect to the equilibrium point? I think that there is a simple answer already hinted at in the last sections: boosts of the whole spring preserve the relevant relational quantities of the system, and that is precisely why they preserve how it looks from inside the spaceship. In particular, consider the following answer to the main question:

> Relational answer (RA): What is essential in the explanation for why the block (attached to the spring) behaves the same way under boosts of the spaceship is that the boost is taken to "act" on the whole spring (and the whole spaceship) simultaneously and in the same way. *Any* transformation that acts on the spring in such a manner will automatically preserve the spring's internal behavior because it will preserve the relevant relational properties (e.g., it preserves the spring's amplitude and the spring's velocity with respect to equilibrium).

To better see the motivation behind RA, note that if the boost were to act only on a proper part of the spring, such as its endpoint, then it would not leave the internal behavior of the spring invariant, as we discussed in prior sections (and this is not because a transformation acting on the endpoint can't be a symmetry of the spring, for $x \mapsto x + \cos t$ is a symmetry even when acting only on the endpoint). Hence, the crucial point in the explanation is not that the transformation itself (the boost) is a dynamical symmetry (it is not in the external interpretation) but simply the fact that the transformation is assumed to act on all the parts of the system simultaneously and in the same way.[11]

To put the point more explicitly, consider the following attempt to justify RA: Boosting the frame preserves the relative positions between the parts of the spring (it shifts all the coordinates equally). But boosting the frame is equivalent to boosting the *whole* spring in the opposite direction. Therefore, boosting the whole spring preserves the relative positions between the parts of the spring. Because the internal behavior of the spring only depends on these relative positions, the boost will preserve such internal behavior. Note that this reasoning also works for *any* transformation of the frame in one dimension, not just boosts. This is why, from the perspective of RA, there is nothing particularly problematic with a proliferation of transformations (symmetries or not) that preserve internal behaviors. For example, according to RA, there is nothing surprising in realizing that a frame transformation given by $x \mapsto x + \cos t + t^3 + t^6 + exp(t)$ preserves internal behaviors because it will preserve the relative coordinate positions of all the parts of the system as a function of time. And because this is not surprising, it is also not surprising that the active version preserves such behaviors as well.

To get a clearer sense of RA and how it compares to DSE, consider the following scenario. Say that we want to understand the relative motion of two subsystems, S_1

[11] In future work, it would be worth looking into how the arguments in this article connect to the ones developed by Saunders (2013).

and S_2 (they could be different parts of the same system, but they do not have to be). Say that we use differential equation DEQ1 to represent the motion of S_1 with respect to an external frame R, and we use differential equation DEQ2 to represent the motion of S_2 with respect to that same frame (these equations represent the relevant dynamical laws). From these differential equations (and the initial conditions), we can obtain a trajectory $x_1(t)$ for the motion of S_1 and a trajectory $x_2(t)$ for the motion of S_2 (with respect to the same frame). The relative motion of S_1 with respect to S_2 will be given by $x_r(t) = x_1(t) - x_2(t)$. Now consider a generic one-dimensional transformation $x \mapsto x + g(t)$ that acts on all the objects in the same way; $x_1 \mapsto x_1 + g(t)$ and $x_2 \mapsto x_2 + g(t)$. Clearly, this transformation will preserve $x_r(t) = x_1(t) - x_2(t)$ regardless of what $g(t)$ is simply because it affects S_1 and S_2 in the exact same manner (or in the passive case, it shifts all the coordinates equally). So if we ask why it is that $g(t)$ does not affect how S_1 behaves with respect to S_2 (this is analogous to asking why boosts preserve how the spring behaves with respect to the detector), RA would simply note that such a transformation preserves $x_r(t) = x_1(t) - x_2(t)$ when acting on the two systems. Crucially, and unlike DSE, RA does not require that the transformation is a symmetry of DEQ1 or DEQ2 (notice that nothing in the explanation just given hinges on $g(t)$ being a symmetry of these equations).

To illustrate, take S_1 to be the endpoint of the spring and S_2 to be the equilibrium point. And say that the spaceship is initially at rest with respect to an external frame R whose origin initially coincides with the equilibrium point. For the behavior of the endpoint with respect to the frame, we use $x_f = -\frac{d^2}{dt^2} x_f$ (with units so that $\omega = 1$). Because the equilibrium point has a constant velocity (zero, in this frame), we can model it using equation $\frac{d^2}{dt^2} x_{\text{eq}} = 0$. The solution of the first equation is $x_f(t) = A \cos t + B \sin t$ (for some specific A and B, depending on the initial conditions), and that of the second one is $x_{\text{eq}}(t) = vt + d = 0$ ($v = d = 0$ in this frame). The displacement of the endpoint with respect to equilibrium is given by the relational variable $x_r(t) = x_f(t) - x_{\text{eq}}(t)$. In this case, $x_r(t) = A \cos t + B \sin t$ (because $x_{\text{eq}} = 0$). So far, this is just the familiar displacement of a spring found in textbooks. But now consider a boost that acts on both subsystems according to $x_f \mapsto x_f + vt$ and $x_{\text{eq}} \mapsto x_{\text{eq}} + vt$. Notice that the boost will be a symmetry of $\frac{d^2}{dt^2} x_{\text{eq}} = 0$ but *not* a symmetry of $x_f = -\frac{d^2}{dt^2} x_f$. The boost will take $x_f(t) = A \cos t + B \sin t$ into $x_f^*(t) = A \cos t + B \sin t - vt$ (which is not the solution of a harmonic oscillator) and $x_{\text{eq}}(t) = 0$ into $x_{\text{eq}}^*(t) = -vt$ (which is a solution for $\frac{d^2}{dt^2} x_{\text{eq}} = 0$). However, $x_r(t) = x_f(t) - x_{\text{eq}}(t)$ will stay just the same because both $x_f(t)$ and $x_{\text{eq}}(t)$ are shifted by vt; $x_r^*(t) = x_f(t)^* - x_{\text{eq}}(t)^* = \left(x_f(t) + vt\right) - \left(x_{\text{eq}}(t) + vt\right) = x_r(t)$. According to RA, this last fact is what explains why the behavior of the spring with respect to the detector remains the same under boosts. As another example, take a ball in free fall inside the cabin of a ship moving uniformly in the sea. As seen from the shore, we can represent the ball's behavior with $\mathbf{g} = \frac{d^2}{dt^2} \mathbf{r_b}$, and the floor's motion can be represented with $\frac{d^2}{dt^2} \mathbf{r_f} = \mathbf{0}$ (because the ship is moving with constant velocity). Then, consider a constant acceleration of both objects, $\mathbf{r_b} \mapsto \mathbf{r_b} + 0.5\mathbf{a}t^2$ and $\mathbf{r_f} \mapsto \mathbf{r_f} + 0.5\mathbf{a}t^2$. Such acceleration is *not* a symmetry of either equation, and yet we can easily see that it would preserve the relative motion of the ball with respect to the floor (i.e., $\mathbf{r_b} - \mathbf{r_f}$ remains invariant). Once again, we have a case where a transformation preserves relational information even if it does not leave the dynamical laws invariant.

Having said this, it is worth pointing out that this kind of framework still allows symmetries to convey interesting physical information about the system. In particular, in this kind of framework, dynamical symmetries give us meaningful information about the type of external behavior of a system (how it behaves with respect to a frame), but not interesting information about the internal behavior (the latter is wholly a matter of the relational degrees of freedom, which can be preserved with or without symmetries). For example, the fact that the transformation $x \mapsto x + \cos t$ is a symmetry of $x_f = -\frac{d^2}{dt^2} x_f$ (read externally) entails that if we had used an oscillatory frame in order to describe the endpoint x_f, we still would have seen that x_f satisfies Hooke's law, even though the acceleration or amplitude would have looked very different. Hence, the *type* of external motion would have been preserved, but the particular external motion would not have. In contrast, the fact that $x \mapsto x + t^2$ is not a symmetry of the same equation entails that if we had moved from an inertial frame to a frame in constant acceleration, we would have seen the endpoint of the spring no longer obeying Hooke's law (with respect to the frame). Hence, the type of external motion would have been different. In short, only those transformations that are symmetries preserve the type of external behavior of a system, and therein lies an important part of their physical significance.[12]

How do we answer the main question, then? Although I believe RA is significantly better than DSE, it is only the beginning of a better answer. There are other relevant physical facts that are not considered by RA. For example, we should mention the rigidity of the bodies constituting mechanical systems, without which we could not actually boost a real system without deforming it; we should mention the homogeneity and isotropy of space, without which the direction and location of the boost might affect the behavior of the system (e.g., if we lived in an Aristotelian universe, translations of the spring to the center of the universe would greatly disrupt its behavior); we should mention the absence of an ethereal-like substance, whose presence might interfere with the motion of material bodies (including springs); and we should mention the fact (not completely independent from the previous ones) that the laws of mechanical systems do not make explicit reference to absolute positions and absolute velocities. Thus, even though RA is right in noting that preserving the relational quantities is necessary when answering the main question, a more complete answer worth investigating in future work would have to consider some of these other facts (it would also be worth investigating the relationship between the arguments developed in this paper and the Principle of Galilean Relativity).

8 Conclusion

There is a widespread belief in the philosophy literature that there is a strong link between symmetries and observations (e.g., see Ismael and van Fraassen 2003; Roberts 2008; Healey 2009; Baker 2010; Dewar 2015; Dasgupta 2016). The current article uses the case of a simple spring to highlight that the alleged link between symmetries and observations is much weaker than has been recognized.[13] In particular, I have shown

[12] At the very least, this is the case for those transformations of a system that can be defined with respect to another system that can work as a frame (such as spacetime transformations).

[13] See Read and Møller-Nielsen (2020) for detailed criticism of approaches that define symmetries purely epistemically (their criticism is different from, but complementary to, the one developed here).

that even if a transformation is not a symmetry of a system, it can still preserve how the system looks, provided that the observer (detector) is transformed in the same way, and I have shown that a transformation can be a symmetry and, at the same time, fail to preserve how the system looks (as when the endpoint of the spring is transformed according to $x \mapsto x + \cos t$). In short, whether or not a transformation is a symmetry of a system and whether or not it preserves how the system looks from the perspective of an observer (detector) that is also transformed are two rather independent matters. Thus, contrary to what has been suggested rather often, dynamical symmetries should not be understood as telling us how things look when both the system and the detector are transformed together (as happens inside the cabin of Galileo's ship or in the Leibniz–Clarke correspondence, where all objects are transformed together). Rather, they should be understood in an external fashion, as telling us how things look when only the system or only the detector is transformed (as when we look at the behavior of a boosted system in a ship from a fixed shore). For example, the fact that $x \mapsto x + \cos t$ is a symmetry of springs but $x \mapsto x + vt$ is not tells us that springs still look like springs when described from the point of view of an external oscillatory frame, but not when described from a frame in uniform motion.

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