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ON A HOMOTOPY INVARIANT FOR SIMPLEXES AND ITS APPLICATION

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Based on a congruence relation of K. Fan in the integer labelling of pseudomanifolds, we follow an idea of H. Sies and define a mod 2 homotopy invariant $\mathbf{d}$ for a class of continuous functions defined on an n-simplex into $\mathbb{R}^{n+1} \setminus \{0\}$ satisfying a certain boundary condition. Using the homotopy invariance of $\mathbf{d}$, generalised coincidence theorems are proved which unify and extend some existence theorems of K. Fan. Our results also contain Kakutani's fixed point theorem and a new covering property of simplexes of Shapley's type.

1. INTRODUCTION

Let $I = \{1, \dots, n+1\}$. For $x \in \mathbb{R}^{n+1}$, the components x_i $(i \in I)$ of x are indicated by subscripts. We denote $\mathbb{R}^{n+1}_+ = \{x \in \mathbb{R}^{n+1} : x_i \geqslant 0 \text{ for } i \in I\}$, denote by $\Delta^n = \{x \in \mathbb{R}^{n+1}_+ : \sum_{i \in I} x_i = 1\}$ the standard n-simplex, and by $\partial\Delta^n$ and $\operatorname{int}\Delta^n$ the boundary and the interior of Δ^n relative to Δ^n, respectively. Let $\mathcal{F}$ be the class of continuous functions $f = (f_1, \dots, f_{n+1})$ defined on Δ^n into $\mathbb{R}^{n+1} \setminus \{0\}$ satisfying the boundary condition:

(1) for each $x \in \partial\Delta^n$, there exists $i \in I$ such that $f_i(x) \geqslant 0$.

Based on a congruence relation of Fan [4] which is fundamental in the integer labelling of pseudomanifolds, Sies defined in [**10**], in the most interesting case, for each $f \in \mathcal{F}$ an integer $\mathbf{d}(f) \in \{0, 1\}$ which is a mod 2 homotopy invariant for the class $\mathcal{F}$. The approach of Sies is analogous to the one by Krasnoselskii [9] who defined the Brouwer's degree by means of integer labelling. The purpose of this paper is to refine and improve the idea of Sies. Interesting mapping properties of $f \in \mathcal{F}$ can be deduced if the parity of $\mathbf{d}(f)$ is known. Relying on our improvements, conditions sufficient for $\mathbf{d}(g) \neq \mathbf{d}(-f)$ are obtained when both g and $-f$ are in $\mathcal{F}$. One of our main results is Theorem 3 in Section 2 which is proved using the well-known Sperner's lemma and its dual form formulated by Fan [4] as boundary conditions. By the homotopy invariance of

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d, another useful result is given in Theorem 5 which unifies and extends some existence theorems of Fan [**2**, **3**, **4**]. In Section 3, Theorem 5 is generalised to the set-valued setting in two forms which contain the fixed point theorem of Kakutani [**8**] and a new covering property of simplexes of Shapley's type (see Aubin [**1**, Chapter 9]).

Here is some more notation. For $x \in \partial\Delta^n$, we denote $I_+(x) = \{i \in I : x_i > 0\}$, $I_0(x) = \{i \in I : x_i = 0\}$. The metric d on $\mathbb{R}^{n+1}$ is defined by $d(x,y) = \max\{|x_i - y_i| : i \in I\}$.

2. A Homotopy Invariant for Simplexes

We first recall some results of Fan on the integer labelling of pseudomanifolds, on which our results are based. For further references to these we refer to Fan [**4**].

Let M^n be a triangulation of Δ^n. An admissible labelling ϕ of M^n assigns to each vertex v of M^n an integer $\phi(v)$ satisfying

$$\phi(v) \in \{\pm 1, \dots, \pm(n+1)\}; \tag{2}$$

$$\phi(v) + \phi(w) \neq 0 \quad \text{if } v, w \text{ are adjacent vertices;} \tag{3}$$

$$\phi(v) > 0 \quad \text{if } v \text{ is a vertex in } \partial\Delta^n. \tag{4}$$

For any combination $(\varepsilon_1, \dots, \varepsilon_{n+1})$ of the signs $\varepsilon_i = \pm 1$, we denote by $\alpha(\varepsilon_1 1, \dots, \varepsilon_{n+1}(n+1))$ the number of those n-simplexes of M^n, each of which is labelled under ϕ by $\varepsilon_1 1, \dots, \varepsilon_{n+1}(n+1)$ at its vertices, and we denote by $\beta(1, \dots, n)$ the number of those boundary $(n-1)$-simplexes of M^n, each of which is labelled under ϕ by $1, \dots, n$ at its vertices. The congruence relation of Fan referred to in Section 1 asserts that for any combination $(\varepsilon_1, \dots, \varepsilon_{n+1})$ of the signs $\varepsilon_i = \pm 1$ with at least one $\varepsilon_i = -1$,

$$\alpha(\varepsilon_1 1, \dots, \varepsilon_{n+1}(n+1)) \equiv \alpha(1, \dots, n+1) + \beta(1, \dots, n) \mod 2. \tag{5}$$

This fundamental result plays an essential role in two key instances below.

Let $f \in \mathcal{F}$. We define $\mathbf{d}(f)$ in the following way. There exist ε, $\delta > 0$ such that $d(f(\Delta^n), 0) > 3\varepsilon$ and $d(f(x), f(y)) < \varepsilon$ whenever $d(x,y) < \delta$. Let M^n be a triangulation of Δ^n with mesh less than δ and let v be a vertex of M^n. The star of v, denoted by $\operatorname{st} v$, is the union of all n-simplexes of M^n with v as a vertex. We first define a labelling ϕ on M^n. If $v \in \operatorname{int}\Delta^n$, then there exists $i \in I$ such that $|f_i(v)| > 3\varepsilon$ and so $|f_i(x)| > 2\varepsilon$ for $x \in \operatorname{st} v$. If $v \in \partial\Delta^n$, then there exists $i \in I$ such that $f_i(v) \geqslant 0$ and so $f_i(x) > -\varepsilon$ for $x \in \operatorname{st} v$. We define

$$\phi(v) = \begin{cases} i & \text{if } v \in \operatorname{int}\Delta^n \quad \text{and } f_i > 2\varepsilon \text{ on } \operatorname{st} v; \\ -i & \text{if } v \in \operatorname{int}\Delta^n \quad \text{and } f_i < -2\varepsilon \text{ on } \operatorname{st} v; \\ i & \text{if } v \in \partial\Delta^n \quad \text{and } f_i > -\varepsilon \text{ on } \operatorname{st} v. \end{cases} \tag{6}$$

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Obviously ϕ is an admissible labelling of M^n. Though the choice of i or $-i$ in (6) is not necessarily unique, the parity of $\alpha(-1,\dots,-(n+1))$ does not depend on the triangulation M^n of Δ^n with mesh fine enough, nor on the admissible labelling ϕ of M^n defined as in (6). For the proof of this statement we refer to Sies [**10**, Lemmas 2.2 and 3.2], which work as well in the present case (see also Krasnoselskii [**9**], pp.84–87). We remark that (5) is essential for the proof of the second assertion above. For a triangulation M^n of Δ^n with mesh fine enough, we shall call a labelling ϕ of M^n defined as in (6) for some $\varepsilon > 0$ an admissible labelling generated by f on M^n. Now we define

$$\mathbf{d}(f) = \begin{cases} 0 & \text{if } \alpha(-1,\dots,-(n+1)) \quad \text{is even;} \\ 1 & \text{if } \alpha(-1,\dots,-(n+1)) \quad \text{is odd.} \end{cases}$$

Theorem 1. *Let $H : \Delta^n \times [0,1] \to \mathbb{R}^{n+1}$ be a continuous function and $f^s(x) = H(x,s)$. If $f^s \in \mathcal{F}$ for each $0 \leqslant s \leqslant 1$, then $\mathbf{d}(f^0) = \mathbf{d}(f^1)$, that is, $\mathbf{d}$ is a homotopy invariant for the class $\mathcal{F}$.*

Proof: There exist ε, $\delta > 0$ such that $d(H(\Delta^n \times [0,1]),0) > 3\varepsilon$ and $d(H(x,s), H(y,t)) < \varepsilon$ whenever $d(x,y) + |s-t| < 2\delta$. Let M^n be a triangulation of Δ^n with mesh less than δ, and let v be a vertex of M^n, $0 \leqslant s \leqslant 1$. For $|s-t| \leqslant \delta$, if $v \in \operatorname{int} \Delta^n$, then there exists $i \in I$ such that $|H_i(x,t)| > 2\varepsilon$ for $x \in \operatorname{st} v$; if $v \in \partial\Delta^n$, then there exists $i \in I$ such that $H_i(x,t) > -\varepsilon$ for $x \in \operatorname{st} v$. Thus f^s and f^t can generate the same admissible labelling on M^n and so $\mathbf{d}(f^s) = \mathbf{d}(f^t)$ whenever $|s-t| \leqslant \delta$. The theorem now follows. $\square$

A direct consequence of the homotopy invariance of $\mathbf{d}$ is the following

Theorem 2. *Let g, $-f \in \mathcal{F}$ satisfy*

(7) *for each $x \in \partial\Delta^n$, $\mu > 0$, there exists $k \in I$ such that $\mu f_k(x) \leqslant g_k(x)$.*

If $\mathbf{d}(g) \neq \mathbf{d}(-f)$, then there exist $\widehat{x} \in \Delta^n$, $\lambda > 0$ such that

$$\lambda f(\widehat{x}) = g(\widehat{x}). \tag{8}$$

Proof: Let $H : \Delta^n \times [0,1] \to \mathbb{R}^{n+1}$ be defined by $H(x,s) = (1-s)g(x) - sf(x)$, $g^s(x) = H(x,s)$. Then (7) implies that g^s satisfies the boundary condition (1) for each $0 < s < 1$. Since $\mathbf{d}(g) \neq \mathbf{d}(-f)$, by Theorem 1 this can happen only when there exists $\widehat{x} \in \Delta^n$ such that $H(\widehat{x}, s) = 0$ for some $0 < s < 1$. The result follows by taking $\lambda = s/(1-s)$. $\square$

We note that (7) is trivially satisfied if $g \geqslant 0$ or $f \leqslant 0$ componentwise on $\partial\Delta^n$.

One of our main results is the following Theorem 3, which gives conditions sufficient for $\mathbf{d}(g) \neq \mathbf{d}(-f)$. For its proof we make use of Sperner's lemma and its dual form by Fan which we formulate in terms of boundary conditions (9) and (10) below. Let M^n be a triangulation of Δ^n, ϕ be an admissible labelling of M^n. We consider the following conditions:

$$\phi(v) \in I_+(v) \qquad \text{for all vertices } v \text{ of } M^n \text{ in } \partial\Delta^n; \tag{9}$$

$$\phi(v) \in I_0(v) \qquad \text{for all vertices } v \text{ of } M^n \text{ in } \partial\Delta^n. \tag{10}$$

If (9) holds, then it follows from Sperner's lemma (see Fan [4, Corollary 1]) that

$$\beta(1,\dots,n) \equiv 1 \qquad \text{mod } 2. \tag{11}$$

If we assume in addition to (10) that M^n restricted to $\partial\Delta^n$ is a further subdivision of the (first) barycentric subdivision of $\partial\Delta^n$, then (11) is also true as shown in Fan [4, Corollary 2]. In what follows, we shall suppose this additional assumption whenever (10) is assumed. Thus if ϕ satisfies either (9) or (10), then by the Fan's congruence relation (5),

$$\alpha(-1,\dots,-(n+1)) + \alpha(1,\dots,n+1) \equiv 1 \qquad \text{mod } 2. \tag{12}$$

THEOREM 3. *Let g, $-f \in \mathcal{F}$ satisfy*

(13) *for each $x \in \Delta^n$, there exists $i \in I$ such that $f_i(x)g_i(x) > 0$;*

(14) *for each $x \in \partial\Delta^n$, there exist $i \in I_+(x)$ ($i \in I_0(x)$, respectively), $j \in I$ such that $f_i(x) \leqslant 0$, $f_j(x) > 0$ and $g_j(x) \geqslant 0$. Then $\mathrm{d}(g) \neq \mathrm{d}(-f)$.*

PROOF: There exists $\varepsilon > 0$ with the property that for each $x \in \Delta^n$, there exists $i \in I$ such that either $f_i(x) > 3\varepsilon$ and $g_i(x) > 3\varepsilon$, or $f_i(x) < -3\varepsilon$ and $g_i(x) < -3\varepsilon$; for each $x \in \partial\Delta^n$, there exists $j \in I$ such that $f_j(x) > 3\varepsilon$. Let $\delta > 0$ be such that $d(f(x), f(y)) < \varepsilon$, $d(g(x), g(y)) < \varepsilon$ whenever $d(x,y) < \delta$ and let M^n be a triangulation of Δ^n with mesh less than δ. We define two labellings ϕ and ψ of M^n. Let v be a vertex of M^n. If $v \in \operatorname{int}\Delta^n$, then there exists $i \in I$ such that either $f_i(x) > 2\varepsilon$ and $g_i(x) > 2\varepsilon$ for $x \in \operatorname{st} v$, or $f_i(x) < -2\varepsilon$ and $g_i(x) < -2\varepsilon$ for $x \in \operatorname{st} v$. We define

$$\phi(v) = \begin{cases} i & \text{if } g_i > 2\varepsilon \text{ on } \operatorname{st} v \\ -i & \text{if } g_i < -2\varepsilon \text{ on } \operatorname{st} v \end{cases}, \qquad \psi(v) = -\phi(v).$$

If $v \in \partial\Delta^n$, then there exist $i \in I_+(v)$ ($i \in I_0(v)$, respectively), $j \in I$ such that $f_i(x) < \varepsilon$, $f_j(x) > 2\varepsilon$ and $g_j(x) > -\varepsilon$ for $x \in \operatorname{st} v$. We define

$$\phi(v) = j, \qquad \psi(v) = i.$$

It is easy to verify that ϕ and ψ are admissible labellings generated by g and $-f$ on M^n, respectively. To prove $\mathrm{d}(g) \neq \mathrm{d}(-f)$, since ψ satisfies (9) ((10), respectively), by (12) it suffices to show that

$$\alpha^{\phi}(-1,\dots,-(n+1)) = \alpha^{\psi}(1,\dots,n+1), \tag{15}$$

where the superscript ϕ or ψ indicates the labelling with respect to which the counting is performed. Obviously it follows from the definition above that (15) is true if the equality sign is replaced by the sign $\leqslant$. On the other hand, if σ is an n-simplex of M^n labelled under ψ by $1,\dots,n+1$ at its vertices, then none of the vertices of σ is in $\partial\Delta^n$. Indeed, let σ have a vertex v in $\partial\Delta^n$, and let w be any vertex of σ, possibly v itself, $j=\phi(v)$, $i=\psi(w)$. If $w\in\operatorname{int}\Delta^n$, then $g_j>-\varepsilon$, $g_i<-2\varepsilon$ on σ and so $\phi(v)\neq\psi(w)$; if $w\in\partial\Delta^n$, then $f_j>2\varepsilon$, $f_i<\varepsilon$ on σ and so again $\phi(v)\neq\psi(w)$. It would follow that $\phi(v)\notin\{1,\dots,n+1\}$ which is absurd. Thus σ is labelled under ϕ by $-1,\dots,-(n+1)$ at its vertices. This proves the equality in (15). ∎

We may apply Theorems 2, 3 directly to solve (8) without knowing the exact parity of $\mathbf{d}(g)$ or $\mathbf{d}(-f)$. We may also use Theorem 3 to determine one of $\mathbf{d}(g)$ and $\mathbf{d}(-f)$ if the other is known. Clearly if $f\in\mathcal{F}$ satisfies

$$\text{(16)}\qquad \text{for each } x\in\Delta^n,\ \text{there exists } i\in I \text{ such that } f_i(x)>0,$$

then $\mathbf{d}(f)=0$. This is the case if f maps Δ^n into $\mathbb{R}^{n+1}_+\setminus\{0\}$, in particular if f is the identity function $f(x)=x$ or a constant function $f(x)=a$ with $a\in\mathbb{R}^{n+1}_+\setminus\{0\}$.

Corollary 4. *If $f\in\mathcal{F}$ satisfies one of the following conditions:*

(17) *for each $x\in\partial\Delta^n$, there exist $i\in I_+(x)$ ($i\in I_0(x)$, respectively), $j\in I$ such that $f_i(x)\leqslant 0$ and $f_j(x)>0$;*

(18) *for each $x\in\partial\Delta^n$, there exist $i\in I_+(x)$ ($i\in I_0(x)$, respectively), $j\in I$ such that $f_i(x)>0$ and $f_j(x)\leqslant 0$,*

then $\mathbf{d}(f)\neq\mathbf{d}(-f)$.

Proof: Let $f=g$ in Theorem 3. If (17) holds, then (13) is is trivially satisfied and so the result is a special case of Theorem 3. The proof when (18) is assumed is similar and can be obtained by a modification of the proof of Theorem 3. ∎

Theorem 5. *If $f\in\mathcal{F}$ satisfies (16) and either (17) or (18), then for any continuous function $g:\Delta^n\to\mathbb{R}^{n+1}_+\setminus\{0\}$, there exist $\widehat{x}\in\Delta^n$, $\lambda>0$ such that $\lambda f(\widehat{x})=g(\widehat{x})$.*

Proof: Clearly $\mathbf{d}(g)=0$ and by Corollary 4 $\mathbf{d}(-f)=1$. The result now follows from Theorem 2. ∎

Theorem 5 extends Fan [**2**, Theorem 1] on the equilibrium value of a finite system of convex and concave functions. We refer to Aubin [**1**, Chapter 11] for related results. We can also obtain the coincidence theorems of Fan [**3**, Theorem 2] and [**4**, Lemma] using Theorem 5 by assuming first $g>0$ componentwise and then using a limiting process.

3. Generalised Coincidence Theorems

We fix a compact convex set Y in $\mathbb{R}_+^{n+1} \setminus \{0\}$. Let $\mathcal{G}$ denote the class of upper semicontinuous set-valued maps G defined on Δ^n such that for each $x \in \Delta^n$, the value $G(x)$ is a nonempty closed convex subset of Y. The well-known fixed point theorem of Kakutani [8] states that if $G \in \mathcal{G}$ with $Y = \Delta^n$, then there exists $\widehat{x} \in \Delta^n$ such that $\widehat{x} \in G(\widehat{x})$. Its proof in Kakutani [8], involving approximation by piecewise linear functions, can be adapted to some more general situations (see Ha [5] for one). We obtain in the following Theorem 6 a common generalisation of Kakutani's fixed point theorem and to the set-valued setting of Fan [2, Theorem 1] (see Aubin [1, p.328] for another extension), [3, Theorem 2] and [4, Lemma]. We shall only sketch its proof and refer for further details to Kakutani [8] and Ha [5].

Theorem 6. *Let $G \in \mathcal{G}$, and let $f \in \mathcal{F}$ satisfy (16) and either (17) or (18). Then there exist $\widehat{x} \in \Delta^n$, $\lambda > 0$ such that $\lambda f(\widehat{x}) \in G(\widehat{x})$.*

Proof: For each $m = 1, 2, \ldots$, let T_m be the mth barycentric subdivision of Δ^n. For each vertex v^m of T_m, we choose $w^m \in G(v^m)$ arbitrarily and define $g^m(v^m) = w^m$. By extending g^m linearly inside each n-simplex of T_m, we obtain a continuous function, denoted still by g^m, on Δ^n into $\mathbb{R}_+^{n+1} \setminus \{0\}$. Clearly $\mathrm{d}(g^m) = 0$. By (16) and either (17) or (18), $\mathrm{d}(-f) = 1$ and so by Theorem 5 there exist $x^m \in \Delta^n$, $\lambda_m > 0$ such that $\lambda_m f(x^m) = g^m(x^m)$. For each $m = 1, 2, \ldots$, there exists an n-simplex σ_m of T_m such that $x^m \in \sigma_m$ and $\tau_m = g^m(\sigma_m)$ is a simplex (possibly of lower dimension) in Y. Since Y is compact and f is bounded away from 0, by passing to subsequences if necessary, we may assume that $\{x^m\}$ converges to a point $\widehat{x} \in \Delta^n$, $\lambda_m \to \lambda$ in $\mathbb{R}_+$, and that there exists a simplex $\widehat{\tau}$ in Y whose vertices are the limits of the corresponding sequences of vertices of τ_m, and $\{g^m(x^m)\}$ converges to a point in $\widehat{\tau}$ as $m \to \infty$. It follows from the upper semicontinuity of G and the convexity of $G(\widehat{x})$ that $\widehat{\tau} \subset G(\widehat{x})$. Hence $\lambda f(\widehat{x}) \in G(\widehat{x})$. ∎

By exchanging the roles of f and G, we obtain another generalisation to the set-valued setting of Theorem 5.

Theorem 7. *Let $G \in \mathcal{G}$ satisfying one of the following conditions:*

(19) *for each $x \in \partial\Delta^n$, there exists $y \in G(x)$ such that $y_i = 0$ for all $i \in I_0(x)$;*

(20) *for each $x \in \partial\Delta^n$, there exists $y \in G(x)$ such that $y_i = 0$ for all $i \in I_+(x)$.*

Then for any continuous function $f : \Delta^n \to \mathbb{R}_+^{n+1} \setminus \{0\}$, there exist $\widehat{x} \in \Delta^n$, $\lambda > 0$ such that $\lambda f(\widehat{x}) \in G(\widehat{x})$.

Proof: We show that if g is a piecewise linear approximation of G obtained similarly as in the proof of Theorem 6, then there exist $\widetilde{x} \in \Delta^n$, $\lambda > 0$ such that $\lambda f(\widetilde{x}) \in g(\widetilde{x})$. Let T be a barycentric subdivision of order $\geqslant 1$, and let v be a vertex of T. If $v \in \operatorname{int} \Delta^n$, we choose $w \in G(v)$ arbitrarily and define $g(v) = w$. If $v \in \partial\Delta^n$, we choose $w \in G(v)$ such that $w_i = 0$ for all $i \in I_0(v)$ when (19) holds, and $w_i = 0$ for all $i \in I_+(v)$ when (20) holds, and define $g(v) = w$. By extending g linearly inside each n-simplex of T, we obtain a piecewise linear approximation to G, denoted still by g, which is a continuous function on Δ^n into $\mathbb{R}_+^{n+1} \setminus \{0\}$. Let $x \in \partial\Delta^n$ and let $v^1, \dots, v^m$ be the vertices of T in $\partial\Delta^n$ such that $x = \sum_{k=1}^{m} \alpha_k v^k$, where $m \leqslant n+1$, $\alpha_k > 0$ for $1 \leqslant k \leqslant m$ and $\sum_{k=1}^{m} \alpha_k = 1$. Thus $g(x) = \sum_{k=1}^{m} \alpha_k w^k$. If (19) is assumed, then for $i \in I_0(x)$, $v_i^k = 0$ and so $w_i^k = 0$ for $1 \leqslant k \leqslant m$, that is, $g_i(x) = 0$. If (20) is assumed, then by the construction of the barycentric subdivision T, there exists $i \in I_+(x)$ such that $v_i^k > 0$ and so $w_i^k = 0$ for $1 \leqslant k \leqslant m$, that is, $g_i(x) = 0$. In either case by Theorem 5 (with the rôles of f and g exchanged) there exist $\tilde{x} \in \Delta^n$, $\lambda > 0$ such that $\lambda f(\tilde{x}) = g(\tilde{x})$. The theorem now follows from the same limiting process as in the proof of Theorem 6. $\square$

Theorem 7 is closely related to Fan [**3**, Theorem 2] and [**4**, Lemma], from which some interesting covering properties of simplexes were deduced. A covering property of simplexes important in game theory is the generalisation by Shapley of the classical theorem of Knaster-Kuratowski-Mazurkiewicz (see Aubin [**1**, Theorem 9.15]). As a simple consequence of Theorem 7 when $Y = \Delta^n$ and (20) is assumed, we obtain a new covering property of simplexes of Shapley's type. We denote by $e^1 = (1, \dots, 0)$, $\dots$, $e^{n+1} = (0, \dots, 1)$ the standard basis of $\mathbb{R}^{n+1}$. Let $\mathcal{A}$ be the family of nonempty subsets of I. For $\alpha \in \mathcal{A}$, we denote by Δ_α the face of Δ^n which is the convex hull of $\{e^i : i \in \alpha\}$ so that $\Delta^n = \Delta_I$, and denote by c_α the barycenter of Δ_α.

Theorem 8. *If $\{A_\alpha : \alpha \in \mathcal{A}\}$ is a closed covering of Δ^n such that for each $\alpha \in \mathcal{A} \setminus \{I\}$*

$$\Delta_\alpha \subset \bigcup\{A_\beta : \beta \subset I \setminus \alpha\},$$

then there exists a subfamily $\mathcal{B}$ of $\mathcal{A}$ such that

$$\bigcap\{A_\beta : \beta \in \mathcal{B}\} \neq \emptyset$$

and c_I is in the convex hull of $\{c_\beta : \beta \in \mathcal{B}\}$.

Proof: For $x \in \Delta^n$, let $J(x) = \{\alpha \in \mathcal{A} : x \in A_\alpha\}$. We define

$$G(x) = \operatorname{conv}\{c_\alpha : \alpha \in J(x)\},$$

and define f to be the constant function $f(x) = c_I$. The complement U of $\bigcup\{A_\alpha : \alpha \notin J(x)\}$ in Δ^n is a neighbourhood of x. Moreover, for any $y \in U$, $J(y) \subset J(x)$ and so $G(y) \subset G(x)$. This shows that G is upper semicontinuous on Δ^n. Obviously

(20) is satisfied and so by Theorem 7 there exists $\widehat{x} \in \Delta^n$ such that $f(\widehat{x}) \in G(\widehat{x})$. The theorem now follows by defining $\mathcal{B} = J(\widehat{x})$. $\square$

The well-known Sperner covering property of simplexes as formulated by Fan [3] says that if $\{A_i : i \in I\}$ is a closed covering of Δ^n such that $\Delta_{I\setminus\{i\}} \subset A_i$ for all $i \in I$, then $\bigcap_{i\in I} A_i \neq \emptyset$. Thus Theorem 5 reduces to Sperner's theorem when A_α is not empty only when α is a singleton in $\mathcal{A}$. We refer to Ha [6] and Ichiishi [7] for related results. Finally we note that Shapley's theorem follows similarly from Theorem 7 when (19) is assumed.

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