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THIRD-ENGEL 2-GROUPS ARE SOLUBLE

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ABSTRACT. It is shown that a 3rd-Engel group is an extension of a soluble group by a group of exponent 5.

We shall use standard notation (see for instance Hanna Neumann [5]). Throughout this note G will be a 3rd-Engel group i.e. satisfying the identity $[x, y, y, y]=1$. Our calculations are based on the following four results of Heineken [3]:

(1) G is locally nilpotent (Haupsatz 2);

(2) If G has no element of order 2 or 5 then it is nilpotent of class at most 4 (Haupsatz 1);

(3) Every 2-generator subgroup of G is nilpotent of class at most 4 (Satz 1); and

(4) G satisfies the identity $[x, y, x, y]^2=1$ (Lemma 4).

By (1) and (2) the 5-th term of the lower central series of G is a $\{2, 5\}$-group, so that for some (suitable) integer $n\geq 5$, G satisfies the identity

$$[x_1, \ldots, x_n]^{2.10^n} = 1. \tag{5}$$

Now from (3) and (4) it follows that for any pair of integers $\alpha, \beta\equiv 0(20)$, G satisfies the identity

$$[x^\alpha, y^\beta] = [x, y]^{\alpha\beta}[x, y, y]^{\alpha\binom{\beta}{2}}[x, y, x]^{\binom{\alpha}{2}\beta}. \tag{6}$$

A simple induction on $m\geq 1$ using (6) shows that

(7) For all $m\geq 1$, the identity $[x_1^{20}, \ldots, x_m^{20}]=1$ is a consequence of the identity $[x_1, \ldots, x_m]^{2.10^m}=1$;

and in particular using (5) we have

(8) G satisfies the identity $[x_1^{20}, \ldots, x_n^{20}]=1$ for some $n\geq 5$.

Thus G is an extension of a nilpotent group by a group of exponent 20. By (1) the top group is a direct product of an exponent-4 group by an exponent-5 group. But 3rd-Engel groups of exponent 4 are soluble of length at most 5 (Gupta-Weston [2, Corollary 3]). It follows that G is an extension of a soluble group by a group of exponent 5.

REMARK 1. Heineken's results (1)–(4) are proved using right normed commutator notation. However in 3rd-Engel groups the left-normed and right-normed notations are equivalent (see [4], Lemma 1).

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REMARK 2. Since a soluble 3rd-Engel 5-group is nilpotent (Gruenberg [1, Theorem 1.10]), the Macdonald-Neumann conjecture in [4] is, by our result, equivalent to the existence of a nonsoluble 3rd-Engel group of exponent 5.

REMARK 3. Let H be a group of type $(2n\to 4n-1)$ i.e. all $2n$-generator subgroups are nilpotent of class at most $4n-1$. It is easy to see that H in particular is an extension of a nilpotent-of-class-$(n-1)$ group by a 3rd-Engel group and hence is an extension of a soluble group by a 3rd-Engel group of exponent 5. We also remark that using a slightly different approach it can be shown that groups of type $(n\to 2n-1)$ are soluble-by-exponent-5 and of type $(n\to 2n-2)$ are soluble (nilpotent-by-abelian).

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