Transposed algebras

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§ 1. *Introduction.*

A linear algebra[1] of order n, in general non-commutative and non-associative, may be regarded as being determined by the "cubic matrix" consisting of its n^3 constants of multiplication, and conversely. This requires that the n basis elements (units) of the algebra should be specified, and should be given in a definite order. Then the various "transpositions" of the cubic matrix induce corresponding "transpositions" of the algebra, for which a notation is given in § 2.

In § 3 it is shown that various equivalence relationships between linear algebras, definable as relations between their cubic matrices, are preserved under transposition. These include isotopy and orthogonal equivalence, but not linear equivalence in general. Such a relation may be denoted $=$. In § 4 the operations of direct addition and direct multiplication of algebras (denoted $+$ and $\times$) are also shown to be invariant under transposition, if attention is paid to the order of basis elements in a direct sum or product. Hence (§ 5), when $+$, $\times$ and a suitably defined $=$ relation are used, any given algebra, or any set of algebras over the same field, generates a system called a *direct algebra* which is unchanged in form by transposition of the generating algebras.

§ § 6-9 contain illustrations of these ideas, in which some well known linear associative algebras are transposed. The resulting algebras are non-associative, but have the property of commuting indices

$$x^{ab} = x^{ba},$$

x being any element of the transposed algebra, a and b being the

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[1] For the definition of this and other terms used, L. E. Dickson, "Linear algebras," Cambridge Tract No. 16, Cambridge, 1930, may be consulted; or the same author's "Algebren und ihre Zahlentheorie," Zürich, 1927.

indices of arbitrary powers. (See[1] N.C. or N.A.M.I.C. for the conventions used to indicate powers in a non-associative algebra). Algebras with this property will be called *palintropic*. The property is invariant under linear transformations of an algebra, but not generally under transposition; and if two or more algebras possess it, it is easily shown that their direct sum also possesses it, but not necessarily their direct product.

A type of non-associative algebra for which this property was previously proved is examined in § 10. Its transposes are also palintropic, and the corresponding direct algebra has a particularly simple form.

Unlike the examples of palintropic algebras given in C.T.A. and N.A.M.I.C., most of those occurring in the present paper are non-commutative as regards multiplication. Thus in the arithmetic of the indices—or as I have called it, the logarithmetic of these algebras—

$$a+b \neq b+a, \qquad ab = ba, \qquad a(b+c) = ab+ac,$$
$$(a+b)+c \neq a+(b+c), \quad ab.c = a.bc, \quad (b+c)a = ba+ca.$$

This fills an obvious gap in the list of logarithmetics given in N.A.M.I.C.

Some connections with the theory of quasi-groups are given in § 11.

§ 2. *Transposition*.

Let X be a linear algebra with basis $(a_1, \ldots . a_n)$ over a given field F. The commutative and associative laws of multiplication are not assumed. Then X is determined when its multiplication table[2]

$$a_i\, a_j = \Sigma \gamma_{ijk}\, a_k$$

is given, and the constants of multiplication γ_{ijk} can be any fixed elements of F. The algebra is characterised by the cubic matrix $\gamma = (\gamma_{ijk})$, and we shall consider two algebras X, Y identical $(X \equiv Y)$ if they have the same γ and are over the same field, though the symbols used for their bases may be different.

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[1] I refer by initials to my earlier papers: N.C. = "On non-associative combinations," *Proc. Roy. Soc. Edinburgh*, **59** (1939), 153-162; C.T.A. = "Commutative train algebras of ranks 2 and 3," *Journ. London Math. Soc.*, **15** (1940), 136-149; N.A.M.I.C. = "Some non-associative algebras in which the multiplication of indices is commutative," *ibid.*, **16** (1941), 48-55.

[2] The suffixes run from 1 to n. It is to be understood throughout that Greek letters with suffixes denote elements of the field F, and that Σ indicates summation with respect to repeated suffixes (here with respect to k).

If γ is transposed we obtain another linear algebra X^*, which will be called a *transpose* of X. Its multiplication table is

$$a_i\, a_j = \Sigma \gamma^*_{ijk}\, a_k$$

where (including X itself among its transposes) γ^*_{ijk} is equal to

$$\gamma_{ijk}, \quad \gamma_{ikj}, \quad \gamma_{kji}, \quad \gamma_{jik}, \quad \gamma_{jki}, \text{ or } \gamma_{kij}.$$

The corresponding algebras will be denoted

$$X, \quad X', \quad X^{\backprime}, \quad X^{-}, \quad X^{\vee}, \quad X^{\wedge}$$

and called the identical, right, left, straight, left-right and right-left transposes of X. A star (*) will be used for any one of the six operations of transposition. These operations form a group, the symmetric group of order 3!, whose multiplication table may be written in the form

$$\begin{array}{llllll}
X \equiv & X^{\prime\prime} & \equiv X^{\backprime\backprime} & \equiv X^{--} & \equiv X^{\vee\wedge} & \equiv X^{\wedge\vee}, \\
 & X' & \equiv X^{\backprime\vee} & \equiv X^{-\wedge} & \equiv X^{\vee-} & \equiv X^{\wedge\backprime}, \\
 & X^{\backprime} & \equiv X^{\prime\wedge} & \equiv X^{-\vee} & \equiv X^{\vee\prime} & \equiv X^{\wedge-}, \\
 & X^{-} & \equiv X^{\prime\vee} & \equiv X^{\backprime\wedge} & \equiv X^{\vee\backprime} & \equiv X^{\wedge\prime}, \\
 & X^{\vee} & \equiv X^{\prime-} & \equiv X^{\backprime\prime} & \equiv X^{-\backprime} & \equiv X^{\wedge\wedge}, \\
 & X^{\wedge} & \equiv X^{\prime\backprime} & \equiv X^{\backprime-} & \equiv X^{-\prime} & \equiv X^{\vee\vee}.
\end{array}$$

Given the multiplication table of X, that of a transpose is easily read off; for if in X (for fixed i and j)

$$\begin{aligned}
a_i\, a_1 &= \ldots. + \gamma_1\, a_j + \ldots., \\
a_i\, a_2 &= \ldots. + \gamma_2\, a_j + \ldots., \\
&\;\vdots \\
a_i\, a_n &= \ldots. + \gamma_n\, a_j + \ldots.,
\end{aligned}$$

then in X': $a_i\, a_j = \gamma_1\, a_1 + \gamma_2\, a_2 + \ldots. + \gamma_n\, a_n$.

To find $a_i\, a_j$ in $X^{\backprime}$, we examine similarly the products $a_1\, a_j$, $a_2\, a_j$, $\ldots.\ a_n\, a_j$ in X. Also X^{-} is derived from X by merely reversing the order of writing all products, and the same relation holds between $X^{\vee}$ and X', and between $X^{\wedge}$ and $X^{\backprime}$. For any particular (non-commutative) algebra X, we should therefore only require to study two of the transposes, say X' and $X^{\backprime}$.

If γ_{ijk} is symmetric in one pair of suffixes, the six transposes occur in identical pairs and one pair is commutative. If X is commutative,

$$X \equiv X^{-}, \qquad X' \equiv X^{\wedge}, \qquad X^{\backprime} \equiv X^{\vee};$$

in this case it would be sufficient to study one transposed algebra, say the right transpose; this is in general non-commutative, and since

$X^{\backprime} \equiv X^{\vee} \equiv X'^{-}$ the left transpose is obtained from it by reversing the order of all products.

If γ_{ijk} is cyclic in its suffixes[1], then

$$X \equiv X^{\wedge} \equiv X^{\vee}, \qquad X^{-} \equiv X' \equiv X^{\backprime}.$$

Returning to the general linear algebra X, it is known that every element $x = \Sigma \alpha_i a_i$ of it satisfies the two characteristic equations[2]

$$x \mid \Sigma \alpha_i \gamma_{ipq} - x \delta_{pq} \mid = 0, \tag{L}$$
$$x \mid \Sigma \alpha_i \gamma_{piq} - x \delta_{pq} \mid = 0. \tag{R}$$

Here $\delta = (\delta_{pq})$ is the unit matrix, and in (L) powers of x are interpreted as left principal powers

$$x, \quad x^2, \quad x^{1+2}, \quad x^{1+(1+2)}, \quad x^{1+\{1+(1+2)\}}, \quad \ldots . ;$$

in (R) they are interpreted as right principal powers

$$x, \quad x^2, \quad x^{2+1}, \quad x^{(2+1)+1}, \quad x^{\{(2+1)+1\}+1}, \quad \ldots . .$$

If along with (L) and (R) we consider a third equation

$$x \mid \Sigma \alpha_i \gamma_{pqi} - x \delta_{pq} \mid = 0, \tag{S}$$

it will be seen that the characteristic equations of the algebras

$$X, \qquad X', \qquad X^{\backprime}, \qquad X^{-}, \qquad X^{\vee}, \qquad X^{\wedge}$$

are respectively

$$L, R; \qquad L, S; \qquad S, R; \qquad R, L; \qquad S, L; \qquad R, S.$$

(It must not be inferred that a similar statement holds regarding the rank equations, or equations of lowest degree satisfied identically by the left and right principal powers of X. A counter example is provided by the algebra with multiplication table $a_1^2 = a_1$, $a_1 a_2 = a_2 a_1 = \frac{1}{2} a_2$, $a_2^2 = 0$.)

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[1] (*Added* 30 *June*, 1944.) An example is the algebra W of order m^2 with multiplication table (where suffixes run from 1 to m) :

$$W: \qquad e_{ij} e_{jk} = e_{ki}, \qquad e_{ij} e_{hk} = 0 \qquad (h \neq j).$$

Albert has recently introduced, for algebras having an involution J, an operation which in some cases coincides with transposition. See A. A. Albert, "Algebras derived by non-associative matrix multiplication," *Amer. Journ. Math.*, 66 (1944), 30-40. If M is the total matric algebra of order m^2 with multiplication table

$$M: \qquad e_{ij} e_{jk} = e_{ik}, \qquad e_{ij} e_{hk} = 0 \qquad (h \neq j),$$

and J is matrix transposition ($e_{ij} J = e_{ji}$), then the algebras (in Albert's notation) $M_\rho(J)$, $M_\kappa(J)$, $M_{\kappa\rho}(J)$ are respectively $M^{\backprime}$, M' and the above algebra W.

[2] L. E. Dickson, "Linear Algebras," *loc. cit.*, §15, Theorem 3. In case X contains a modulus, the initial factors x in (L) and (R) can be omitted and the constant terms interpreted as multiples of the modulus ; but we suppose that they are in any case included, since even if X contains a modulus its transposes may not.

§ 3. *Equivalence relations.*

Consider two linear algebras X, Y of the same order n, over the same field, with multiplication tables

$$X: \qquad a_i\, a_j = \Sigma \gamma_{ijk}\, a_k,$$
$$Y: \qquad b_p\, b_q = \Sigma \eta_{pqr}\, b_r.$$

We shall define various equivalence relations (reflexive, symmetric and transitive) which may hold between X and Y, with corresponding relations between the cubic matrices γ, η, and consider which relations are preserved when the algebras are transposed in the same way. (3.1), (3.3) and (3.5) are analogous to the "equivalent," "congruent," and "similar" transformations HAK, HAH', HAH^{-1} of a square matrix A.

(a) X and Y are *isotopic* if there exist non-singular (square) matrices $\theta = (\theta_{ij})$, $\phi = (\phi_{ij})$, $\psi = (\psi_{ij})$, such that

$$\eta_{pqr} = \Sigma\, \theta_{pi}\, \phi_{qj}\, \psi_{rk}\, \gamma_{ijk}. \tag{3.1}$$

This is equivalent to Albert's definition[1].
Evidently if X is isotopic to Y, X^* is isotopic to Y^*; but it is *not the same* isotopy, since the order of the transforming matrices θ, ϕ, ψ is changed.

It may be noted in passing that an algebra and its transposes may be isotopes. A sufficient condition for X and X' to be isotopic is that the n square matrices (γ_{1jk}), (γ_{2jk}), ... should be symmetrical about their secondary diagonals. For then if we take θ to be the unit matrix, and take $\phi = \psi$ to be the mirror image of the unit matrix, having 1's down the secondary diagonal, (3.1) becomes

$$\begin{aligned} \eta_{pqr} &= \Sigma\, \theta_{pi}\, \phi_{qj}\, \psi_{rk}\, \gamma_{ijk} \\ &= \Sigma\, \delta_{pi}\, \delta_{q,\, n+1-j}\, \delta_{r,\, n+1-k}\, \gamma_{i,\, n+1-k,\, n+1} \\ &= \gamma_{prq}. \end{aligned}$$

A similar condition regarding (γ_{i1k}), (γ_{i2k}), ... suffices for isotopy of X and $X^{\backprime}$.—The algebra of complex numbers with basis 1, i satisfies these conditions and is therefore isotopic with its transposes.

(b) X and Y will be called *congruent* if there exists a non-singular

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[1] A. A. Albert, "Non-associative algebras, I. Fundamental concepts and isotopy," *Ann. Math.*, 43 (1942), 685-707; § 11. My X, Y, θ, ϕ, ψ' correspond to Albert's $\mathfrak{A}$, $\mathfrak{A}_0$, P, Q, C.

(square) matrix $\theta = (\theta_{ij})$ and a non-zero quantity κ ($< F$) such that

$$\eta_{pqr} = \kappa \Sigma \theta_{pi} \theta_{qj} \theta_{rk} \gamma_{ijk}. \tag{3.2}$$

The factor κ can be omitted without loss of generality if the coefficient-field F is sufficiently extended, for with a change of notation we can replace $\kappa^{1/3} \theta_{ij}$ by θ_{ij} and obtain

$$\eta_{pqr} = \Sigma \theta_{pi} \theta_{qj} \theta_{rk} \gamma_{ijk}. \tag{3.3}$$

But if for example F is the rational field, (3.2) is more general than (3.3).

In this case X^* is congruent to Y^*, with the *same* transforming matrix θ and scalar κ.

(c) It is usual to call X and Y *equivalent* if one can be derived from the other by a non-singular linear transformation of the basis. If the transformation is

$$b_p = \Sigma \lambda_{pi} a_i, \quad a_i = \Sigma \Lambda_{ip} b_p, \tag{3.4}$$

the relation between γ and η is

$$\eta_{pqr} = \Sigma \lambda_{pi} \lambda_{qj} \Lambda_{kr} \gamma_{ijk}. \tag{3.5}$$

In this case X^* is isotopic but not necessarily equivalent to Y^*.

However, by imposing suitable restrictions on λ, the matrix of the linear transformation, we can obtain equivalence relations, special cases of (c), which are also special cases of (b) with θ restricted, and which are therefore invariant under transposition. This is achieved, for example, if the transformation (3.4) is restricted to being one of the following:—

(d) orthogonal: then

$$\lambda' = \lambda^{-1}, \quad \lambda_{pi} = \Lambda_{ip}, \quad \eta_{pqr} = \Sigma \lambda_{pi} \lambda_{qj} \lambda_{rk} \gamma_{ijk};$$

(e) a permutation of the basis: λ is a permutation matrix: γ is transformed by applying the same permutation to its i-, j- and k-planes;

(f) a change of scale, $b_i = \kappa a_i$; λ is a scalar matrix; we shall say that X is *proportional* to Y;

(g) an orthogonal transformation followed by a change of scale: $\lambda\lambda'$ is a scalar matrix: we shall say that X is *conformal* to Y: this includes (d) (e) (f) (h) (i) as special cases;

(h) a permutation with change of scale;

(i) a change of notation: λ is the unit matrix, $\gamma_{ijk} = \eta_{ijk}$, $X \equiv Y$.

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It will be convenient to use " $=$ " to indicate any one of these nine relations between linear algebras. With any one of these meanings *except equivalence*

$$X = Y \text{ implies } X^* = Y^*. \tag{3.6}$$

§ 4. *Direct sums and products.*

By addition $(+)$ and multiplication $(\times)$ applied to square matrices, cubic matrices or linear algebras, we shall understand direct addition and direct multiplication. Equality $(=)$ between linear algebras has any fixed one of the meanings explained in § 3, except that sometimes particular meanings will be specially excluded.

If X, Y are linear algebras over F with multiplication tables

$$X: \qquad a_i\, a_j = \Sigma\, \gamma_{ijk}\, a_k \qquad (i,\ j,\ k = 1, \ldots . n),$$
$$Y: \qquad b_\alpha\, b_\beta = \Sigma\, \eta_{\alpha\beta\gamma}\, b_\gamma \qquad (\alpha,\ \beta,\ \gamma = 1, \ldots . \nu),$$

their direct sum and direct product are the linear algebras over F with the following multiplication tables.

$$X + Y: \quad a_i\, a_j = \Sigma\, \gamma_{ijk}\, a_k,\ \ b_\alpha\, b_\beta = \Sigma\, \eta_{\alpha\beta\gamma}\, b_\gamma,\ \ a_i\, b_\alpha = b_\alpha\, a_i = 0. \tag{4.1}$$
$$\text{(Basis: } a_1, \ldots . a_n,\ b_1, \ldots . b_\nu.\text{)}$$

Or with change of notation:

$$\left.\begin{aligned} & a_A\, a_B = \Sigma\, \Gamma_{ABC}\, a_C && (A, B, C = 1, \ldots, n + \nu) \\ \text{where}\quad & \Gamma_{ABC} = \gamma_{ABC} && \text{if } A, B, C \leqq n, \\ & \phantom{\Gamma_{ABC}} = \eta_{A-n,\, B-n,\, C-n} && \text{if } A, B, C > n, \\ & \phantom{\Gamma_{ABC}} = 0 && \text{otherwise.} \end{aligned}\right\} \tag{4.2}$$

The cubic matrix Γ is the direct sum $\gamma + \eta$.

$$X \times Y: \qquad a_{i\alpha}\, a_{j\beta} = \Sigma\, \gamma_{ijk}\, \eta_{\alpha\beta\gamma}\, a_{k\gamma}. \tag{4.3}$$
$$\text{(Basis: } a_{11}, \ldots . a_{n\nu} \text{ in lexical order.)}$$

The cubic matrix of the constants of multiplication is the direct product $\gamma \times \eta$.

If $X = X^\circ$ and $Y = Y^\circ$, then

$$X + Y = X^\circ + Y^\circ, \qquad X \times Y = X^\circ \times Y^\circ. \tag{4.4}$$

For example, if $=$ indicates isotopy we may suppose that the multiplication tables of X°, Y° are

$$X^\circ: \qquad a_p\, a_q = \Sigma\, \theta_{pi}\, \phi_{qj}\, \psi_{rk}\, \gamma_{ijk}\, a_r,$$
$$Y^\circ: \qquad b_\pi\, b_\rho = \Sigma\, \lambda_{\pi\alpha}\, \mu_{\rho\beta}\, \nu_{\sigma\gamma}\, \eta_{\alpha\beta\gamma}\, b_\sigma,$$

where the matrices θ, ϕ, ψ, λ, μ, ν are non-singular. Then that of $X^\circ + Y^\circ$ consists of these equations together with $a_i\, b_\alpha = b_\alpha\, a_i = 0$; or with change of notation (replacing b_π by $a_{n+\pi}$)

$$\begin{aligned} X^\circ + Y^\circ:\quad a_A a_B &= \Sigma\, \Gamma^\circ_{ABC}\, a_C && (A, B, C = 1, \ldots, n+\nu) \\ \text{where}\quad \Gamma^\circ_{ABC} &= \Sigma\, \theta_{Ai}\, \phi_{Bj}\, \psi_{Ck}\, \gamma_{ijk} && \text{if } A, B, C \leqq n, \\ &= \Sigma\, \lambda_{A-n,\alpha}\, \mu_{B-n,\beta}\, \nu_{C-n,\gamma}\, \eta_{\alpha\beta\gamma} && \text{if } A, B, C > n, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Comparing with $X + Y$, it is seen that these equations can be written

$$\Gamma^\circ_{ABC} = \Theta_{AD}\, \Phi_{BE}\, \Psi_{CF}\, \Gamma_{DEF}$$

where Θ, Φ, Ψ, Γ are the direct sums $\theta + \lambda$, $\phi + \mu$, $\psi + \nu$, $\gamma + \eta$, and Θ, Φ, Ψ are non-singular. This proves the first of equations (4.4) for this interpretation of $=$. There is no difficulty in completing the proof for the other interpretations and for the direct product.

To form the right transpose of (4.3), we interchange k, γ with j, β in the constants of multiplication and obtain

$$(X \times Y)': \qquad a_{i\alpha}\, a_{j\beta} = \Sigma\, \gamma_{ikj}\, \eta_{\alpha\gamma\beta}\, a_{k\gamma},$$

i.e., the direct product of X' and Y'. Similarly with the other transposes, and similarly with the direct sum. Thus

$$(X + Y)^* \equiv X^* + Y^*, \quad (X \times Y)^* \equiv X^* \times Y^*. \tag{4.5}$$

If an algebra A is a direct sum or product in the sense that its multiplication table can be put into the form (4.1) or (4.3) *by a linear transformation*, it does not follow that A^* is a direct sum or product unless the linear transformation involved is of the type (g) in § 3. *With equivalence excluded*, however,

$$Z = X + Y \text{ implies } Z^* = X^* + Y^*, \tag{4.6}$$

$$Z = X \times Y \text{ implies } Z^* = X^* \times Y^*. \tag{4.7}$$

§ 5. *Direct algebras of algebras.*

Since $X + Y$ and $Y + X$ are in general neither identical nor proportional, their bases being differently ordered, and the same applies to $X \times Y$ and $Y \times X$, it will now be convenient to *exclude the cases when $=$ means identity or proportionality.* Then the symbols $+$, $\times$, $=$ obey the commutative, associative and distributive laws. Write mX, X^m for the direct sum and direct product of m identical algebras; interpret $0.X$ as a null algebra (algebra of order zero), and X^0 as the field F (algebra of order 1, with multiplication table $1^2 = 1$); and consider "polynomials" in X (algebras equal to direct linear combinations of powers of X with positive integer coefficients), forming a totality $P(X)$, a subset of a polynomial ring. The powers of X may be all linearly independent; but this is not necessarily the case, and X may satisfy an algebraic equation with integer coefficients, the minimum equation of X as an element of the polynomial ring. By enlargement of the coefficient domain of P to a field, P becomes a polynomial algebra, of infinite or finite order, which will be called the *direct algebra* of X for the relation $=$.

Similarly any set of linear algebras X, Y, ... over the same field generates a system $P(X, Y, \ldots.)$, a subset of a commutative ring, consisting of all algebras equal to polynomials in X, Y, ...; and all algebras belonging to P can be expressed linearly with positive integer coefficients in terms of some infinite or finite set of direct powers and products. This basic set will have a multiplication table with positive integer constants of multiplication. By enlargement of the coefficient domain of P to a field, P becomes a commutative associative linear algebra of infinite or finite order. It will be called the *direct algebra* of the algebras X, Y, ... for the relation $=$; it is defined except for linear transformations of itself, since its basis is not uniquely determined.

We have already excluded the meanings *identity* and *proportionality* for $=$; *if equivalence also is excluded*, then (4.6), (4.7) show that the multiplication table of P is invariant for transposition of the generating algebras; that is:

$$P(X, Y, \ldots.) \equiv P(X^*, Y^*, \ldots.). \tag{5.1}$$

Of course the direct algebra $P(X, Y, \ldots.)$ for equivalence may be identical with the direct algebra $P(X, Y, \ldots.)$ for conformity; and if the same is true of $P(X^*, Y^*, \ldots.)$ then (5.1) holds for equivalence. On the other hand the direct algebra for equivalence may have a smaller basis than that for conformity, and then (5.1) may not hold. Both possibilities are exemplified in § 8.

If X is of order n and Y is of order ν, then $X + Y$ is of order $n + \nu$ and $X \times Y$ is of order $n\nu$. It follows that the direct algebra $P(X, Y, \ldots.)$ is always a baric algebra, each generating algebra having weight equal to its order. (See C. T. A.) This suggests that its multiplication table may usually be simplified by a change of scale, taking all basis elements to be of unit weight (*e.g.*, X/n, Y/ν, etc.—these of course are not interpreted as algebras).

§ 6. *Complex numbers transposed.*

Let Z be the algebra of complex numbers with basis 1, i. The coefficient field F is the field of real numbers. The following are the multiplication tables of Z and its transposes.

$$\begin{array}{c|cc} & 1 & i \\ \hline 1 & 1 & i \\ i & i & -1 \end{array} \qquad \begin{array}{c|cc} & 1 & i \\ \hline 1 & 1 & i \\ i & -i & 1 \end{array} \qquad \begin{array}{c|cc} & 1 & i \\ \hline 1 & 1 & -i \\ i & i & 1 \end{array} \tag{6.1}$$
$$Z,\ Z^{-} \qquad\qquad Z',\ Z^{\wedge} \qquad\qquad Z^{\backprime},\ Z^{\vee}$$

The general element or (hyper)complex number in any of these algebras can be taken as

$$x = \alpha 1 + \beta i = \rho\,(1 \cos\theta + i \sin\theta)$$

where $\alpha, \beta, \rho, \theta$ are real. Writing $\bar{x}$ for the "conjugate" element $\alpha 1 - \beta i$, it is easily seen that the value of a product xy in Z' is the same as that of $\bar{x}y$ in Z; and its value in $Z^{\backprime}$ is the same as that of $x\bar{y}$ in Z. Hence if we write

$$\operatorname{mod} x = \rho, \quad \arg x = \theta,$$

$$\left.\begin{array}{ll} \text{then in } Z: & \arg\ xy = \arg x + \arg y; \\ \text{in } Z': & \arg\ xy = \arg y - \arg x; \\ \text{in } Z^{\backprime}: & \arg\ xy = \arg x - \arg y; \\ \text{in } Z, Z', Z^{\backprime}: & \operatorname{mod}\ xy = (\operatorname{mod} x)(\operatorname{mod} y). \end{array}\right\} \tag{6.2}$$

(Equations involving "arg" are to be understood as congruences modulo 2π.)

Consider now any power x^s, of degree δ_s. Multiplication in Z' and $Z^{\backprime}$ being non-commutative and non-associative, s must be specified as an integer in an arithmetic with non-commutative and non-associative addition (*cf.* N.C.), and δ_s, the number of factors x in this power, is equal to the number obtained by evaluating s as if it were an integer in ordinary arithmetic. From (6.2) it follows that in each algebra arg x^s will be some multiple of θ depending on s, say $\epsilon_s\theta$, and we have

$$\operatorname{mod} x^s = \rho^{\delta_s}, \quad \arg x^s = \epsilon_s\theta. \tag{6.3}$$

In Z, $\qquad \epsilon_s = \delta_s$. (De Moivre's theorem.)

In $Z^{\backprime}$, *e.g.*,
$$\arg x^2 = 0,$$
$$\left.\begin{array}{l}\arg x^{1+2} = \arg\,(x.x^2) = \theta, \\ \arg x^{2+1} = \arg\,(x^2.x) = -\theta,\end{array}\right\}$$
$$\left.\begin{array}{l}\arg x^{(1+2)+1} = \arg\,(x^{1+2}.x) = 0, \\ \arg x^{1+(2+1)} = \arg\,(x.x^{2+1}) = 2\theta.\end{array}\right\}$$

In Z', $\arg x^2 = 0$, and the values bracketed above are interchanged.

The three characteristic equations mentioned in § 2 are here

$$\left.\begin{array}{l} L: \\ R: \end{array}\right\} \qquad 0 = x \begin{vmatrix} \alpha - x, & -\beta \\ \beta\ , & \alpha - x \end{vmatrix} = x^3 - 2\alpha x^2 + (\alpha^2 + \beta^2)\,x,$$

$$S: \qquad 0 = x \begin{vmatrix} \alpha - x, & \beta \\ \beta\ , & -\alpha - x \end{vmatrix} = x^3 - (\alpha^2 + \beta^2)\,x.$$

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So for example in $Z^{\backprime}$ we may verify that

S, R: $$0 = x^{1+2} - (\alpha^2 + \beta^2)\,x = x^{2+1} - 2\alpha\,x^2 + (\alpha^2 + \beta^2)\,x.$$

To prove the palintropic property, let a, b be the indices of two powers, so that x^{ab} means $(x^a)^b$ and x^{ba} means $(x^b)^a$. Then

$$\arg x^{ab} = \epsilon_b \arg x^a = \epsilon_b\,\epsilon_a\,\theta = \arg x^{ba},$$
$$\operatorname{mod} x^{ab} = (\operatorname{mod} x^a)^{\delta_b} = \rho^{\delta_a \delta_b} = \operatorname{mod} x^{ba},$$

and hence $$x^{ab} = x^{ba}. \tag{6.4}$$

It will be seen that

$$\left.\begin{array}{lll} \text{in } Z^{\backprime}: & \epsilon_{a+b} = \epsilon_a - \epsilon_b, & \epsilon_{ab} = \epsilon_a\,\epsilon_b\,; \\ \text{in } Z': & \epsilon_{a+b} = \epsilon_b - \epsilon_a, & \epsilon_{ab} = \epsilon_a\,\epsilon_b. \end{array}\right\} \tag{6.5}$$

Hence in $Z^{\backprime}$, ϵ_s may be evaluated by treating all the + signs in the expression for s as $-$ signs. Thus for $2 = 1 + 1$

$$\epsilon_2 = 1 - 1 = 0, \quad \delta_2 = 1 + 1 = 2, \quad x^2 = \rho^2\,\mathbf{1}\,;$$

and to take a more complicated example, if

$$s = \{1 + (2 + 1)\}\,(2.2 + 1) + (1 + 2),$$

then

$$\epsilon_s = \{1 - (0 - 1)\}\,(0 - 1) - (1 - 0) = -3,$$
$$\delta_s = 4.5 + 3 = 23,$$
$$x^s = \rho^{23}\,(\mathbf{1}\cos 3\theta - i \sin 3\theta).$$

To evaluate the same power x^s in Z', apply the same process to s read backwards:

$$\epsilon_s = (0 - 1) - (1 - 0)\,\{(1 - 0) - 1\} = -1, \quad \delta_s = 23,$$
$$x^s = \rho^{23}\,(\mathbf{1}\cos\theta - i \sin\theta). \tag{6.6}$$

If Z is taken to be the algebra with multiplication table as given in (6.1), over any field F, the methods we have used will not be generally applicable since ρ and θ may not exist; but the results obtained by these methods will be valid if suitably interpreted. *E.g.*, the last result concerning Z', (6.6), may be written

$$x^s = (\alpha^2 + \beta^2)^{11}\,(\alpha\,\mathbf{1} - \beta\,i);$$

this result, proved when F is the real field, must be verifiable by direct calculation from (6.1); and this verification will continue to hold good for a general coefficient field. Similarly the palintropic property (6.4) holds generally.

As an illustration of the fact observed in § 3 (*c*), suppose that F includes the complex field, and apply the transformation

$$b_1 = \tfrac{1}{2}\,(\mathbf{1} + \iota\,i), \quad b_2 = \tfrac{1}{2}\,(\mathbf{1} - \iota\,i). \tag{6.7}$$

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(ι denotes the element $\sqrt{-1}$ of the field F, i that of the algebra Z.) The multiplication table of Z becomes

$$b_1^2 = b_1, \quad b_2^2 = b_2, \quad b_1 b_2 = b_2 b_1 = 0. \tag{6.8}$$

Calling this algebra Y,

$$Y \equiv Y^- \equiv Y' \equiv Y^{\backprime} \equiv Y^{\wedge} \equiv Y^{\vee}. \tag{6.9}$$

On the other hand, if we apply the same transformation to Z', we obtain the algebra with multiplication table

$$b_1^2 = b_2^2 = 0, \quad b_1 b_2 = b_2, \quad b_2 b_1 = b_1. \tag{6.10}$$

This is a special case of the algebras to be considered in § 10, where it is denoted $A_2^{\backprime}$; being equivalent to Z' it is palintropic, and we shall require to use this fact in § 9.

§ 7. *Quaternions transposed.*

Let Q be the algebra of quaternions, with basis $1, i, j, k$, over the real field. It will be shown that Q' is palintropic, and a similar proof applies to $Q^{\backprime}$. Q^-, $Q^{\vee}$ and $Q^{\wedge}$, being the straight transposes of Q, Q' and $Q^{\backprime}$, have the same property.

The multiplication tables are

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1,

Q

	1	i	j	k
1	1	i	j	k
i	$-i$	1	$-k$	j
j	$-j$	k	1	$-i$
k	$-k$	$-j$	i	1.

Q'

We wish to prove that, in Q', $q^{ab} = q^{ba}$, where

$$q = \alpha 1 + \beta i + \gamma j + \delta k.$$

This is certainly true when $q = \alpha 1 + \beta i$, for the multiplication table shows that all such q's belong to a subalgebra of Q' identical with the palintropic algebra Z' of § 6; thus the subalgebra of Q' with basis $(1, i)$ is palintropic.

Apply to Q the linear transformation $Q \rightarrow Q^{\circ}$ which represents geometrically some rotation of axes in the ijk-space, such that one of the new axes has the direction of the vector $\beta i + \gamma j + \delta k$. The equations of transformation are of the form

$$\begin{aligned} 1 &= 1, \\ I &= (\beta^2 + \gamma^2 + \delta^2)^{-\frac{1}{2}} (\beta i + \gamma j + \delta k) \\ &= \lambda_{11} i + \lambda_{12} j + \lambda_{13} k, \\ J &= \lambda_{21} i + \lambda_{22} j + \lambda_{23} k, \\ K &= \lambda_{31} i + \lambda_{32} j + \lambda_{33} k, \end{aligned}$$

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where (λ_{ij}) is an orthogonal matrix, and the (four-rowed) matrix of this transformation is therefore also orthogonal. The transformed algebra Q° with basis $(1, I, J, K)$ has a multiplication table of the same form as Q; that is $Q \equiv Q^\circ$, and hence $Q' \equiv (Q^\circ)'$. Moreover, since the transformation is an orthogonal one, the *same* orthogonal transformation carries Q' into $(Q^\circ)'$. The general element of Q' with which we started belongs to the palintropic subalgebra $(1, I)$; thus in Q'

$$q^{ab} = q^{ba},$$

and we have shown that Q' is palintropic.

§ 8. *Complex numbers: the direct algebra.*

Starting with the multiplication table of Z in (6.1) and forming the direct square, we obtain the multiplication table

$$Z^2: \quad \begin{array}{c|cccc} & a_{11} & a_{12} & a_{21} & a_{22} \\ \hline a_{11} & a_{11} & a_{12} & a_{21} & a_{22} \\ a_{12} & a_{12} & -a_{11} & a_{22} & -a_{21} \\ a_{21} & a_{21} & a_{22} & -a_{11} & -a_{12} \\ a_{22} & a_{22} & -a_{21} & -a_{12} & a_{11}. \end{array}$$

This may be transformed into a direct sum, namely

$$2Z: \quad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & a & b & 0 & 0 \\ b & b & -a & 0 & 0 \\ c & 0 & 0 & c & d \\ d & 0 & 0 & d & -c, \end{array}$$

by applying the transformation

$$a = \tfrac{1}{2}(a_{11} + a_{22}), \quad b = \tfrac{1}{2}(a_{12} - a_{21}), \quad c = \tfrac{1}{2}(a_{11} - a_{22}), \quad d = \tfrac{1}{2}(a_{12} + a_{21}).$$

This is a combination of an orthogonal transformation $a_1 = (a_{11} + a_{22}) / \surd 2$, etc., and a change of scale $a = a_1 / \surd 2$, etc. Thus, *using* $=$ *in the sense of conformity* (§ 3 g),

$$Z^2 = 2\,Z.$$

It follows that the transposed algebras Z', $Z^\backprime$ have the same property. It also follows that if m is any positive integer

$$(Z^*)^m = 2^{m-1}\,Z^*.$$

Now Z^0 is the real field F, and $Z \mp 2F$; thus the direct algebra $P(Z)$ is of order 2 and has multiplication table

$$F^2 = F, \qquad F \times Z = Z \times F = Z, \qquad Z^2 = 2Z;$$

and of course $P(Z^*)$ has the same form. Any algebra belonging to $P(Z^*)$ is equal to a direct sum $\alpha F + \beta Z^*$ and is therefore palintropic.

If $=$ *indicates equivalence,* the same facts $(Z^*)^2 = 2Z^*$, $Z^* \neq 2F$ hold; so in this case also, as remarked in § 5, $P(Z) \equiv P(Z^*)$. On the other hand, if we take the coefficient field F of Z to be the complex field, $=$ still denoting equivalence, (6.8) shows that $Z = 2F$, $Z^n = 2^n F$. In this case $P(Z)$ is of order 1, that is to say it is isomorphic with its coefficient field; whereas $P(Z')$, $P(Z`)$ are still of order 2. In either case, any algebra belonging to $P(Z^*)$ is equal to a direct sum $\alpha F + \beta Z^*$ and is therefore palintropic.

If $=$ *signifies congruence or isotopy and* F *is the complex field,* then again $P(Z)$ is of order 1 since $Z = 2F$, but now of course $P(Z) \equiv P(Z^*)$. Any algebra belonging to $P(Z^*)$ is equal to a direct sum αF; it does not follow that it is palintropic.

§ 9. *Abelian group algebras transposed.*

The group algebra of a finite group means the linear algebra, over some field F, whose basis consists of the elements of the group in some definite order, and which has the same multiplication table as the group.

(a) We consider first the cyclic group of order n, and denote the group algebra C_n. It will be shown that C_n^* is palintropic. To prove this, it will evidently be sufficient if we assume that F includes the complex field and prove the theorem for C_n'.

The multiplication table of C_n may be written

C_n: $$a_i a_j = a_{i+j}$$

where the suffixes run from 1 to n, or 0 to $n-1$, and a_r is identified with a_s if $r \equiv s \pmod{n}$. Hence in C_n', $a_i a_{i+j} = a_j$; that is

C_n': $$a_i a_j = a_{j-i}.$$

Take ω to be a primitive nth root of unity, and apply to C_n' the linear transformation

$$b_j = \frac{1}{n}(a_0 + \omega^j a_1 + \ldots + \omega^{sj} a_s + \ldots + \omega^{(n-1)j} a_{n-1}).$$

Then, in C_n', $a_s b_j = \omega^{sj} b_j$. Hence

$$\begin{aligned} b_i b_j &= \frac{1}{n} \sum_s \omega^{si} a_s b_j = \frac{1}{n} \sum_s \omega^{s(i+j)} b_j \\ &= b_j \text{ if } i + j \equiv 0 \pmod{n}, \\ &= 0 \text{ otherwise.} \end{aligned}$$

C_n' is therefore equivalent to the algebra of order n with multiplication table

$$\left.\begin{array}{ll} b_0^2 = b_0, & b_1 b_{n-1} = b_{n-1}, \; \ldots . \; b_r b_{n-r} = b_{n-r}, \; \ldots . , \\ & b_{n-1} b_1 = b_1, \quad \ldots . \; b_{n-r} b_r = b_r, \quad \ldots . , \\ b_i b_j = 0 & (i + j \neq 0 \text{ or } n). \end{array}\right\}.$$

Thus C_n' is equivalent to the direct sum of one or two algebras isomorphic with F and $\frac{1}{2}(n-1)$ or $\frac{1}{2}(n-2)$ algebras identical with the algebra $A_2^{\backprime}$ (*cf.* 6.10). These are all palintropic algebras, the direct sum of a number of such algebras is palintropic, and the theorem is proved.

It may be remarked that the linear transformation used is not orthogonal or conformal, so that from $C_n' = \alpha F + \beta A_2^{\backprime} = \alpha F + \beta Z'$, we cannot deduce $C_n = \alpha F + \beta A_2{}^{\mathrm{v}} = \alpha F + \beta Z$.

(*b*) An abelian group is a direct product of cyclic groups. The group algebra G is therefore the direct product of a number of algebras of the form C_n, say $G \equiv \Pi\, C_{n_i}$. Each C_{n_i}' is equivalent to an algebra of the form $\alpha_i F + \beta_i A_2^{\backprime}$. Hence, using $=$ in the sense of equivalence,

$$G' \equiv \Pi\, C_{n_i}' = \Pi\,(\alpha_i F + \beta_i A_2^{\backprime}) = \Pi\,(\alpha_i F + \beta_2 Z').$$

It follows that G' belongs to $P(Z')$, can be expressed as $\alpha F + \beta Z'$, and is palintropic.

§ 10. *A class of palintropic algebras.*

Let A_n denote the commutative non-associative linear algebra of order n with multiplication table

$$A_n: \quad \left\{\begin{array}{l} a_1^2 = a_2, \quad a_2^2 = a_3, \quad \ldots . \quad a_{n-1}^2 = a_n, \quad a_n^2 = a_1, \\ \qquad a_i a_j = 0 \quad (i \neq j). \end{array}\right.$$

Such algebras were considered in N.A.M.I.C. § 3 and shown to be palintropic. The right and left transposes are:

$$A_n': \quad \left\{\begin{array}{l} a_1 a_2 = a_1, \quad a_2 a_3 = a_2, \quad \ldots . \quad a_n a_1 = a_n, \\ \qquad a_i a_j = 0 \quad (j \not\equiv i + 1, \bmod n). \end{array}\right.$$

$$A_n^{\backprime}: \quad \left\{\begin{array}{l} a_2 a_1 = a_1, \quad a_3 a_2 = a_2, \quad \ldots . \quad a_1 a_n = a_n, \\ \qquad a_i a_j = 0 \quad (j \not\equiv i - 1, \bmod n). \end{array}\right.$$

It may be shown by the same method of proof as was used for A_n that these algebras also are palintropic; and it will be noticed that A_2', $A_2^{\backprime}$ are complex linear transforms of $Z^{\backprime}$, Z' (*cf.* end of § 6).

Form the direct product of two such algebras A_m, A_n. Take the basis as $(a_{\alpha i})$ where α and i take positive integer values, $a_{\alpha i}$ being identified with $a_{\beta j}$ if $\alpha \equiv \beta \pmod{m}$ and $i \equiv j \pmod{n}$. The multiplication table is:

$$A_m \times A_n: \quad \begin{cases} a_{\alpha i}^2 = a_{\alpha+1, i+1}, & (10.1) \\ a_{\alpha i}\, a_{\beta j} = 0 \text{ if } a_{\alpha i} \neq a_{\beta j}. & (10.2) \end{cases}$$

Let p = the h.c.f. of m and n, and q = their l.c.m., so that $pq = mn$. Then the mn equations (10.1) can be rearranged in p cycles of q equations, each cycle being of the form

$$x_1^2 = x_2, \qquad x_2^2 = x_3, \qquad \ldots . \qquad x_q^2 = x_1.$$

Thus with a suitable change of the order of its basis elements, $A_m \times A_n$ becomes the direct sum of p algebras A_q. If we interpret $=$ in the sense of § 3 (e), we have

$$A_m \times A_n = p\, A_q; \tag{10.3}$$

and this is the multiplication table of the direct algebra $P(A_1, A_2, A_3, \ldots .)$. It follows that $A_m^* \times A_n^* = p\, A_q^*$.

As suggested in § 5, the direct algebra may be simplified by a change of scale. Putting $A_n' = n\, B_n$, we have

$$mB_m \times nB_n = A_m \times A_n = pA_q = pqB_q,$$

so that

$$B_m \times B_n = B_q. \tag{10.4}$$

§ 11. *Postscript on quasi-groups.* (Added 13th November 1944).

A *quasi-group*[1] is a multiplicative system of elements $a, b, c, \ldots .$ in which an equation of the form $ab = c$ is always uniquely soluble for each of a, b and c, given the other two. (Thus an associative quasi-group is a group.) If finite, of order n, its multiplication table is a latin square, and is characterised by a cubic matrix of special type, analogous to a (square) permutation matrix, such that every row, column and file contains one 1 and $n - 1$ zeros.

The idea of transposing a quasi-group was mentioned but not developed by Bruck[2]. He referred to it as "slight broadening of the concept of isotopy .... by admitting to the equivalence class of a quasi-group five other quasi-groups formed essentially by permutation of the letters a, b, c in the relation $ab = c$." This comes to the

[1] B. A. Hausmann and Oystein Ore, "Theory of quasi-groups," *Amer. Journ. of Math.*, **59** (1937), 983-1004.

[2] R. H. Bruck, "Some results in the theory of quasi-groups," *Trans. Amer. Math. Soc.*, **55** (1944), 19-52.

same thing as transposing the cubic matrix. It may be noted that if $ab = c$ is written $c \div b = a$, we obtain the left transpose by interpreting $\div$ as multiplication.

Murdoch[1] defines an *abelian* quasi-group as one in which the law $ab.cd = ac.bd$ holds; and his Corollary to Theorem 10 is equivalent to the statement that an abelian quasi-group has the palintropic property $a^{mn} = a^{nm}$. Now it is easily shown that the transposes of an abelian group are abelian (and therefore palintropic) quasi-groups; *e.g.*, the law $ab.cd = ac.bd$ in the left transpose is the same as $(ab^{-1})(cd^{-1})^{-1} = (ac^{-1})(bd^{-1})^{-1}$ in the abelian group. It follows that in a transposed abelian group algebra (G^*) the property $ab.cd = ac.bd$ holds among the basis elements, and therefore generally.

Now the proofs of Murdoch's Theorem 10 and Corollary apply with only a minor verbal change ("cyclic groupoid" for "cyclic quasi-group" in the definition of a *power*) to linear algebras. Thus if a linear algebra has the property $ab.cd = ac.bd$, it is palintropic. This gives therefore an alternative proof of my result (§ 9 *b*) that the transposes of an abelian group algebra are palintropic algebras.

It may be observed further that for the proof of Murdoch's Corollary (though not for the Theorem 10 from which it is derived) it is sufficient to assume $a^p a^q.a^r a^s = a^p a^r.a^q a^s$ in place of the stronger abelian law. If this weaker assumption holds for a particular element a and arbitrary powers p, q, r, s, it then follows that $a^{mn} = a^{nm}$; if it holds for all a, we have the palintropic property. This applies to linear algebras as well as quasi-groups. Now in all the palintropic algebras which I have obtained (here and in C.T.A. and N.A.M.I.C), I have verified that powers of an arbitrary element obey this weaker assumption, in other words that

$$(p + q) + (r + s) = (p + r) + (q + s)$$

holds in the logarithmetic. The question arises, which I must leave here unanswered, whether palintropic algebras exist for which this is not the case.

[*Added in proof.*]

Transposed algebras have been considered by R. H. Bruck in a recently published paper, "Some results in the theory of linear non-

[1] D. C. Murdoch, "Quasi-groups which satisfy certain generalized associative laws," *Amer. Journ. of Math.*, 61 (1939), 509-522.

associative algebras," *Trans. Amer. Math. Soc.*, **56** (1944), 141-199, where they are called *associated* algebras. Bruck uses trilinear forms in place of cubic matrices, and refers to R. M. Thrall, "On projective equivalence of trilinear forms," *Ann. Math.*, **42** (1941), 469-485, for a bibliography. He observes the invariance of isotopy under transposition (my § 3a). He also shows that the property of being a division algebra is unaffected by transposition, and uses this in a study of non-associative division algebras.

Bruck has referred me in correspondence to a paper, encountered by him since his own was written, in which transposed algebras were introduced thirty years ago. See J. B. Shaw, "On parastrophic algebras," *Trans. Amer. Math. Soc.*, **16** (1915), 361-370. Among other things, Shaw considers under what circumstances multiplication in the transposed algebras can be defined in terms of multiplication in the original algebra, as in my § 6. He shows that the algebra must be a Dedekind (*i.e.*, semi-simple) algebra. The property in question is not always invariant for a change of basis, but holds if the Dedekind algebra is in its canonical form. There is an obvious misprint in Shaw's definitions, p. 362, where the terms *antipreparastrophic* and *antipostparastrophic* (corresponding to my $X^{\vee}$, $X^{\wedge}$) are reversed: doubt about this is dispelled in § 7, p. 365.

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