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INTEGER-VALUED CONTINUOUS FUNCTIONS II

BY
H. SUBRAMANIAN

We follow [6] and [7] for all terminologies. The purpose of this note is to prove

THEOREM 1. *Let X and Y be any two integer-compact spaces. The following are equivalent*:

(1) *X is homeomorphic to Y.*

(2) *$C(X, \mathbf{Z})$ and $C(Y, \mathbf{Z})$ are isomorphic as rings.*

(3) *$C(X, \mathbf{Z})$ and $C(Y, \mathbf{Z})$ are isomorphic as lattices.*

(4) *$C(X, \mathbf{Z})$ and $C(Y, \mathbf{Z})$ are isomorphic as p.o. groups.*

(5) *$C(X, \mathbf{Z})$ and $C(Y, \mathbf{Z})$ are isomorphic as multiplicative semigroups.*

When X and Y are real-compact spaces, the above result is known with $\mathbf{Z}$ replaced by $\mathbf{R}$ [3]. The above theorem itself was proved in [7] under the assumptions that X and Y are compact.

We digress a little in order to prove the theorem. Let R be a commutative l-semisimple f-ring with unit element, and $\mathscr{M}(R)$ its space of maximal l-ideals with hull-kernel topology. A (proper) lattice-prime ideal P of R is said to be [6] associated with a point $M \in \mathscr{M}(R)$ if

$$y(M) < x(M), x \in P \Rightarrow y \in P, \quad \text{for every } x, y \in R,$$

where $r(M)$ stands for the canonical homomorphic image of $r \in R$ in R/M. Let $[M]$ denote all the lattice-prime ideals of R which are associated with $M \in \mathscr{M}(R)$. We assemble below some known results to facilitate convenient reading.

(A) $\{[M] \mid M \in \mathscr{M}(R)\}$ defines a partition of the set of all lattice-prime ideals of R [6].

(B) The equivalence relation generated by set inclusion gives the same partition [6].

(C) $\mathscr{M}(R)$ is determined by the lattice R [6].

(D) If $R = C(X, \mathbf{Z})$, $M \in \mathscr{M}(R)$, R/M is either isomorphic to $\mathbf{Z}$ or has no countable cofinal subset [1].

(E) If $R = C(X, \mathbf{R})$, $M \in \mathscr{M}(R)$, R/M is either isomorphic to $\mathbf{R}$ or has no countable cofinal subset [2].

(F) A subset S of $C(X, \mathbf{Z})$ is a maximal l-ideal if and only if it is a minimal prime ideal [7].

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Let R be just a commutative ring with unit element in (G) and (H).

(G) A prime ideal M of R is minimal prime if and only if for every $x \in R$, $x \in M$ implies that there exists $y \in R$ such that $y \notin M$ and xy is nilpotent [4].

(H) An ideal of the multiplicative semigroup R is minimal prime if and only if it is a minimal prime ring ideal [4].

Considering each $[M]$, $M \in \mathscr{M}(R)$ as a p.o. set (by set inclusion), we prove

THEOREM 2. *The following are equivalent for any* $M \in \mathscr{M}(R)$.

(1) $[M]$ *has a countable cofinal subset.*

(2) $[M]$ *has a countable subset, whose set union is* R.

(3) R/M *has a countable cofinal subset.*

Proof. For any $r \in R$, let $P(r)=\{x \in R \mid x(M) \leq r(M)\}$. Then $r \in P(r)$ and $P(r) \in [M]$.

(1) $\Rightarrow$ (2). Let $\{P_n\}_{n \in \mathbf{N}}$ be cofinal in $[M]$. For every $r \in R$, we have some $n \in \mathbf{N}$ such that $P(r) \subseteq P_n$. Thus $\bigcup_{n \in \mathbf{N}} P_n = R$.

(2) $\Rightarrow$ (3). Let $\{P_n\}_{n \in \mathbf{N}} \subseteq [M]$ be such that $\bigcup_{n \in \mathbf{N}} P_n = R$. For each $n \in \mathbf{N}$, take some $x_n \in R$ such that $x_n \notin P_n$. Now for any $x \in R$, $x \in P_n$ for some $n \in \mathbf{N}$; then, $x(M) \leq x_n(M)$. Otherwise, $x_n(M) < x(M)$ will imply that $x_n \in P_n$, because $x \in P_n$. Thus $\{x_n(M)\}_{n \in \mathbf{N}}$ is cofinal in R/M.

(3) $\Rightarrow$ (1). Let $\{x_n(M)\}_{n \in \mathbf{N}}$ be cofinal in R/M. Then $\{P(x_n)\}_{n \in \mathbf{N}}$ is cofinal in $[M]$. For, if $P \in [M]$, take some $y \notin P$. We see, as in (2) $\Rightarrow$ (3), that for every $x \in P$, $x(M) \leq y(M)$. Now $y(M) \leq x_n(M)$ for some $n \in \mathbf{N}$. So $P \subseteq P_n$ for that particular $n \in \mathbf{N}$.

Using the results (A)–(E) quoted before, we have

COROLLARY 1. *If X is an integer-compact space, the lattice $C(X, \mathbf{Z})$ determines X.*

COROLLARY 2. *If X is a real-compact space, the lattice $C(X, \mathbf{R})$ determines X.*

REMARK 1. Corollary 1 was a problem unanswered in [7]. Corollary 2 answers the problem in [6], viz. whether this result of Shirota [5] follows from the main result of [6].

REMARK 2. Similar to Theorem 2, we have proved earlier [7] that R/M is non-archimedian if and only if there exists $P \in [M]$ such that P contains all of $\{1, 2, \ldots, n, \ldots\}$. It should be noted however that the archimedian property of R/M as such is not characterized by the lattice R. The required counter-example is the lattice isomorphism between $\mathbf{Q}$ and $\mathbf{Q}[X]$.

Let now R be a commutative ring with unit element and without nonzero nil-potent elements; and, M a given minimal prime ideal of R. We have

THEOREM 3. *The multiplicative semigroup R/M is determined by the multiplicative semigroup R.*

Proof. Consider any $x, y \in R$. By using (G), it can be shown that $x \equiv y$ modulo M if and only if there exists some $r \in R$ such that $r \notin M$ and $xr = yr$. The desired result is immediate.

Taking $R = C(X, \mathbf{Z})$ and $M \in \mathcal{M}(R)$, we see, using (D), that R/M is either isomorphic to $\mathbf{Z}$ or uncountable. (F) and (H) now imply

COROLLARY. *If X is an integer-compact space, the multiplicative semigroup $C(X, \mathbf{Z})$ determines X.*

We conclude with

Proof of Theorem 1. Clearly (1) implies all the other conditions. (2) ⇒ (1) is known [7]. (3) ⇒ (1) and (5) ⇒ (1) are proved above. (4) ⇒ (3) because any order-group isomorphism between two l.o. groups preserves the lattice structures also.

References

1. N. L. Alling, *Rings of continuous integer-valued functions and nonstandard arithmetic*, Trans. Amer. Math. Soc. **118** (1965), 498–525.
2. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960.
3. M. Henriksen, *On the equivalence of the ring, lattice, and semi-group of continuous functions*, Proc. Amer. Math. Soc. **7** (1956), 959–960.
4. J. Kist, *Minimal prime ideals in commutative semigroups*, Proc. London Math. Soc. **13** (1963), 31–50.
5. T. Shirota, *A generalization of a theorem of I. Kaplansky*, Osaka J. Math. **4** (1952), 121–132.
6. H. Subramanian, *Kaplansky's theorem for f-rings*, Math. Ann. **179** (1968), 70–73.
7. ———, *Integer-valued continuous functions*, Bull. Soc. Math. France **97** (1969).

STATE UNIVERSITY OF NEW YORK,
AMHERST, NEW YORK

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