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DENSE SUBALGEBRAS OF LEFT HILBERT ALGEBRAS

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Let M be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and assume that M has a separating and cyclic vector ω in $\mathscr{H}$. Then it can happen that M contains a proper von Neumann subalgebra N for which ω is still cyclic. Such an example was given by Kadison in [**4**]. He considered $M = \mathscr{B}(\mathscr{H}) \otimes \mathscr{B}(\mathscr{H}) \otimes 1$ and $N = \mathscr{B}(\mathscr{H}) \otimes 1 \otimes 1$ acting on $\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}$ where $\mathscr{H}$ is a separable Hilbert space. In fact by a result of Dixmier and Maréchal, M, M' and N have a joint cyclic vector [**3**]. Also Bratteli and Haagerup constructed such an example ([**2**], example 4.2) to illustrate the necessity of one of the conditions in the main result of their paper. In fact this situation seems to occur rather often in quantum field theory (see [**1**] Section 24.2, [**3**] and [**4**]).

While investigating the different conditions in our fixed point theorem [**7**] we ran into the same type of problem, but in the more general framework of left Hilbert algebras. The question was this: given a left Hilbert algebra $\mathscr{A}$, is it possible to construct a left Hilbert subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{B}$ is dense in $\mathscr{A}$ for the Hilbert space norm but such that the left von Neumann algebra $\mathscr{L}(\mathscr{B})$ of $\mathscr{B}$ is properly contained in the left von Neumann algebra $\mathscr{L}(\mathscr{A})$? It is clear that this occurs if $\mathscr{A} = M\omega$ and $\mathscr{B} = N\omega$ with M, N and ω as above.

From the theory of left Hilbert algebras we know that the problem can also be stated as follows. For a given left Hilbert algebra $\mathscr{A}$, is there a left Hilbert subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{B}$ is dense in $\mathscr{A}$ for the Hilbert space norm but not for the norm in $\mathscr{D}^\#$ given by

$$\|\xi\|_\# = (\|\xi\|^2 + \|\xi^\#\|^2)^{1/2} \quad \text{for } \xi \in \mathscr{A}$$

(see [**6**], page 24)? This is important because it shows that it will not be possible to find such a subalgebra when the $^\#$-operation is bounded. And in particular if ω is a separating and cyclic trace vector for the von Neumann algebra M, it cannot be cyclic anymore for a proper von Neumann subalgebra (see also [**5**], Section 4).

If the $^\#$-operation is not bounded it is not so difficult to construct subspaces of $\mathscr{A}$, dense for the Hilbert space norm, but not for the $^\#$-norm. The situation is not so simple if the subspace is required to be a subalgebra of $\mathscr{A}$. In this note we prove a result in this direction. We only

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consider the case where $\mathcal{L}(\mathcal{A})$ is isomorphic to the algebra of all bounded operators on a separable Hilbert space. This was precisely the situation we had to deal with in the study of the conditions in our fixed point theorem. And we believe it may also be useful in other situations to construct counter examples.

Our theorem is proved using a result on non-singular positive self-adjoint unbounded operators, but we must say that all together we were very much inspired by the example of Bratteli and Haagerup. We will work with a Hilbert space $\mathcal{H}$ and we will consider the conjugate space $\overline{\mathcal{H}}$ as the set $\mathcal{H}$ with the same addition but with scalar multiplication $(\lambda, \xi) \rightarrow \bar{\lambda}\xi$ and scalar product $(\xi, \eta) \rightarrow \overline{\langle \xi, \eta \rangle}$. Then the following result is standard.

1. PROPOSITION. *Let h be a non-singular positive self-adjoint operator on $\mathcal{H}$ and $\mathcal{A}$ the subspace of $\mathcal{H} \otimes \overline{\mathcal{H}}$ spanned by the vectors $\xi \otimes \eta$ where $\xi \in \mathcal{D}(h)$ and $\eta \in \mathcal{D}(h^{-1})$. Then $\mathcal{A}$ is a left Hilbert algebra if the product and the involution are defined by*

$$(\xi_1 \otimes \eta_1)(\xi_2 \otimes \eta_2) = \langle \xi_2, h^{-1}\eta_1 \rangle \xi_1 \otimes \eta_2$$
$$(\xi \otimes \eta)^\# = h^{-1}\eta \otimes h\xi$$

when $\xi, \xi_1, \xi_2 \in \mathcal{D}(h)$ and $\eta, \eta_1, \eta_2 \in \mathcal{D}(h^{-1})$.

It is also well known (and easy to check) that $\mathcal{L}(A) = \mathcal{B}(\mathcal{H}) \otimes 1$ and that the operators J and $\Delta^{1/2}$ are given by

$$J(\xi \otimes \eta) = \eta \otimes \xi \quad \text{and} \quad \Delta^{1/2}(\xi \otimes \eta) = h\xi \otimes h^{-1}\eta$$

if $\xi \in \mathcal{D}(h)$ and $\eta \in \mathcal{D}(h^{-1})$.

We will prove the following result.

2. THEOREM. *Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{A}$ be the left Hilbert algebra associated with a non-singular positive self-adjoint operator h as above. Then there exists a left Hilbert subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ is dense in $\mathcal{A}$ for the Hilbert space norm but $\mathcal{L}(\mathcal{B})$ is properly contained in $\mathcal{L}(\mathcal{A})$ if and only if h^{-1} is unbounded.*

Our proof will be based on the following property of a non-singular positive self-adjoint operator. This result is essentially contained in [**8**].

3. PROPOSITION. *If h is a non-singular positive self-adjoint operator on a separable Hilbert space $\mathcal{H}$ such that h^{-1} is unbounded, then there exist two subspaces $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\mathcal{D}(h)$ such that $\mathcal{D}_1 \perp \mathcal{D}_2$ and $\mathcal{D}_1 + \mathcal{D}_2$ is dense, and such that $h\mathcal{D}_1$ and $h\mathcal{D}_2$ also are dense.*

Proof. Since details can be found in [**8**] we only give the main steps. Using spectral theory we can write $h = ak$ where k is a non-singular positive self-adjoint operator with pure point spectrum and a is a bounded self-adjoint operator such that $1 \leqq a \leqq 2$. And if $\mathcal{D}_1$ and $\mathcal{D}_2$ are subspaces of $\mathscr{D}(k)$ such that $\mathscr{D}_1 \perp \mathscr{D}_2$ and $\mathscr{D}_1 + \mathscr{D}_2$ is dense, and such that also $k\mathscr{D}_1$ and $k\mathscr{D}_2$ are dense, they also will be a required pair for h. Therefore we may assume that h has pure point spectrum.

So suppose that $\{e_n | n = 1, 2, \ldots\}$ is an orthonormal basis for $\mathscr{H}$ such that $he_n = p_n e_n$ where $p_n > 0$ for all n. Then we can partition the basis vectors into a sequence $\{e_{nj} | j \in \mathbf{Z}\}$ with the property that $he_{nj} = p_{nj} e_{nj}$ for all n and j and such that $p_{nj} \leqq \exp(-j^2)$ if $j \neq 0$ for all n. Here it is needed that h^{-1} is unbounded. Now suppose that we can find a pair of subspaces $\mathscr{D}_1^n$ and $\mathscr{D}_2^n$ for each of the restrictions h_n of h to the subspaces spanned by the vectors $\{e_{nj} | j \in \mathbf{Z}\}$ satisfying our requirements with respect to h_n. Then it is clear that the spaces spanned by $\{\mathscr{D}_1^n | n = 1, 2, \ldots\}$ and $\{\mathscr{D}_2^n | n = 1, 2, \ldots\}$ respectively will be the right spaces for h.

So we may suppose that there exists an orthonormal basis $\{e_n | n \in \mathbf{Z}\}$ such that h is given by $he_n = p_n e_n$ where $p_n > 0$ for all n and $p_n \leqq \exp(-n^2)$ if $n \neq 0$. If now

$$\mathscr{D}_1 = \left\{ \sum_{n \in \mathbf{Z}} \lambda_n e_n \,\middle|\, \lambda \in l^2(\mathbf{Z}),\ \sum_{n \in \mathbf{Z}} \lambda_n e^{in\theta} = 0 \text{ a.e. in } [-\pi, 0] \right\}$$

$$\mathscr{D}_2 = \left\{ \sum_{n \in \mathbf{Z}} \lambda_n e_n \,\middle|\, \lambda \in l^2(\mathbf{Z}),\ \sum_{n \in \mathbf{Z}} \lambda_n e^{in\theta} = 0 \text{ a.e. in } [0, \pi] \right\}$$

then it is clear that $\mathscr{D}_1$ and $\mathscr{D}_2$ are closed and each others' orthogonal complement. To prove that $h\mathscr{D}_1$ is dense, assume

$$\mu \in l^2(\mathbf{Z}) \quad \text{and} \quad \sum_{n \in \mathbf{Z}} \mu_n e_n \perp h\mathscr{D}_1.$$

Then

$$\sum_{n \in \mathbf{Z}} \mu_n p_n e_n \perp \mathscr{D}_1$$

so that

$$\sum_{n \in \mathbf{Z}} \mu_n p_n e_n \in \mathscr{D}_2 \quad \text{and} \quad \sum_{n \in \mathbf{Z}} \mu_n p_n e^{in\theta} = 0 \text{ a.e. in } [0, \pi].$$

However

$$f(z) = \sum_{n \in \mathbf{Z}} \mu_n p_n e^{inz}$$

defines an analytic function in all of $\mathbf{C}$. And because it is zero almost everywhere in $[0, \pi]$ it must be zero by the identity theorem. This proves that $\sum_{n \in \mathbf{Z}} \mu_n e_n = 0$. Similarly also $h\mathscr{D}_2$ is dense.

We now come to the proof of our theorem.

Proof of Theorem 2. First suppose that h^{-1} is unbounded. Then by Proposition 3 we can find subspaces $\mathscr{D}_1$ and $\mathscr{D}_2$ of $\mathscr{D}(h)$ such that $\mathscr{D}_1 \perp \mathscr{D}_2$ and $\mathscr{D}_1 + \mathscr{D}_2$ as well as $h\mathscr{D}_1$ and $h\mathscr{D}_2$ are dense. Let $\mathscr{B}$ be the subspace of $\mathscr{A}$ of linear combinations of vectors of the form $\xi_1 \otimes h\eta_1$ with $\xi_1, \eta_1 \in \mathscr{D}_1$ and of the form $\xi_2 \otimes h\eta_2$ with $\xi_2, \eta_2 \in \mathscr{D}_2$.

Because $(\xi \otimes h\eta)^\# = \eta \otimes h\xi$ for all $\xi, \eta \in \mathscr{D}(h)$ we have that $\mathscr{B}$ is closed under the $^\#$-operation. Because also

$$(\xi_1 \otimes h\eta_1)(\xi_2 \otimes h\eta_2) = \langle \xi_2, \eta_1 \rangle \xi_1 \otimes h\eta_2$$

whenever $\xi_1, \eta_1, \xi_2, \eta_2 \in \mathscr{D}(h)$ and because $\mathscr{D}_1 \perp \mathscr{D}_2$ we also have that $\mathscr{B}$ is a subalgebra.

To prove that $\mathscr{B}$ is dense in $\mathscr{A}$ for the Hilbert space norm, consider $\xi \otimes \eta$ with $\xi, \eta \in \mathscr{H}$. Because $\mathscr{D}_1 + \mathscr{D}_2$ is dense we can approximate $\xi \otimes \eta$ by sums of vectors $\xi_1 \otimes \eta$ and $\xi_2 \otimes \eta$ with $\xi_1 \in \mathscr{D}_1$ and $\xi_2 \in \mathscr{D}_2$. And since $h\mathscr{D}_1$ is dense we can approximate $\xi_1 \otimes \eta$ by vectors $\xi_1 \otimes h\eta_1$ with $\xi_1, \eta_1 \in \mathscr{D}_1$ and because also $h\mathscr{D}_2$ is dense we can approximate $\xi_2 \otimes \eta$ by vectors $\xi_2 \otimes h\eta_2$ with $\xi_2, \eta_2 \in \mathscr{D}_2$.

On the other hand, if $\xi, \eta \in \mathscr{D}(h)$, then left multiplication by $\xi \otimes h\eta$ is of the form $x \otimes 1$ where x is the rank one operator on $\mathscr{H}$ defined by $x\zeta = \langle \zeta, \eta \rangle \xi$. So if p is the projection onto the closure $\overline{\mathscr{D}}_1$ of $\mathscr{D}_1$ then

$$\pi(\xi_1 \otimes h\eta_1) \in p\mathscr{B}(\mathscr{H})p \otimes 1 \quad \text{if } \xi_1, \eta_1 \in \mathscr{D}_1 \quad \text{and}$$
$$\pi(\xi_2 \otimes h\eta_2) \in (1 - p)\mathscr{B}(\mathscr{H})(1 - p) \otimes 1 \quad \text{if } \xi_2, \eta_2 \in \mathscr{D}_2.$$

So we will have

$$\mathscr{L}(\mathscr{B}) = (p\mathscr{B}(\mathscr{H})p + (1 - p)\mathscr{B}(\mathscr{H})(1 - p)) \otimes 1$$

while

$$\mathscr{L}(\mathscr{A}) = \mathscr{B}(\mathscr{H}) \otimes 1.$$

This proves one direction.

To prove the other implication assume that h^{-1} is bounded. We claim that then the operator norm is majorized by a multiple of the Hilbert space norm. Then of course if $\mathscr{B}$ is dense in $\mathscr{A}$, also $\pi(\mathscr{B})$ will be dense in $\pi(\mathscr{A})$ so that $\mathscr{L}(\mathscr{B}) = \mathscr{L}(\mathscr{A})$. To prove the claim let $\sum_{i=1}^n \xi_i \otimes \eta_i$ be any element in $\mathscr{A}$, so $\xi_i \in \mathscr{D}(h)$ and $\eta_i \in \mathscr{H}$ (as h^{-1} is bounded). Left multiplication by this element is given by $x \otimes 1$ where

$$x\zeta = \sum_{i=1}^n \langle \zeta, h^{-1}\eta_i \rangle \xi_i$$

for all $\zeta \in \mathscr{H}$. We may assume that $\{\xi_1, \xi_2, \dots, \xi_n\}$ is an orthonormal subset in $\mathscr{D}(h)$ and we obtain

$$\|x\zeta\| = \left(\sum_{i=1}^n |\langle \zeta, h^{-1}\eta_i \rangle|^2 \right)^{1/2} \leqq \|\zeta\| \, \|h^{-1}\| \left(\sum_{i=1}^n \|\eta_i\|^2 \right)^{1/2}$$
$$= \|\zeta\| \, \|h^{-1}\| \left\| \sum_{i=1}^n \xi_i \otimes \eta_i \right\|.$$

Therefore

$$\|x\| \leqq \|h^{-1}\| \left\| \sum_{i=1}^{n} \xi_i \otimes \eta_i \right\|$$

and the proof is complete.

The theorem shows that the unboundedness of the $^{\#}$-operation is not sufficient for the existence of such a subalgebra.

We have formulated our result only for separable Hilbert spaces. It is clear that in the non-separable case the last part of the proof is still valid. The other direction however uses the result of Proposition 3, and the proof of this proposition depends highly on the separability condition. In fact as we show below, Proposition 3 is wrong in the non-separable case, and it is not at all clear what happens with the theorem in that case. To see that Proposition 3 is wrong in the non-separable case, assume that $\mathscr{H}$ is a non-separable Hilbert space and that h is a non-singular positive self-adjoint operator such that there is a projection E with separable range and such that $h(1 - E) = 1 - E$; in other words $h = 1$ except on a separable subspace. Assume that there are subspaces $\mathscr{D}_1$ and $\mathscr{D}_2$ of $\mathscr{D}(h)$ such that $\mathscr{D}_1 \perp \mathscr{D}_2$ and $\mathscr{D}_1 + \mathscr{D}_2$, $h\mathscr{D}_1$ and $h\mathscr{D}_2$ are dense. Denote by F the projection on the closure of $\mathscr{D}_1$. Also let G_1 be the support projection of EF and G_2 the support projection of $E(1 - F)$. Then $G_1 \leqq F$ and $G_2 \leqq 1 - F$ and $G = G_1 + G_2$ defines another projection. Because E has separable range, also G_1 and G_2 must have separable range, and in particular $G \neq 1$. By definition $EF(1 - G_2) = 0$ and also $EF(1 - G) = 0$ as $G_2 \leqq G$. Similarly $E(1 - F)(1 - G) = 0$ and by adding the two equalities we get $E(1 - G) = 0$. Because $h\mathscr{D}_1$ is dense in $\mathscr{H}$ we have $(1 - G)h\mathscr{D}_1$ is dense in $(1 - G)\mathscr{H}$. But

$$(1 - G)h = (1 - G)(1 - E)h \leqq 1 - G$$

so that

$$(1 - G)h\mathscr{D}_1 \subseteq (1 - G)F\mathscr{H}$$

and we obtain that $(1 - G)F\mathscr{H}$ is dense in $(1 - G)\mathscr{H}$. Because F and G commute it follows that $(1 - G) \leqq F$. Similarly because $h\mathscr{D}_2$ is dense we will obtain $(1 - G) \leqq 1 - F$ so that $1 - G = 0$ and $G = 1$. This is a contradiction.

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