



Arithmetic of Degenerating Principal Variations of Hodge Structure: Examples Arising From Mirror Symmetry and Middle Convolution

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Abstract. We collect evidence in support of a conjecture of Griffiths, Green, and Kerr on the arithmetic of extension classes of limiting mixed Hodge structures arising from semistable degenerations over a number field. After briefly summarizing how a result of Iritani implies this conjecture for a collection of hypergeometric Calabi–Yau threefold examples studied by Doran and Morgan, the authors investigate a sequence of (non-hypergeometric) examples in dimensions $1 \leq d \leq 6$ arising from Katz’s theory of the middle convolution. A crucial role is played by the Mumford–Tate group (which is G_2) of the family of 6-folds, and the theory of boundary components of Mumford–Tate domains.

1 Introduction

Absolutely irreducible \mathbb{Q} -local systems can underlie at most one polarized variation of Hodge structure, which suggests that the asymptotics of such variations at a puncture should exhibit interesting arithmetic. For variations of motivic origin, one envisions arithmetic constraints on the extension classes (periods) of the limiting mixed Hodge structures, (*cf.* Conjecture 2.4). The Mumford–Tate group G of the VHS imposes its own algebraic constraints upon these extensions, which can simplify the form of the conjecture. Given a G -rigid local system, the middle convolutions of Katz [Ka] give some hope for constructing a family of motives with the local system (and VHS) appearing in its cohomology.

This paper was motivated by the desire to check the conjectural property for some local systems on the thrice-punctured sphere underlying motivic VHS of type $(1, 1, 1, 1)$ and $(1, 1, 1, 1, 1, 1, 1)$ at a point of maximal unipotent monodromy. Variations with extremal Hodge numbers 1 have for some time been called *Calabi–Yau*; when all the Hodge numbers are 1, terminology from representation theory (“principal sl_2 ”) is used.

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suggests calling them *principal*. Robles’s recent classification [Ro] of the corresponding Hodge representations rules out all exceptional groups except for G_2 as Mumford–Tate group, which itself can only occur in the weight/level 6 case. Moreover, the effects of G_2 on the limiting MHS are well understood via boundary components [KP], a story which is briefly reviewed in Section 3.

Our first main point is that a recent result of Iritani in mirror symmetry [Ir] allows one to compute the limiting extension classes for many of the $(1, 1, 1, 1)$ examples classified by Doran and Morgan [DM]. If X° is a complete intersection CY threefold in a weighted projective space with rank 4 even cohomology, we use Iritani’s $\widehat{\Gamma}$ integral structure on its quantum cohomology to give a straightforward computation of the (large complex structure) limiting period matrix Ω_{lim} of the VHS arising from H^3 of the mirror family (cf. (4.1)). In particular, the nontorsion extension class $\varepsilon \in \mathbb{C}/\mathbb{Q}(3) \cong \text{Ext}_{\text{MHS}}^1(\mathbb{Q}(-3), \mathbb{Q}(0))$ is given by $(\int_{X^\circ} c_3(X^\circ))\zeta(3)$.

The second point is that using middle convolution, one may construct interesting motivic variations not treatable by mirror symmetry, but where the limiting periods may be computed directly by a residue method. This approach, which we apply in Section 5 to two examples (including one with G_2 monodromy) from the work of Dettweiler and Reiter [DR1], shows promise more generally for cyclic covers branched along a union of hyperplanes. Moreover, it gives a clearer picture of the origin of the zeta values in limiting extension classes, which is buried in a deep mirror theorem in Section 4. The main idea is that Katz’s method builds a sequence of families $X_d(t)$ ($d \geq 1$) of the form $w^2 = f(x_1, \dots, x_d, t)$, whose $\log^d t$ period in a neighborhood of the point $t = 0$ of maximal unipotence is given by the iterative formula

$$\pi_{2j}(t) := i \int_t^1 \frac{\pi_{2j-1}(x) dx}{\sqrt{x(1-x)(1-\frac{t}{x})}}, \quad \pi_{2j+1}(t) := i \int_t^1 \frac{\pi_{2j}(x) dx}{x\sqrt{1-\frac{t}{x}}},$$

where $\pi_1(t) := 2 \int_1^{\frac{1}{t}} \frac{dx}{\sqrt{(1-tx)(1-x)x}}$. The top weight graded piece of $H^d(X_d(t))$ contains a principal variation (at least, for $d \leq 7$), and all the data of Ω_{lim} for this VHS is contained in the asymptotics of $\pi_d(t)$.

For $d = 3$, we are able to completely determine these asymptotics (Theorem 5.6), and hence the extension class $\varepsilon = -48\zeta(3) \in \mathbb{C}/\mathbb{Q}(3)$. Our luck did not hold out for $d = 6$ (Section 5.5), where we were only able to compute “part” of the integral; however, this piece does contain a term of the form $-72\zeta(5)$ as expected (“towards” the extension class in $\text{Ext}_{\text{MHS}}^1(\mathbb{Q}(-5), \mathbb{Q}(0))$), and Conjecture 5.7 represents an educated guess at the entire thing. Moreover, our partial computation contains a ravishing number-theoretic tidbit (Lemma 5.5), which we call the G_2 -identity:

$$\begin{aligned} & \sum' \frac{\binom{\frac{1}{2}}{k_1} \binom{\frac{1}{2}}{k_2} \binom{\frac{1}{2}}{a} \binom{\frac{1}{2}}{b}}{(b-a+\frac{1}{2})(b+k_1+\frac{1}{2})(a+k_1)(a+k_2)} \\ & + \sum' \frac{\binom{\frac{1}{2}}{k_1} \binom{\frac{1}{2}}{k_2} \binom{\frac{1}{2}}{a} \binom{\frac{1}{2}}{b}}{(b-a+\frac{1}{2})(b+k_1+\frac{1}{2})(b+k_2+\frac{1}{2})(a+k_1)} \\ & - \sum' \frac{\binom{\frac{1}{2}}{a} \binom{\frac{1}{2}}{b}}{(b+a+\frac{1}{2})(b+\frac{1}{2})a^2} = \frac{64\pi}{3} \log^3 2 + \frac{2\pi^3}{3} \log 2 - 12\pi\zeta(3). \end{aligned}$$

(Here \sum' means to sum over tuples of non-negative integers for which the summand is defined.) This identity is a consequence of the vanishing of the “third extension class” in the maximal unipotent LMHS of a principal variation with G_2 Mumford-Tate group, and (bizarrely) it is needed to finish off the $d = 3$ computation. Our computations also make heavy use of some identities relating hypergeometric special values and Riemann zeta values whose derivation is outlined in the Appendix.

In writing this paper we encountered several questions that merit further investigation. For example, is there a direct construction of “limiting data”, from a quasi-unipotent G -rigid local system \mathbb{V} over $\mathbb{P}^1 \setminus \text{pts.}$, that does not pass through variations of Hodge structure? Here the motivation is that there should exist a unique VHS underlain by \mathbb{V} , (*cf.* Remark 2.3). In the middle convolution case, is there a better approach to computing the LMHS than direct computation of asymptotics of $\pi_d(t)$, perhaps one extending the computation of the LMHS Hodge numbers in [DS]? Finally, can one use mirror symmetry to compute any limiting extension classes in $\text{Ext}_{\text{MHS}}^1(\mathbb{Q}(-5), \mathbb{Q}(0))$? Here the problem is typically that some extension class in $\text{Ext}_{\text{MHS}}^1(\mathbb{Q}(-3), \mathbb{Q}(0))$ is nonzero, and then one of the previous form is not well defined; but even in this case it would still be of interest to compute Ω_{lim} (say, for CY 5- or 6-folds).

2 Local Systems and Limiting Mixed Hodge Structures

Let \mathcal{S} be a complex algebraic manifold and fix a base point $s_0 \in \mathcal{S}$. Given an absolutely irreducible \mathbb{Q} -local system \mathbb{V} over \mathcal{S} , put $V := \mathbb{V}_{s_0}$ and define the monodromy group

$$\Gamma := \text{image}(\rho: \pi_1(\mathcal{S}, s_0) \rightarrow GL(V)).$$

The geometric monodromy group Π is the identity connected component of its \mathbb{Q} -Zariski closure.

Now suppose there exists a polarized variation of Hodge structure (PVHS) $\mathcal{V} = (\mathbb{V}, Q, \mathcal{F}^\bullet)$ over \mathbb{V} of weight n .¹

Proposition 2.1 *Up to Tate twist, \mathcal{V} is unique.*

Proof Given $\mathcal{V}, \mathcal{V}'$ PVHS over \mathbb{V} , $\mathcal{V}^\vee \otimes \mathcal{V}'$ is a PVHS over $\mathbb{V}^\vee \otimes \mathbb{V}$, and by Schur’s lemma $(\mathbb{V}^\vee \otimes \mathbb{V})^\Gamma = \mathbb{Q}\langle \text{id}_{\mathbb{V}} \rangle$. By the Theorem of the Fixed Part [Sc], $\mathbb{Q}\langle \text{id}_{\mathbb{V}} \rangle$ therefore underlies a constant sub-VHS of $\mathcal{V}^\vee \otimes \mathcal{V}'$, rank 1 hence of type (p, p) . ■

As Griffiths puts it, Riemann would be proud: this sort of result goes back to his characterization of the hypergeometric functions by their local monodromy about $0, 1, \infty$. Note that the existence of \mathcal{V} implies that \mathbb{V} is semisimple with quasi-unipotent monodromies.

¹ \mathcal{V} will also sometimes denote, by abuse of notation, the locally free sheaf $\mathbb{V} \otimes \mathcal{O}_{\mathcal{S}}$ (or the corresponding vector bundle).

Assume now that \mathcal{S} has a smooth compactification $\overline{\mathcal{S}}$ with a holomorphic disk embedding

$$\begin{array}{ccccc} \mathcal{S} & \subset & \overline{\mathcal{S}} & \ni & x \\ \cup & & \cup & & \uparrow \\ \Delta^* & \xrightarrow{J} & \Delta & \ni & 0 \end{array}$$

and let s be a choice of local coordinate on Δ . Restricting \mathcal{V} to Δ^* , we assume the local monodromy T is unipotent and set

$$N := \log(T) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} (T - I)^k \in \text{End}(V, Q).$$

There exists a unique increasing filtration $W_\bullet = W(N)_\bullet$ on V such that (for all k),

$$\begin{aligned} N(W_k V) &\subset W_k V, \\ N^\ell: \text{Gr}_{n+k}^W V &\xrightarrow{\cong} \text{Gr}_{n-k}^W V. \end{aligned}$$

Moreover, the “untwisted” local system $\widetilde{\mathcal{V}} := J_* (e^{-\ell(s)N} \mathbb{V})$ (where $\ell(s) := \frac{\log(s)}{2\pi i}$) extends to Δ . By the Nilpotent Orbit Theorem [Sc], the Hodge sheaves $\mathcal{F}^p \subset \mathcal{V}$ extend to locally free subsheaves $\mathcal{F}_e^p \subset \mathcal{V}_e := \widetilde{\mathcal{V}} \otimes \mathcal{O}_\Delta$ on Δ , and the SL_2 -orbit Theorem [op. cit.] implies the following proposition.

Proposition 2.2 $(\psi_s \mathcal{V})_x := (\widetilde{\mathcal{V}}, W_\bullet, \mathcal{F}_e^\bullet)|_x$ is a mixed Hodge structure polarized by N , called the limiting mixed Hodge structure (LMHS).

Writing $F_{\text{lim}}^\bullet := \mathcal{F}_e^\bullet|_x$, we will denote by $\mathcal{V}_{\text{nilp}} := (\mathbb{V}, Q, e^{-\ell(s)N} F_{\text{lim}}^\bullet)$ the associated nilpotent orbit, which is again a VHS over (a possibly smaller) punctured disk Δ^* .

We conclude that to an absolutely irreducible local system on \mathcal{S} , point $x \in \overline{\mathcal{S}}$, and local coordinate $s \in \mathcal{O}(\Delta)$, Schmid’s results associate a MHS. The extension classes inherent in the latter are thereby already in this sense *invariants of the local system*.

Remark 2.3 Of course, this begs the question as to which local systems underlie a PVHS! It is expected (cf. [DR2]) that quasi-unipotency and G -rigidity² (for some $G \leq GL(V)$ containing Γ) suffice for \mathbb{V} to underlie a *motivic* PVHS, that is, one arising from a family of varieties over \mathcal{S} . For $\mathcal{S} = \mathbb{P}^1 \setminus \text{pts.}$ and $G = GL(V)$, this is proved by Katz [Ka] using his middle convolution algorithm, which we touch on in Section 5.

Recall next that a motive over a field $k \subseteq \mathbb{C}$ is, roughly speaking, a bounded complex of smooth quasi-projective varieties with arbitrary maps between them, all defined over k . Through a “realization” process similar to hypercohomology, one can take the various cohomology groups of such a complex, which yields in particular (from de Rham and Betti) a MHS we will say to be k -motivated.

²We say that \mathbb{V} is G -rigid if the G -orbit of the associated monodromy representation ρ is open in $\text{Hom}(\pi_1(\mathcal{S}), G)$.

Now assume that $\mathbb{V} = R^k f_* \mathbb{Q}_{\mathcal{X}}$ arises from the following situation, called a *semi-stable degeneration (SSD)* over k :

$$\begin{array}{ccccc} \mathcal{X} & \hookrightarrow & \overline{\mathcal{X}} & \hookleftarrow & X_0 = \overline{f}^{-1}(x) \\ \downarrow f & & \downarrow \overline{f} & & \downarrow \\ \mathcal{S} & \hookrightarrow & \overline{\mathcal{S}} & \ni & x \\ & & \downarrow g & & \\ & & \mathbb{P}^1 & & \end{array}$$

where \overline{f} is proper and flat, f is smooth, $g(x) = 0$ with $\text{ord}_x(g) = 1$, and $X_0 := (g \circ \overline{f})^{-1}(0) = \cup Y_i$ is a reduced SNCD. (That is, the intersections $Y_I := \cap_{i \in I} Y_i$ are smooth and transversal.) Moreover, the entire diagram, and all inclusions $Y_I \hookrightarrow Y_{I \setminus i}$, are defined over k .

Conjecture 2.4 ([GGK]) *Let s be the restriction of g to a disk about x . Then $(\psi_s \mathbb{V})_x$ is k -motivated. In particular, the extension class $\varepsilon \in \mathbb{C}/\mathbb{Z}(m)$ of any Tate subquotient $0 \rightarrow \mathbb{Z}(0) \rightarrow \mathbb{E} \rightarrow \mathbb{Z}(-m) \rightarrow 0$ belongs to the image of motivic cohomology $H_M^1(\text{Spec}(k), \mathbb{Z}(m))$ under the generalized Abel–Jacobi mapping.*

If $k = \mathbb{Q}$, this says that

$$\varepsilon \equiv \begin{cases} \log(a), a \in \mathbb{Q}^* & (m = 1), \\ q\zeta(m), q \in \mathbb{Q} & (m > 1); \end{cases}$$

note that $\zeta(m) \in \mathbb{C}/\mathbb{Z}(m)$ is torsion if m is even. In what follows we will be working with rational coefficients, and hence interested only in odd m .

Remark 2.5 Several constructions of limiting motives have recently appeared in the literature, for instance [Le]. Conjecture 2.4 would probably follow from the assertion (itself still conjectural) that the Hodge realization of Levine’s motive is the LMHS.

The existence (up to torsion) of a Tate subquotient is an algebraic requirement; it is useful at this point to consider what algebraic conditions might produce it, which brings us to the next section.

3 Mumford–Tate Domains and Boundary Components

Let V be a (finite-dimensional) \mathbb{Q} -vector space, $Q: V \times V \rightarrow \mathbb{Q}$ a $(-1)^n$ -symmetric nondegenerate bilinear form. Writing $S^1 \subset \mathbb{C}^*$ for the unit circle, consider a homomorphism $\phi: S^1 \rightarrow SL(V_{\mathbb{R}})$ with $\phi(-1) = (-1)^n \cdot \text{id}_V$.

Definition 3.1 (i) (V, Q, ϕ) is a *polarized Hodge structure (PHS)*³ if

$$\phi(S^1) \subset \text{Aut}(V, Q) \quad \text{and} \quad Q(v, \phi(i)\bar{v}) > 0 \quad (\forall v \in V_{\mathbb{C}} \setminus \{0\}).$$

(ii) The *Mumford–Tate group (MTG)* G_{ϕ} is the \mathbb{Q} -algebraic group closure of $\phi(S^1)$.

³It is implicit here that the PHS is “pure of weight n ”, even though the definition only records the parity of n .

The $\phi(S^1)$ -fixed points in the tensor spaces $T^{k,l}V := V^{\otimes k} \otimes (V^\vee)^{\otimes l}$ are the Hodge tensors $Hg^{k,l}V$.

Proposition 3.2 ([De]) *G_ϕ is the subgroup of $GL(V)$ fixing $\oplus Hg^{k,l}V$ pointwise.*

The Lie group of real points of the MTG acts by conjugation on ϕ , and the (connected) Mumford–Tate domain (MTD) associated with ϕ is the orbit (under the identity connected component)

$$D := G_\phi(\mathbb{R})^+ \cdot \phi \cong G_\phi(\mathbb{R})^+ / \mathcal{H}_\phi.$$

By taking $V_{\tau_\phi}^{p,n-p} \subset V_{\mathbb{C}}$ to be the z^{p-q} -eigenspace for $'\phi(z)$ ($'\phi \in D$), D can be viewed as (a connected component of) the locus in some period domain on which a finite set of tensors in $\oplus T^{k,l}V$ becomes Hodge. Moreover, using the Hodge flags $F_{\tau_\phi}^\bullet V_{\mathbb{C}} = \oplus_{p \geq \bullet} V_{\tau_\phi}^{p,n-p}$, we may embed D in a product of Grassmanians, where its Zariski closure defines the compact dual

$$\check{D} = G_\phi(\mathbb{C}) \cdot F_\phi^\bullet \cong G_\phi(\mathbb{C}) / P_{F_\phi^\bullet}.$$

For a polarized variation \mathcal{V} as in Section 2, the pointwise MTG is equal to some $G < GL(V)$ on the complement of a countable union of proper analytic subvarieties, and we call this the MTG of \mathcal{V} . Since $\Pi \trianglelefteq G^{\text{der}}$ [An], we obtain (after possibly replacing \mathcal{S} by a finite cover) a period map $\Phi: \mathcal{S} \rightarrow \Gamma \backslash D$. For studying the possible LMHS, it is convenient to replace G by G^{ad} , V by $\mathfrak{g} = \text{Lie}(G^{\text{ad}})$, ϕ by the composition

$$S^1 \xrightarrow{\phi} G \xrightarrow{\text{Ad}} G^{\text{ad}},$$

Q by $-B$ ($B = \text{Killing form}$), and the weight n by 0; then D is unchanged and no information is lost [KP]. Assume now that this change has been made, and let $N \in \mathfrak{g}_{\mathbb{Q}} \setminus \{0\}$ be a nilpotent element (acting on V by ad).

Definition 3.3 ([KP]) *The pre-boundary component⁴*

$$\tilde{B}(N) := \left\{ F^\bullet \in \check{D} \mid \begin{array}{l} NF^\bullet \subset F^{\bullet-1}, \\ \text{Ad}(e^{\tau N})F^\bullet \in D \end{array} \text{ for } \mathcal{I}(\tau) \gg 0 \right\}$$

associated with N and D classifies the possible LMHS $(F^\bullet, W(N)_\bullet)$ of period maps into any quotient $\Gamma \backslash D$. Let $\tilde{B}_{\mathbb{R}}(N) \subset \tilde{B}(N)$ be the subset consisting of \mathbb{R} -split LMHS, and $B(N) := e^{\mathbb{C}N} \backslash \tilde{B}(N)$ the set of nilpotent orbits. The boundary component associated with N , D , and a choice of Γ is then $\bar{B}(N) := \Gamma_N \backslash B(N)$, where $\Gamma_N := \text{Stab}(\mathbb{C}N) \cap \Gamma$.

Remark 3.4 One can think of the quotient by $e^{\mathbb{C}N}$ as eliminating the dependence of the LMHS on the scaling of the local coordinate. We will be interested below in computing the point in $\bar{B}(N)$ associated with LMHS of Hodge–Tate type for principal

⁴It sometimes happens that $\tilde{B}(N)$ as defined is not connected; in this case, one should replace it by a choice of connected component.

VHS \mathcal{V} , so that $\text{Gr}_k^{W(N)} V$ is 0 for k odd and $\mathbb{Q}(-\frac{k}{2})$ or 0 for k even. In this case, there is an easy first step. We can rescale (“canonically normalize”) the local coordinate to eliminate the adjacent extensions (of $\mathbb{Q}(p)$ by $\mathbb{Q}(p+1)$) in V .

The terminology of “boundary component” comes from the appearance of $\bar{B}(N)$ in partial compactifications of $\Gamma \backslash D$, assuming it is nonempty. In this case, let $M \leq \text{Aut}(G, B)$ be the subgroup fixing all Hodge tensors of all LMHS in $\tilde{B}(N)$. (For a given MHS on V , the Hodge tensors are the elements of $\oplus_{p,k,l} \text{Hom}_{\text{MHS}}(\mathbb{Q}(p), T^{k,l} V)$.)

Proposition 3.5 ([DK]) *Let $Z_G(N) \leq G$ denote the centralizer of N . Then $M \leq Z_G(N)$ and $Z_G(N)$ share the unipotent radical $U = \exp\{\text{im}(\text{ad } N) \cap \ker(\text{ad } N)\}$. Writing G_N for a choice of Levi subgroup of $M (= G_N U)$, $M(\mathbb{R})$ (resp. $G_N(\mathbb{R}) \times U(\mathbb{C})$) acts transitively on $\tilde{B}_{\mathbb{R}}(N)$ (resp. $\tilde{B}(N)$). Assuming Γ is neat, there exists an iterated generalized intermediate Jacobian fibration*

$$\bar{B}(N) \rightarrow \cdots \rightarrow \bar{B}(N)_{(k)} \rightarrow \cdots \rightarrow \bar{B}(N)_{(1)} \rightarrow \bar{D}(N) = \Gamma_N \backslash D(N),$$

where $D(N)$ is a MTD for G_N .

We will need a source of PHS ϕ with interesting MTG. Let G be a \mathbb{Q} -simple adjoint group of rank r such that $G_{\mathbb{R}}$ has a compact Cartan subgroup. Let $\theta \in \text{Aut}(G_{\mathbb{R}})$ be a Cartan involution, $K < G_{\mathbb{R}}$ the corresponding maximal compact subgroup, and $T_{\mathbb{R}} < K$ a Cartan subgroup (of dimension r). Writing \mathcal{R} for the root lattice, Δ (resp. Δ_c, Δ_n) for the roots (resp. compact, noncompact roots), we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Delta_c} \mathfrak{g}_{\alpha} \right) \oplus \left(\bigoplus_{\beta \in \Delta_n} \mathfrak{g}_{\beta} \right).$$

Proposition 3.6 ([GGK2]) *Given a homomorphism $\pi : \mathcal{R} \rightarrow 2\mathbb{Z}$ with $\pi(\Delta_c) \in 4\mathbb{Z}$, $\pi(\Delta_n) \subset 4\mathbb{Z} + 2$, there exists a unique weight 0 PHS $(\mathfrak{g}, -B, \phi)$ with $d\phi/dz(1)|_{\mathcal{R}} = \pi$. Moreover, provided $T_{\mathbb{R}}$ (hence also θ) is sufficiently general, the MTG $G_{\phi} = G$. The Hodge numbers of the Hodge structures on \mathfrak{g} parametrized by $D = G(\mathbb{R})^+ \cdot \phi$ are then*

$$h^{j,-j} = \begin{cases} |\pi^{-1}(2j)|, & j \neq 0 \\ |\pi^{-1}(0)| + r, & j = 0. \end{cases}$$

In some cases, these constructions “lift to a standard representation”. Here are two examples where this occurs, together with choices of N (with “maximally unipotent” $T = e^N$) that produce nonempty boundary components and the pictures of (p, q) types for the resulting LMHS.

Example 3.7 ($\mathfrak{g} = \mathfrak{sp}_4$) With π as shown in Figure 1, D (of dimension 4) parametrizes weight 3 PHS (of rank 4) on the standard representation V with Hodge numbers $(1, 1, 1, 1)$.

The large dots on the left-hand side are roots, with the boxed ones in Δ_c ; the small dots indicate the Cartan subalgebra \mathfrak{t} . On the right, the dots are weights of V . Moreover, there exist $\phi \in D$ and $N \in \mathfrak{g}_{\mathbb{Q}}$ such that F_{ϕ}^{\bullet} belongs to $\tilde{B}(N)$, and the $\mathfrak{g}_{\phi}^{-1,1}$ -component $N^{-1,1}$ is a linear combination of root vectors as shown on the left-hand side

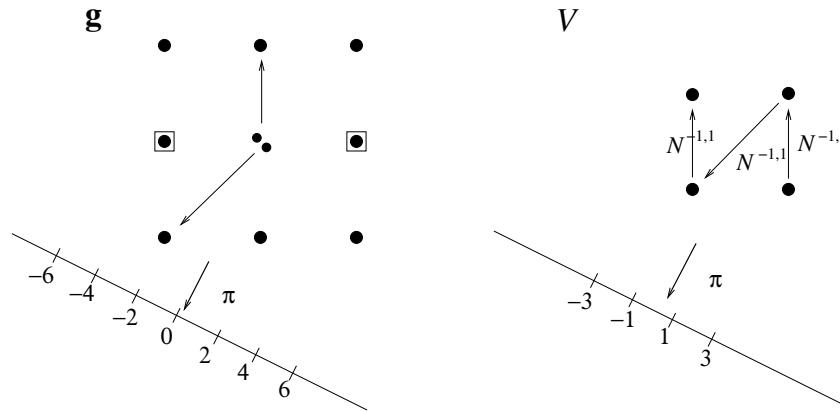


Figure 1

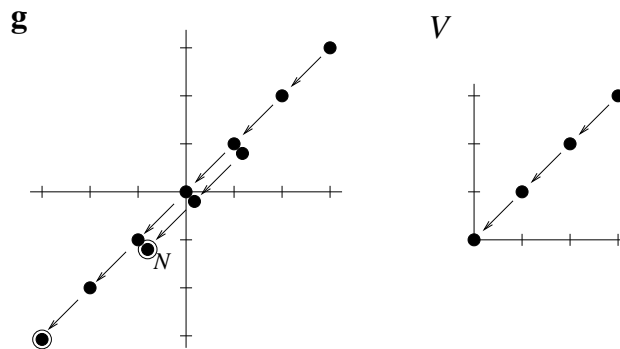


Figure 2

(the arrows point to the corresponding roots) and operates on V as described by the arrows on the right-hand side.

The LMHS $(F_\phi^\bullet, W(N)_\bullet)$ on \mathfrak{g} and the induced LMHS on V take the form in Figure 2, where the arrows describe the action of N . From the figure, one sees that $\mathfrak{m}_\mathbb{Q}/\mathbb{Q}N$ ($\mathfrak{m} = \text{Lie}(M)$) corresponding to the circled types) is pure of rank one and type $(-3, -3)$; according to [KP] it follows that $\bar{B}(N) \cong \text{Ext}_{\text{MHS}}^1(\mathbb{Z}(-3), \mathbb{Z}(0)) \cong \mathbb{C}/\mathbb{Z}(3)$. (Dividing out by $(2\pi i)^3$, this is just \mathbb{C}/\mathbb{Z} .)

Turning to an exceptional group, we have the following example.

Example 3.8 ($\mathfrak{g} = \mathfrak{g}_2$) See Figure 3. Here D is of dimension 5, parametrizing weight 6 PHS on the 7-dimensional irreducible representation V of G_2 with Hodge numbers $(1, 1, 1, 1, 1, 1, 1)$.

For the LMHS, see Figure 4, from which $\bar{B}(N) \cong \text{Ext}_{\text{MHS}}^1(\mathbb{Z}(-5), \mathbb{Z}(0)) \cong \mathbb{C}/\mathbb{Z}(5)$.

Henceforth we will be interested in the Hodge structures (and LMHS) on V rather than \mathfrak{g} . Note that in both examples, these PHS are “Calabi–Yau” in the sense that the leading Hodge number is 1. Moreover, $\mathfrak{g}^{-1,1}$ has rank 2 and is nonabelian; therefore Griffiths transversality forces the image of a period map into $\Gamma \backslash D$ to be a curve.

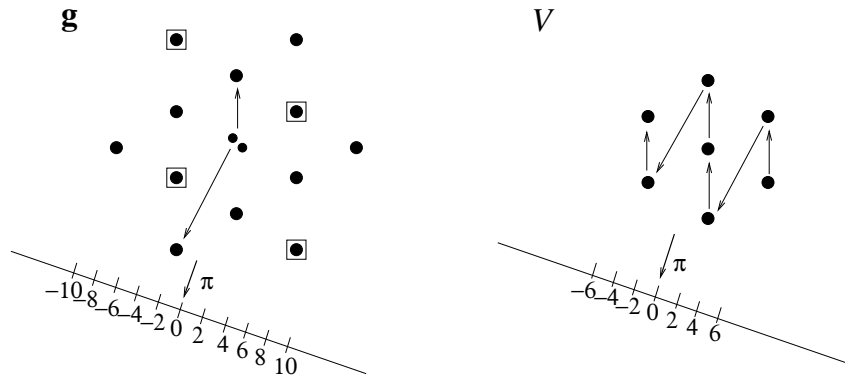


Figure 3

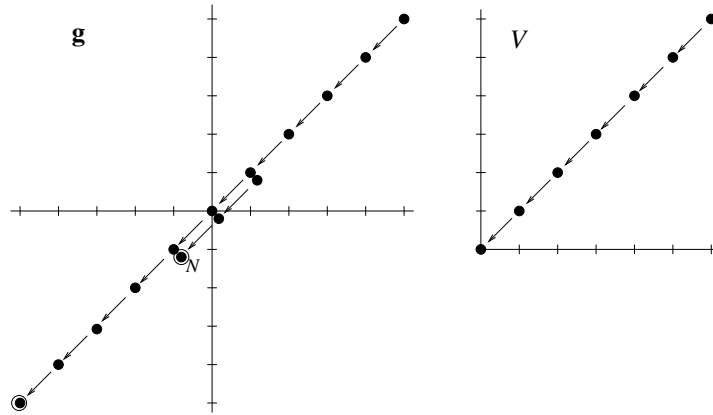


Figure 4

Most importantly, they each give fertile testing-grounds for Conjecture 2.4. In Sections 4 and 5, we will verify it (at $x = 0$) for some VHS over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ arising from SSD's over \mathbb{Q} , which have MTG Sp_4 or G_2 and maximal unipotent monodromy about 0. In both cases this boils down to checking that a single limiting period $\xi \in \mathbb{C}/\mathbb{Q}$ takes a particular form.

In the Sp_4 case, we can assume given a symplectic basis, so that the polarization takes the form

$$(3.1) \quad Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

After conjugating by $Sp_4(\mathbb{Q})$ to have

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & e & -a & 0 \end{pmatrix}$$

and canonically normalizing the local coordinate at 0, one knows that (cf. [GGK]) the limiting period matrix takes the form

$$\Omega_{\text{lim}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{f}{2a} & \frac{e}{a} & 1 & 0 \\ \xi & \frac{f}{2a} & 0 & 1 \end{pmatrix} \quad (\xi \in \mathbb{C}).$$

The entries other than ξ are rational and correspond to torsion extension classes. The LMHS is \mathbb{Q} -motivated if and only if $\xi = q \frac{\zeta(3)}{(2\pi i)^3}$ ($q \in \mathbb{Q}$).

For G_2 , again after appropriate normalizations, one has

$$\Omega_{\text{lim}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 \\ \textcircled{*} & * & * & 1 & 0 & 0 & 0 \\ * & \textcircled{*} & * & * & 1 & 0 & 0 \\ \xi & * & \textcircled{*} & * & * & 1 & 0 \\ * & \xi & * & \textcircled{*} & * & 0 & 1 \end{pmatrix},$$

where $*$ denotes rational numbers. In the \mathbb{Q} -motivated scenario,

$$\xi = q \frac{\zeta(5)}{(2\pi i)^5} \quad (q \in \mathbb{Q}).$$

For the same type of LMHS on V but with the larger M-T group $SO(3, 4)$ (instead of G_2), the third extension class need not be trivial. That is, it is G_2 , which forces the circled entries to be rational.

4 Calabi–Yau Variations from Mirror Symmetry

In this section we shall briefly describe how a recent result of Iritani [Ir] allows one to systematically compute LMHS of variations arising from families of anticanonical toric complete intersections. We will carry this out for the 1-parameter, $h^{2,1} = 1$ hypergeometric families of complete intersection C-Y threefolds classified in [DM]. Each family yields a semistable degeneration over \mathbb{Q} (cf. Section 2) with X_0 the (suitably blown-up) “large complex structure limit” fiber.

Until recently, toric mirror symmetry (e.g., as described in [CK] or [Mo]) only identified complex variations of Hodge structure arising from the A-model and B-model, because the Dubrovin connection on quantum cohomology merely provides a \mathbb{C} -local system on the A-model side. Iritani’s mirror theorem says that the integral structure on this local system provided by the $\widehat{\Gamma}$ -class (in the sense described below) completes the A-model \mathbb{C} -VHS to a \mathbb{Z} -VHS matching the one arising from H^3 of fibers on the B-model side. The upshot is that to compute Ω_{lim} (at 0) for a 1-parameter family of toric complete intersection Calabi-Yau 3-folds $X_t \subset \mathbb{P}_\Delta$ over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, we can use what boils down to characteristic class data from the mirror $X^\circ \subset \mathbb{P}_{\Delta^\circ}$.

In each case, $V := H^{\text{even}}(X^\circ, \mathbb{C}) = \bigoplus_{j=0}^3 H^{j,j}(X^\circ)$ is a vector space of rank 4, $\mathbb{P} := \mathbb{P}_{\Delta^\circ} = \mathbb{W}\mathbb{P}(\delta_0, \dots, \delta_{3+r})$ is a weighted projective space⁵ (with $\delta_0 = \delta_1 = 1$), and $X^\circ \subset \mathbb{P}$ is smooth⁶ of multidegree $(d_k)_{k=1}^r$ with $\sum d_k = \sum \delta_i =: m$. Let H denote the intersection with X° of the vanishing locus of the weight 1 homogeneous coordinate X_0 ; write $\tau[H] \in H^{1,1}(X^\circ)$ for the Kähler class and $q = e^{2\pi i \tau}$ for the Kähler parameter. We now give a general recipe (following [DK, sec. 1]) for constructing a polarized \mathbb{Z} -VHS, over $\Delta^* : 0 < |q| < \epsilon$, on $\mathcal{V} := V \otimes \mathcal{O}_{\Delta^*}$.

The easy parts are the Hodge filtration and polarization. Indeed, we simply put

$$F^p := \bigoplus_{j \leq 3-p} H^{j,j} \subset V \quad \text{and} \quad \mathcal{F}_e^p := F^p \otimes \mathcal{O}_\Delta \subset V \otimes \mathcal{O}_\Delta =: \mathcal{V}_e.$$

Similarly, Q on \mathcal{V}_e is induced from the form on V given by the direct sum of pairings $Q_j: H^{j,j} \times H^{3-j,3-j} \rightarrow \mathbb{C}$ defined by $Q_j(\alpha, \beta) := (-1)^j \int_{X^\circ} \alpha \cup \beta$. A Hodge basis $e = \{e_i\}_{i=0}^3$ of H^{even} , with $e_i \in H^{3-i,3-i}(X^\circ)$ and $[Q]_e$ of the form (3.1), is given by $e_3 = [X^\circ]$, $e_2 = [H]$, $e_1 = -[L]$, and $e_0 = [p]$. Here L is a copy of \mathbb{P}^1 (parametrized by $[X_0 : X_1]$) in X° with $L \cdot H = p$, and $[H] \cdot [H] = m[L]$. The $\{e_i\}$ give a Hodge basis⁷ for \mathcal{V}_e .

For the local system, we consider the generating series⁸ $\Phi_h(q) := \frac{1}{(2\pi i)^3} \sum_{d \geq 1} N_d q^d$ of the genus-zero Gromov–Witten invariants of X° , and define the small quantum product on V by $e_2 * e_2 := -(m + \Phi_h'''(q))e_1$ and $e_i * e_j := e_i \cup e_j$ for $(i, j) \neq (2, 2)$.

⁵Technically, there are three exceptions to this amongst the examples we consider, which are weighted projective spaces $\mathbb{W}\mathbb{P}(\delta_0, \dots, \delta_n)$ for which the convex hull of $\{e_1, \dots, e_n, -\sum \delta_i e_i\}$ is not reflexive. As described in [DM], taking Δ to be the convex hull of this set together with $-e_n$ yields a reflexive polytope, and $\mathbb{P}_{\Delta^\circ}$ is the blow-up of the $\mathbb{W}\mathbb{P}$ at a point not meeting (hence not affecting) the complete intersections we consider. Hence we may take $X^\circ \subset \mathbb{P} = \mathbb{W}\mathbb{P}(\delta_1, \dots, \delta_n)$.

⁶The codimension of the singular locus in \mathbb{P} is at least 4 in every case, so does not meet a sufficiently general X° .

⁷Note: in all bases we shall run the indices backwards ($e = \{e_3, e_2, e_1, e_0\}$, etc.) for purposes of writing matrices.

⁸derivatives $\Phi_h^{(k)}$ will be taken with respect to $\tau (= \ell(q))$

This gives rise to the Dubrovin connection

$$\nabla := \text{id}_V \otimes d + (e_2^*) \otimes d\tau,$$

which we view as a map from $\mathcal{V} \cong V \otimes \mathcal{O}_{\Delta^*} \rightarrow V \otimes \Omega_{\Delta^*}^1 \cong \mathcal{V} \otimes \Omega_{\Delta^*}^1$, and the \mathbb{C} -local system $\mathbb{V}_{\mathbb{C}} := \ker(\nabla) \subset \mathcal{V}$.

Now define a map $\tilde{\sigma} : V \rightarrow V \otimes \mathcal{O}(\Delta)$ by

$$\tilde{\sigma}(e_0) := e_0, \quad \tilde{\sigma}(e_1) := e_1, \quad \tilde{\sigma}(e_2) := e_2 + \Phi_h'' e_1 + \Phi_h' e_0, \quad \tilde{\sigma}(e_3) := e_3 + \Phi_h' e_1 + 2\Phi_h e_0.$$

For any $\alpha \in V$, one easily checks that

$$\sigma(\alpha) := \tilde{\sigma}(e^{-\tau[H]} \cup \alpha) := \sum_{k \geq 0} \frac{(-1)^k \tau^k}{k!} \tilde{\sigma}([H]^k \cup \alpha)$$

satisfies $\nabla \sigma(\alpha) = 0$, and hence yields an isomorphism $\sigma : V \xrightarrow{\cong} \Gamma(\mathfrak{H}, \rho^* \mathbb{V}_{\mathbb{C}})$ (where $\rho : \mathfrak{H} \rightarrow \Delta^*$ sends $\tau \mapsto q$). Writing⁹

$$\widehat{\Gamma}(X^\circ) := \exp\left(-\frac{1}{24} ch_2(X^\circ) - \frac{2\zeta(3)}{(2\pi i)^3} ch_3(X^\circ)\right) \in V,$$

the image of

$$\begin{aligned} \gamma : K_0^{\text{num}}(X^\circ) &\longrightarrow \Gamma(\mathfrak{H}, \rho^* \mathbb{V}_{\mathbb{C}}) \\ \xi &\longmapsto \sigma(\widehat{\Gamma}(X^\circ) \cup ch(\xi)) \end{aligned}$$

defines Iritani's \mathbb{Z} -local system \mathbb{V} underlying $\mathbb{V}_{\mathbb{C}}$. The filtration $W_\bullet := W(N)_\bullet$ associated with its monodromy $T(\gamma(\xi)) = \gamma(\mathcal{O}(-H) \otimes \xi)$ satisfies

$$W_k \mathcal{V}_e = \left(\bigoplus_{j \geq 3-k/2} H^{j,j} \right) \otimes \mathcal{O}_\Delta.$$

In order to compute the limiting period matrix of this \mathbb{Z} -VHS over Δ^* , we now require a (multivalued) basis $\{\gamma_i\}_{i=0}^3$ of \mathbb{V} satisfying $\gamma_i \in W_{2i} \cap \mathbb{V}$, $\gamma_i \equiv e_i \pmod{W_{2i-2}}$, and $[Q]_\gamma = [Q]_e$. The corresponding \mathbb{Q} -basis of $\widetilde{\mathbb{V}}|_{q=0} =: V_{\text{lim}}$ is given by $\gamma_i^{\text{lim}} := \widetilde{\gamma}_i(0)$, where $\widetilde{\gamma}_i := e^{-\tau N} \gamma_i \in \Gamma(\Delta, \widetilde{\mathbb{V}})$. Of course, the e_i are another basis of $V_{\text{lim}, \mathbb{C}}$, and $\Omega_{\text{lim}} = \gamma_{\text{lim}}[\text{id}]_e$. Note that since $N_{\text{lim}} = -(2\pi i) \text{Res}_{q=0}(\nabla) = -(e_2^*)|_{q=0} = -(e_2 \cup)|_{q=0}$, we have

$$[N_{\text{lim}}]_e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

A basis of the form we require is obtained by considering the Mukai pairing

$$\langle \xi, \xi' \rangle := \int_{X^\circ} ch(\xi^\vee \otimes \xi') \cup \text{Td}(X^\circ)$$

on $K_0^{\text{num}}(X^\circ)$. Since $\langle \xi, \xi' \rangle = Q(\gamma(\xi), \gamma(\xi'))$, any Mukai-symplectic¹⁰ basis of $K_0^{\text{num}}(X^\circ)$ of the form

$$(4.1) \quad \begin{aligned} \xi_1 &= \mathcal{O} + A\mathcal{O}_H + B\mathcal{O}_L + C\mathcal{O}_p, & \xi_2 &= \mathcal{O}_H + D\mathcal{O}_L + E\mathcal{O}_p, \\ \xi_3 &= -\mathcal{O}_L + F\mathcal{O}_p, & \xi_4 &= \mathcal{O}_p, \end{aligned}$$

⁹cf. [DK, §1] for the more general definition of $\widehat{\Gamma}(X^\circ)$

¹⁰That is, $\langle \xi_i, \xi_{3-j} \rangle = 0$ unless $i = j$, in which case it is $+1$ for $i = 0, 1$ and -1 for $i = 2, 3$.

will produce $\gamma_i := \gamma(\xi_i)$ satisfying the above hypotheses. In this case, taking

$$\sigma_\infty(\alpha) := \lim_{q \rightarrow 0} \tilde{\sigma}(\alpha), \quad \gamma_\infty(\xi) := \sigma_\infty(\widehat{\Gamma}(X^\circ) \cup ch(\xi)),$$

we have $\gamma_i^{\text{lim}} = \gamma_\infty(\xi_i)$.

We now run this computation. Let $c(X^\circ) = 1 + a[L] + b[p]$ be the Chern class of X° ; note that there is no $[H]$ term due to the fact that X° is Calabi–Yau. Since the Chern character is $ch(X^\circ) = 3 - a[L] + \frac{b}{2}[p]$ and the Todd class is

$$\text{Td}(X^\circ) = 1 + \frac{a}{12}[L], \quad \widehat{\Gamma}(X^\circ) = 1 + \frac{a}{24}[L] - \frac{b\zeta(3)}{(2\pi i)^3}[p].$$

This yields

$$\begin{aligned} \gamma_3^{\text{lim}} &= e_3 + Ae_2 + \left(-B + \frac{m}{2}A - \frac{a}{24}\right)e_1 + \left(C - B + \frac{4m+a}{24}A - b\frac{\zeta(3)}{(2\pi i)^3}\right)e_0, \\ \gamma_2^{\text{lim}} &= e_2 + \left(-D + \frac{m}{2}\right)e_1 + \left(E - D + \frac{4m+a}{24}\right)e_0, \\ \gamma_1^{\text{lim}} &= e_1 + (F+1)e_0, \quad \gamma_0^{\text{lim}} = e_0. \end{aligned}$$

Imposing the symplectic condition produces constraints $1 + F + A = 0$ and

$$\frac{a+2m}{12} - D + E - AD + B = 0.$$

After normalizing¹¹ $A = B = C = D = 0$ ($\Rightarrow F = -1, E = -\frac{a+2m}{12}$) in (4.1), expressing each e_i in terms of $\{\gamma_i^{\text{lim}}\}$ gives the columns of

$$\Omega_{\text{lim}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{24} & -\frac{m}{2} & 1 & 0 \\ \frac{b\zeta(3)}{(2\pi i)^3} & \frac{a}{24} & 0 & 1 \end{pmatrix}.$$

To compute N (with these normalizations), we apply $\mathcal{O}(-H) \otimes$ to the ξ_i in $K_0^{\text{num}}(X^\circ)$; then

$$[T]_y = [\mathcal{O}(-H) \otimes]_\xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & m & 1 & 0 \\ -\frac{a+2m}{12} & m & 1 & 1 \end{pmatrix},$$

whereupon taking log gives

$$[N_{\text{lim}}]_{y^{\text{lim}}} = [N]_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \frac{m}{2} & m & 0 & 0 \\ -\frac{a}{12} & \frac{m}{2} & 0 & 0 \end{pmatrix}.$$

The data required to compute N and Ω_{lim} for the complete intersection Calabi–Yau (CICY) examples from [DM] is displayed in Table 1. Here, for example, “ $\mathbb{P}^5[3, 3]$ ” means that X° is a complete intersection of bidegree $(3, 3)$ in \mathbb{P}^5 .

¹¹ $A = 0$ is the canonical normalization of the local coordinate; the remaining choices are made to simplify the end result.

X°	m	a	b
$\mathbb{P}^4 [5]$	5	50	-200
$\mathbb{P}^5 [2, 4]$	8	56	-176
$\mathbb{P}^5 [3, 3]$	9	54	-144
$\mathbb{P}^6 [2, 2, 3]$	12	60	-144
$\mathbb{P}^7 [2, 2, 2, 2]$	8	64	-128
$\mathbb{W}\mathbb{P}_{1,1,1,2,5}^4 [10]$	10	340	-2880
$\mathbb{W}\mathbb{P}_{1,1,1,1,4}^4 [8]$	8	176	-1184
$\mathbb{W}\mathbb{P}_{1,1,2,2,3,3}^5 [6, 6]$	36	792	-4320
$\mathbb{W}\mathbb{P}_{1,1,1,2,2,3}^5 [4, 6]$	24	384	-1872
$\mathbb{W}\mathbb{P}_{1,1,1,1,2}^4 [6]$	6	84	-408
$\mathbb{W}\mathbb{P}_{1,1,1,1,1,3}^5 [2, 6]$	12	156	-768
$\mathbb{W}\mathbb{P}_{1,1,1,1,2,2}^5 [4, 4]$	16	160	-576
$\mathbb{W}\mathbb{P}_{1,1,1,1,1,2}^5 [3, 4]$	12	96	-312

Table 1

Since X° is smooth, the Chern numbers can be calculated using

$$c(X^\circ) = \frac{c(\mathbb{P})|_{X^\circ}}{c(N_{X^\circ/\mathbb{P}})} = \frac{\prod_{i=0}^{3+r} (1 + \delta_i[H])}{\prod_{k=1}^r (1 + d_k[H])}$$

Remark 4.1 An interesting case not included amongst the CICY examples is the so called “14th case VHS”, labeled “I” in [loc. cit.]. It is shown in [CDLNT] that this VHS arises from the $\text{Gr}_3^W H^3$ of a subfamily contained in the singular locus of a larger family of hypersurfaces in weighted-projective space. The LMHS of this sort of example is probably inaccessible to the above approach. The technique of the next section provides a possible approach to such examples.

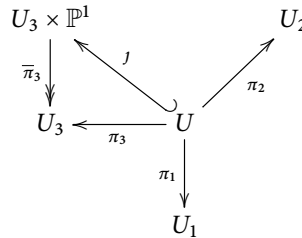
5 Calabi–Yau Variations from Middle Convolutions

Middle convolution is a binary operation on local systems introduced by Katz [Ka] to study the construction of rigid local systems on Zariski open subsets $U \subset \mathbb{P}^1$. Recent work of Dettweiler and others (e.g., [D, DRI, DS]) has demonstrated the Hodge-theoretic importance of this construction, of which we will give only the briefest description. The main point for us is that it yields interesting Calabi–Yau type variation for which the limiting invariant ξ above may be computed directly. In this way we

can see where the rational multiples of $\zeta(3)$ (or $\zeta(5)$) come from, in contrast to the approach of the last section.

5.1 The Variations

If $\{\underline{a}\}$ and $\{\underline{b}\}$ are finite sets of points in \mathbb{A}^1 , we define $\{\underline{c}\} = \{\underline{a}\} * \{\underline{b}\}$ to be the set obtained by taking all sums of pairs $a_j + b_k$ from $\{\underline{a}\}$ and $\{\underline{b}\}$. Let $U_1 = \mathbb{A}_x^1 \setminus \{a_1, \dots, a_m\}$, $U_2 = \mathbb{A}_z^1 \setminus \{b_1, \dots, b_n\}$, and $U_3 = \mathbb{A}_y^1 \setminus \{c_1, \dots, c_p\}$. Let $U \subset \mathbb{A}_{(x,y)}^2$ be the Zariski open where $\prod_j (x - a_j) \prod_k ((y - x) - b_k) \prod_l (y - c_l)$ does not vanish. We have a diagram



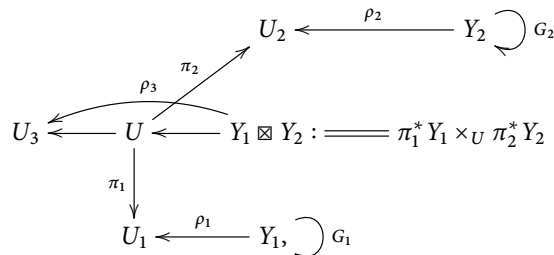
where $\pi_1(x, y) := x$, $\pi_2(x, y) := y - x$, and $\pi_3(x, y) = y$. Given local systems $\mathbb{V}_i \rightarrow U_i$ ($i = 1, 2$), their *middle convolution* is the local system on U_3 defined by

$$\mathbb{V}_1 * \mathbb{V}_2 := R^1(\bar{\pi}_3)_*(J_*(\pi_1^*\mathbb{V}_1 \otimes \pi_2^*\mathbb{V}_2)).$$

Now suppose (following [D, sec. 2.6]) that the local systems are motivic, say

$$\mathbb{V}_i = \text{Gr}_{d_i}^W P_i R^{d_i}(\rho_i)! \mathbb{Q}_{Y_i} \quad (i = 1, 2),$$

where $Y_i \xrightarrow{\rho_i} U_i$ are smooth morphisms and $P_i \in \mathbb{Q}[\text{Aut}(\rho_i)]$ idempotents. The situation is described by the diagram



and the middle convolution is described by

$$\mathbb{V}_1 * \mathbb{V}_2 = \text{Gr}_{d_1+d_2+1}^W (P_1 \times P_2) R^{d_1+d_2+1}(\rho_3)! \mathbb{Q}_{Y_1 \boxtimes Y_2}.$$

By iteratively alternating this construction with quadratic twists as described in [DR1, sec. 2.3-4], we obtain a sequence of VHS \mathcal{V}_d over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ of weight d , with $h^{d,0} = 1$, for each $d \in \mathbb{N}$. From the motivic perspective, for each iteration we begin with a family $\mathcal{X}_{d-1} = \cup_{t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}} X_{d-1}(t)$ of “singular Calabi-Yau” $(d-1)$ -folds (with involution σ_1) over $U_1 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and the \sqrt{z} -double-cover Y_2 (with involution σ_2) over $U_2 = \mathbb{P}^1 \setminus \{0, \infty\}$. Taking Y_1 to be a quadratic twist (by \sqrt{t} or $\sqrt{1-t}$) of $\mathcal{X}_{d-1} \setminus (\mathcal{X}_{d-1})^{\sigma_1}$, we then apply to $Y_1 \boxtimes Y_2$ the “projector” of quotienting by $\sigma_1 \times \sigma_2$,

producing Y_3 . This has a natural compactification to a family \mathcal{X}_d of “singular C-Y” d -folds with involution σ_3 over U_3 , in which

$$Y_3 = \mathcal{X}_d \setminus (\mathcal{X}_d)^{\sigma_3} =: \mathcal{X}_d^-.$$

The local system underlying \mathcal{V}_d is just the σ_3 -anti-invariants in $\text{Gr}_d^W R^d(\rho_3)_! \mathbb{Q}_{Y_3}$.

The $\{X_d(t)\}$ produced by this algorithm (which are singular for $d \geq 2$) all take the form $w^2 = f_d(x_1, \dots, x_d, t)$, and include the following¹²:

$$\begin{aligned} d = 1: & \quad w^2 = (1 - tx)x(x - 1) \\ & \quad \text{(Legendre elliptic curve)} \\ d = 3: & \quad w^2 = (1 - tx_3)x_3(x_2 - x_3)(x_2 - 1)(x_1 - x_2)(x_1 - 1)x_1 \\ & \quad \text{(CY 3-fold family, cf. Remark 5.2)} \\ d = 6: & \quad w^2 = (1 - tx_6)(1 - x_6)(x_5 - x_6)x_5(x_4 - x_5)(1 - x_4) \times \\ & \quad (x_3 - x_4)x_3(x_2 - x_3)(1 - x_2)(x_1 - x_2)x_1(1 - x_1). \end{aligned}$$

Each has an obvious involution σ given by $w \mapsto -w$. Write $\pi_d: \mathcal{X}_d^- \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$ so that $\mathbb{V}_d = (\text{Gr}_d^W R^d(\pi_d)_! \mathbb{Q}_{\mathcal{X}_d^-})^{-\sigma}$.

Proposition 5.1

- (i) For $1 \leq d \leq 6$, \mathbb{V}_d is a VHS of weight d and rank $d + 1$, with Hodge numbers all 1.
- (ii) [DRI] \mathbb{V}_6 has MTG G_2 .

Proof How (ii) follows from the results of [DRI] is explained in [KP, sec. 9], while (i) follows from the proof of [DRI, Theorem 1.3.1]. In particular, the table in that proof (with 0 and ∞ swapped, as our $1/t$ is their \mathbb{P}^1 parameter) shows that the monodromy at $t = 0$ is a single Jordan block $U(d + 1)$, which can only happen for rank $d + 1$ if the Hodge numbers are $(1, 1, \dots, 1)$. ■

Remark 5.2 The $\{X_3(t)\}$ are degree-8 hypersurfaces in $\mathbb{W}\mathbb{P}(1, 1, 1, 1, 4)$ that are C-Y 3-folds after desingularization (or for purposes of computing $(\text{Gr}_3^W R^3(\pi_3)_! \mathbb{Q}_{\mathcal{X}_3^-})^{-\sigma}$). Note that this is not the mirror family for which the LMHS was computed in Section 4. Its LMHS at $t = 0$ does not appear to be accessible by mirror symmetry, since it belongs to the singular locus of a much larger variation and does not meet the large complex structure limit of this larger family. We also note that while for $d = 1, 2$ the vanishing cycle period $\int_{\mu_t} \omega_t$ (Section 5.2) is a hypergeometric function (up to quadratic twist), for $d \geq 3$ this is not the case. So methods of computing LMHS using Meijer G -functions [GL] would also not be applicable.

Remark 5.3 Referring to [DRI, p. 940] and accounting for the inversion and quadratic twists, the monodromies of \mathbb{V}_d are displayed in Table 2, in which $U(n)$ denotes a Jordan block of rank n .

For the stalks, we have (writing $D_t := X_d(t)^\sigma$, $X_d^-(t) := X_d(t) \setminus D_t$)

$$\begin{aligned} \mathbb{V}_{d,t} & \cong \text{Gr}_d^W H_c^d(X_d^-(t), \mathbb{Q})^{-\sigma} \cong \text{Gr}_d^W H_d(X_d^-(t), \mathbb{Q}(-d))^{-\sigma}, \\ \mathbb{V}_{d,t}^\vee & \cong \text{Gr}_{-d}^W H_d(X_d(t), D_t; \mathbb{Q})^{-\sigma} \cong \text{Gr}_{-d}^W H_d(X_d(t), \mathbb{Q})^{-\sigma}. \end{aligned}$$

¹²Note that our parameter t is inverse to that in [DRI]; for odd d , we have also removed a final quadratic twist present in [op. cit.] (to rid f_d of a factor of $(1 - t)$).

	at 0	at 1	at ∞
$d = 1$	$U(2)$	$U(2)$	$-U(2)$
$d = 3$	$U(4)$	$-U(2) \oplus \mathbf{1}^{\oplus 2}$	$(-U(2))^{\oplus 2}$
$d = 6$	$U(7)$	$U(2)^{\oplus 2} \oplus U(3)$	$(-\mathbf{1})^{\oplus 4} \oplus \mathbf{1}^{\oplus 3}$

Table 2

On each $X_d(t)$ there are obvious σ -anti-invariant topological d -cycles consisting of two sheets ($\pm w$) bounding on components of D_t (e.g., μ_t and τ_t below); clearly such cycles span $(\text{Gr}_{-d}^W H_d(X_d(t), \mathbb{Q}))^{-\sigma}$. By a topological argument (omitted here), these may be moved off D_t , and hence belong to the image of $H_d(X_d^-(t)) \rightarrow H_d(X_d(t))$. The resulting isomorphism $\mathbb{V}_{d,t} \xrightarrow{\cong} \mathbb{V}_{d,t}^{\vee}(-d)$ allows us to pair homology cycles in $\text{Gr}_{-d}^W H_d(X_d(t), \mathbb{Q})^{-\sigma}$ and write classes in $\mathbb{V}_{d,t} \otimes \mathbb{C}$ in terms of them. That is, in a sense we may work as if $X_d(t)$ were smooth, which is immensely convenient for the computations that follow.

5.2 Cauchy Residue Method

In each case ($d = 1, 3, 6$), we are after the LMHS at $t = 0$. The idea is to compute $\mathcal{F}^d(\mathbb{V}_d)_{\text{nilp}}$ by Cauchy residue.

More precisely, assume $0 < t \ll 1$ and write

$$\omega_t = \frac{2^{d-1}}{(2\pi i)^d} \frac{dx_1 \wedge \dots \wedge dx_d}{w} \in \Omega^d(X_d(t))$$

for the “holomorphic form”¹³ and

$$\tau_t = \{ (w, x_1, \dots, x_d) \in X_d(\mathbb{R}) \mid 1 \leq x_1 \leq x_2 \leq \dots \leq x_d \leq t^{-1} \}$$

for a family of cycles (with two branches coming from $\pm w$). Note that there exists a family $\mu_t := \{ (w, x_1, \dots, x_d) \in X_d(\mathbb{R}) \mid \frac{1}{t} \leq x_d \leq \dots \leq x_1 < \infty \}$ ($0 < t \ll 1$) of vanishing cycles with $(\mu_t, \tau_t) = 1$ and $\int_{\mu_t} \omega_t \rightarrow 1$ as $t \rightarrow 0$ (for example, using $\int_1^\infty \frac{du}{u\sqrt{u-1}} = \pi$ and the residue approach below). Hence τ_t and μ_t are correctly normalized; that is, they are the extremal members γ_d resp. γ_0 of an integral symplectic basis $\{\gamma_j\}_{j=0}^d$ of \mathbb{V}_d over a punctured disk, in which the monodromy about $t = 0$ takes the form¹⁴

$$[T]_y = \begin{pmatrix} 1 & 0 & \dots & 0 \\ a & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \dots & \dots & \pm a & 1 \end{pmatrix}.$$

¹³The notation means that ω_t pulls back to a holomorphic form on a desingularization of $X_d(t)$; in particular, it gives a class in $F^d \text{Gr}_d^W H_c^d(X_d^-(t), \mathbb{C})^{-\sigma}$.

¹⁴The “ \pm ” is $(-1)^d$ for $d > 1$.

In this section, $\oint_{|t|=\epsilon} dt$ denotes integration counterclockwise from $\arg(t) = -\pi$ to $\arg(t) = \pi$. Recalling the notation $\ell(t) = (\log(t))/(2\pi i)$, integration yields

$$(5.1) \quad \Pi_d(t) := \int_{\tau_t} \omega_t = (-1)^d \sum_{j=0}^d \ell^j(t) \sum_{k \geq 0} a_{jk} t^k,$$

whereupon

$$(5.2) \quad \frac{1}{2\pi i} \oint_{|t|=\epsilon} \frac{dt}{t} \int_{\tau_t} \omega_t = \underbrace{(-1)^d \sum_{j=0}^d a_{j0} \ell^j(\epsilon)}_{=:\Pi_d^{\text{nilp}}(\epsilon)} + \underbrace{\mathcal{O}(\epsilon \log^d \epsilon)}_{\rightarrow 0 \text{ with } \epsilon}.$$

In the rest of this paper, “ \equiv ” shall be used to denote working modulo $\mathcal{O}(\epsilon \log^d \epsilon)$.

If ω_t^{nilp} is the section of $\mathcal{F}^d(\mathcal{V}_d)_{\text{nilp}}$ with period 1 against μ_t , then $\Pi_d^{\text{nilp}}(\epsilon)$ in (5.2) is its period at $t = \epsilon$ against τ_ϵ . Its full period vector takes the form

$$[\omega_\epsilon^{\text{nilp}}]_y = \begin{pmatrix} 1 \\ a_{10}^{(d-1)} \ell(\epsilon) + a_{00}^{(d-1)} \\ \vdots \\ \sum_{j=0}^{d-1} a_{j0}^{(1)} \ell^j(\epsilon) \\ \sum_{j=0}^d a_{j0} \ell^j(\epsilon) \end{pmatrix}.$$

Applying $e^{\ell(\epsilon)N}$ to this must yield

$$(5.3) \quad {}^t(1, a_{00}^{(d-1)}, \dots, a_{00}^{(1)}, a_{00}),$$

from which we deduce that $\mp a_{00}^{(1)} a = a_{10}$ and $a_{10}^{(d-1)} = -a$. So (5.3) is the first column of Ω_{lim} prior to canonically normalizing the local coordinate. To carry out this normalization, we make the substitution $t = \alpha s$ in (5.1), where

$$\ell(\alpha) = -\frac{a_{00}^{(d-1)}}{a_{10}^{(d-1)}} = \frac{a_{00}^{(d-1)}}{a},$$

and rewrite the right-hand side in powers of s and $\ell(s)$. Writing \tilde{a}_{jk} (more generally $\tilde{a}_{jk}^{(i)}$) for the modified coefficients and

$$\tilde{\Pi}_d^{\text{nilp}}(\epsilon) := (-1)^d \sum_{j=0}^d \tilde{a}_{j0} \ell^j(\epsilon)$$

(more generally $\tilde{\omega}_\epsilon^{\text{nilp}}$) for the modified periods, we repeat the above computation with the result that (5.3) is replaced by

$${}^t(1, \tilde{a}_{00}^{(d-1)}, \dots, \tilde{a}_{00}^{(1)}, \tilde{a}_{00}) = {}^t(1, 0, \dots, \mp \frac{\tilde{a}_{10}}{a}, \tilde{a}_{00}),$$

which is now the correct first column of Ω_{lim} . (Equivalently, apply $e^{\ell(\alpha)N}$ to (5.3).) In particular, the extension class $\xi \in \mathbb{C}/\mathbb{Q}$ from the end of Section 3 is given by \tilde{a}_{00} (for $d = 3$) or $-\tilde{a}_{10}/a$ (for $d = 6$). More information is contained in the following proposition.

Proposition 5.4 *The bottom row of the normalized Ω_{lim} is*

$$\left(\tilde{a}_{00}, -\frac{1!\tilde{a}_{10}}{2}, \frac{2!\tilde{a}_{20}}{2^2}, \dots, \frac{(d-1)!\tilde{a}_{d-1,0}}{(-2)^{d-1}}, 1 \right).$$

Moreover, we have $a = 2$ and $\alpha = 4^{d+1}$.

Proof From the calculations we carry out for $d = 1, 3$ and 6 in Sections 5.3–5.5, it is evident that

$$(\tilde{a}_{d0} =) a_{d0} = \frac{2^d}{d!} \quad \text{and} \quad a_{d-1,0} = -\frac{2^d(d+1)}{(d-1)!} \ell(4).$$

In order for ω_t^{nilp} and its derivatives $\nabla_{t \frac{\partial}{\partial t}} \omega_t^{\text{nilp}}$ to be single-valued, we must have

$$(\tilde{a}_{k0}^{(d-k)} =) a_{k0}^{(d-k)} = \frac{(-a)^k}{k!}$$

(for $1 \leq k < d$) and $a_{d0} = \frac{a^d}{d!}$, so $a = 2$.

By Griffiths transversality, the columns of the normalized Ω_{lim} are given by

$$\left\{ \frac{(2\pi i)^k}{k! a_{k0}^{(d-k)}} e^{\ell(t)N} \left(t \frac{\partial}{\partial t} \right)^k [\tilde{\omega}_t^{\text{nilp}}]_Y \right\}_{k=0}^d.$$

In particular, the bottom row has entries

$$\frac{\tilde{a}_{k0}}{a_{k0}^{(d-k)}} = \frac{k! \tilde{a}_{k0}}{(-2)^k} \quad (1 \leq k < d).$$

Since normalization kills the $k = d - 1$ entry, we must have $\tilde{a}_{d-1,0} = 0$. In order for replacing $\ell(t)$ by $\ell(s) + \ell(\alpha)$ to eliminate the $\ell^{d-1}(t)$ term of $\Pi_d^{\text{nilp}}(t)$, we need $\ell(\alpha) = -\frac{a_{d-1,0}}{a_{d0}d} = (d+1)\ell(4)$. ■

The main point is that Π_d^{nilp} contains all the information in Ω_{lim} , and the normalization can be carried out using Π_d^{nilp} alone: one just makes the substitution that kills the $\ell^{d-1}(t)$ term. In our computations, this will simply mean replacing $\Pi_d^{\text{nilp}}(\epsilon)$ by $\tilde{\Pi}_d^{\text{nilp}}(\epsilon) = \Pi_d^{\text{nilp}}(4^{d+1}\epsilon)$.

5.3 Computing the Extension Classes

For $d = 1$, the (normalized) \tilde{a}_{00} is zero, but Conjecture 2.4 still has content: it says that the *unnormalized* $a_{00} \in \mathbb{C}/\mathbb{Q}$ should be (a rational multiple of) $\ell(q)$ for some $q \in \mathbb{Q}^*$. While the conjecture is known for elliptic curves (cf. [GGK, (III.B.11)]), checking it gives an initial feasibility test for our Cauchy residue approach to Ω_{lim} , and motivates what will take place in higher dimension. Referring to (5.2),

$$\frac{1}{2\pi i} \oint_{|t|=\epsilon} \frac{dt}{t} \int_{\tau_t} \omega_t = \frac{2}{(2\pi i)^2} \oint_{|t|=\epsilon} \frac{dt}{t} \int_1^{\frac{1}{t}} \frac{dx}{\sqrt{x(x-1)(1-tx)}}.$$

Substituting $u = \frac{(x-1)t}{1-t}$ yields

$$\begin{aligned} &= \frac{2}{(2\pi i)^2} \oint_{|t|=\epsilon} \frac{dt}{t} \int_0^1 \frac{du}{\sqrt{u(1-u)(u+(1-u)t)}} \\ &= \frac{1}{\pi i} \int_0^1 \left(\oint_{|t|=\epsilon} \frac{1}{\sqrt{u+(1-u)t} 2\pi i t} \right) \frac{du}{\sqrt{u(1-u)}}. \end{aligned}$$

Let $\eta = \frac{\epsilon}{1+\epsilon}$. Then we have the power series expansions

$$\frac{1}{\sqrt{u+(1-u)t}} = \frac{1}{\sqrt{u}} \sum_{m \geq 0} t^m \binom{-\frac{1}{2}}{m} \frac{(1-u)^m}{u^m}$$

valid for $u \in [\eta, 1]$, and

$$\frac{1}{\sqrt{u+(1-u)t}} = \sum_{m \geq 0} t^{-m-\frac{1}{2}} \binom{-\frac{1}{2}}{m} \frac{u^m}{(1-u)^{m+\frac{1}{2}}}$$

valid for $u \in [0, \eta]$. In the former expansion, \oint annihilates all but the $m \geq 0$ term; for the latter, we use¹⁵

$$(5.4) \quad \oint_{|t|=\epsilon} t^{-(m+\frac{1}{2})} \frac{dt}{2\pi i t} = \frac{(-1)^m \epsilon^{-(m+\frac{1}{2})}}{\pi(m+\frac{1}{2})}.$$

Altogether, the above

$$= \frac{1}{\pi i} \int_{\eta}^1 \frac{du}{u\sqrt{1-u}} + \frac{1}{\pi^2 i} \sum_{m \geq 0} \frac{|\binom{-\frac{1}{2}}{m}| \epsilon^{-(m+\frac{1}{2})}}{(m+\frac{1}{2})} \int_0^{\eta} \frac{u^{m-\frac{1}{2}}}{(1-u)^{m+1}} du.$$

Working modulo $\mathcal{O}(\epsilon \log \epsilon)$, this becomes

$$\begin{aligned} &\equiv \frac{1}{\pi i} \left\{ \int_{\eta}^1 \frac{du}{u} + \sum_{k \geq 1} \left| \binom{-\frac{1}{2}}{k} \right| \int_{\eta}^1 u^{k-1} du \right\} + \frac{1}{\pi^2 i} \sum_{m \geq 0} \frac{|\binom{-\frac{1}{2}}{m}| \epsilon^{-(m+\frac{1}{2})}}{(m+\frac{1}{2})} \int_0^{\eta} u^{m-\frac{1}{2}} du \\ &\equiv -2\ell(\epsilon) + \underbrace{\frac{1}{\pi i} \left\{ \sum_{k \geq 1} \frac{|\binom{-\frac{1}{2}}{k}|}{k} + \frac{1}{\pi} \sum_{m \geq 0} \frac{|\binom{-\frac{1}{2}}{m}|}{(m+\frac{1}{2})^2} \right\}}_{-a_{00}} \end{aligned}$$

A short computation now shows that $a_{00} = \frac{-2}{2\pi i} \{\log 4 + \log 4\} = \ell(\frac{1}{4^4})$, and certainly $\frac{1}{4^4} \in \mathbb{Q}^*$.

Essentially, the same thing happens in general: in computing Π_d^{nilp} , one has to face (for example, generalizing the \int_{η}^1 integral above)

$$(5.5) \quad \sum_{k_1, \dots, k_d \geq 0} \left(\prod_j \left| \binom{-\frac{1}{2}}{k_j} \right| \right) \int_{[0,1]^d \cap \{u_1 \dots u_d > \epsilon\}} \left(\prod_j u_j^{k_j-1} du_j \right)$$

¹⁵Note: it is not correct to “go twice around the circle” and kill $t^{-m-\frac{1}{2}}$. The problem is that $\int_0^1 \frac{du}{\sqrt{u(1-u)(u+(1-u)t)}}$ only matches the analytic continuation of $\int_{\tau_t} \omega_t$ for $\arg(t) \in (-\pi, \pi)$.

and hence (as a byproduct) the constants

$$(5.6) \quad \gamma_n := \sum_{k \geq 1} \frac{\left| \binom{-\frac{1}{2}}{k} \right|}{k^n} = \frac{1}{2^{n+2}} F_{n+1} \left(\begin{matrix} 1, \dots, 1, \frac{3}{2} \\ 2, \dots, 2 \end{matrix} \mid 1 \right)$$

for $n \geq 1$. The key observation about them is that (while $\gamma_1 = \log 4$)

$$\gamma_n = q_n \zeta(n) + \text{“degenerate” terms,}$$

where $q_2 = 1, q_3 = 2, q_4 = \frac{9}{4}, q_5 = 6, q_6 = \frac{79}{16}$, etc. The generalization of the \int_0^1 integral above is more complicated, with

$$(5.7) \quad \tilde{\gamma}_n := \frac{1}{\pi} \sum_{k \geq 0} \frac{\left| \binom{-\frac{1}{2}}{k} \right|}{\left(k + \frac{1}{2}\right)^{n+1}} = \frac{2^{n+1}}{\pi} F_{n+1} \left(\begin{matrix} \frac{1}{2}, \dots, \frac{1}{2} \\ \frac{3}{2}, \dots, \frac{3}{2} \end{matrix} \mid 1 \right)$$

as well as some very interesting multiple series appearing. See the Appendix for evaluation and discussion of the γ_n and $\tilde{\gamma}_n$.

5.4 Computing the LMHS of \mathcal{V}_3

For $d = 3$, (5.2) is $\frac{2^3}{(2\pi i)^3}$ times

$$\frac{1}{2\pi i} \oint_{|t|=\epsilon} \frac{dt}{t} \int_1^{x_3} \int_1^{x_2} \int_1^{x_1} \frac{1}{\sqrt{f_3(x_1, x_2, x_3, t)}} dx_1 dx_2 dx_3,$$

which, upon substituting $\tilde{X}_i = \frac{(x_i-1)t}{1-t}$, becomes

$$= \frac{1}{2\pi i} \oint \frac{dt}{t} \int_0^1 \int_0^{\tilde{X}_3} \int_0^{\tilde{X}_2} \frac{\sqrt{1-t}}{\sqrt{\tilde{F}_3(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, t)}} d\tilde{X}_1 d\tilde{X}_2 d\tilde{X}_3,$$

where

$$\tilde{F}_3(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, t) = \prod_{i=1}^3 (1 - \tilde{X}_i) \prod_{i=1}^2 (\tilde{X}_i - \tilde{X}_{i+1}) \prod_{i=1,3} \{ (1 - \tilde{X}_i)t + \tilde{X}_i \}.$$

Note that the region of integration is now independent of t ; moving the \oint inside and performing the further substitutions $\tilde{X}_3 = X_3, \tilde{X}_2 = X_2 X_3, \tilde{X}_1 = X_1 X_2 X_3$, the above integral

$$(5.8) \quad = \iiint_{[0,1]^3} \left(\oint_{|t|=\epsilon} \frac{\sqrt{1-t}}{\sqrt{F_3(X_1, X_2, X_3, t)}} \frac{dt}{2\pi i t} \right) dX_1 dX_2 dX_3$$

where

$$F(X_1, X_2, X_3, t) = \{ (1 - X_3)t + X_3 \} \{ (1 - X_1 X_2 X_3)t + X_1 X_2 X_3 \} X_1 X_2 \prod_{i=1}^3 (1 - X_i).$$

Next, we break $[0, 1]^3$ in (5.8) into 4 regions according to whether (I) $X_1 X_2 X_3 > \eta := \frac{\epsilon}{1+\epsilon}$, (IIa) $X_2 X_3 > \eta > X_1 X_2 X_3$, (IIb) $X_3 > \eta > X_2 X_3$, or (III) $\eta > X_3$. The

expansion of $F_3(X_1, X_2, X_3, t)^{-\frac{1}{2}}$ depends on the region:

$$\begin{aligned}
 \text{(I)} \quad & \sum_{a,b \geq 0} t^{a+b} \binom{-\frac{1}{2}}{a} \binom{-\frac{1}{2}}{b} \frac{X_1^{-a} X_2^{-a} X_3^{-a-b} (1-X_3)^{b-\frac{1}{2}} (1-X_1 X_2 X_3)^a}{\sqrt{(1-X_1)(1-X_2)}}, \\
 \text{(IIa,b)} \quad & \sum_{a,b \geq 0} t^{a-b-\frac{1}{2}} \binom{-\frac{1}{2}}{a} \binom{-\frac{1}{2}}{b} \frac{X_1^b X_2^b X_3^{b-a} (1-X_3)^{a-\frac{1}{2}} (1-X_1 X_2 X_3)^{-b}}{\sqrt{(1-X_1)(1-X_2)}}, \\
 \text{(III)} \quad & \sum_{a,b \geq 0} t^{-a-b} \binom{-\frac{1}{2}}{a} \binom{-\frac{1}{2}}{b} \frac{X_1^a X_2^a X_3^{a+b} (1-X_3)^{-a-\frac{1}{2}} (1-X_1 X_2 X_3)^{-b}}{\sqrt{(1-X_1)(1-X_2)}}.
 \end{aligned}$$

For purposes of working modulo $\mathcal{O}(\epsilon \log^3 \epsilon)$, computation shows that we may replace $\sqrt{1-t}$ and $(1-X_1 X_2 X_3)$ by 1; whereas $(1-X_i)^{-\frac{1}{2}}$ is always expanded as¹⁶ $\sum_{k_i \geq 0} \binom{\frac{1}{2}}{k_i} X_i^{k_i}$. (The special case $(1-X_3)^{a-\frac{1}{2}}$ is expanded when $a=0$ and replaced by 1 when $a > 0$.) We may also replace η by ϵ in the triple integrals, which become

$$\begin{aligned}
 \text{(I)} \quad & \int_{\epsilon}^1 \int_{\frac{\epsilon}{X_3}}^1 \int_{\frac{\epsilon}{X_2 X_3}}^1, \\
 \text{(IIa)} \quad & \int_{\epsilon}^1 \int_{\frac{\epsilon}{X_3}}^1 \int_0^{\frac{\epsilon}{X_2 X_3}}, \\
 \text{(IIb)} \quad & \int_{\epsilon}^1 \int_0^{\frac{\epsilon}{X_3}} \int_0^1, \text{ and} \\
 \text{(III)} \quad & \int_0^{\epsilon} \int_0^1 \int_0^1.
 \end{aligned}$$

Region (III) makes no contribution.

Performing the \oint in region (I) kills all terms except $(a, b) = (0, 0)$. So the portion of (5.8) over region (I) is

$$\begin{aligned}
 \int_{\epsilon}^1 \int_{\frac{\epsilon}{X_3}}^1 \int_{\frac{\epsilon}{X_2 X_3}}^1 \prod_{i=1}^3 X_i^{-1} (1-X_i)^{-\frac{1}{2}} dX_i = \\
 \sum_{k_1, k_2, k_3 \geq 0} \left(\prod_{i=1}^3 \binom{\frac{1}{2}}{k_i} \right) \int_{\epsilon}^1 \int_{\frac{\epsilon}{X_3}}^1 \int_{\frac{\epsilon}{X_2 X_3}}^1 \left(\prod_{j=1}^3 X_j^{k_j-1} dX_j \right),
 \end{aligned}$$

which is now (5.5) with $d=3$. Repeatedly applying the formula

$$- \int_{\mu}^1 (\log^r x) x^{k-1} dx = \begin{cases} \frac{1}{r+1} \log^{r+1} \mu, & k=0 \\ \sum_{\ell=0}^r \frac{(-1)^{\ell} \mu^k r!}{k^{\ell+1} (r-\ell)!} \log^{r-\ell} \mu + \frac{(-1)^{r+1} r!}{k^{r+1}}, & k \neq 0 \end{cases}$$

and throwing out terms with positive powers of ϵ , we arrive at

$$(5.9) \quad -\frac{1}{6} \log^3 \epsilon + \frac{3}{2} \gamma_1 \log^2 \epsilon + (3\gamma_2 - 3\gamma_1^2) \log \epsilon + (\gamma_1^3 - 6\gamma_1 \gamma_2 + 3\gamma_3).$$

For region (IIa,b), applying (5.4) and computing the triple integrals (and simplifying results using (A.5)) yields

$$(5.10) \quad \frac{\gamma_1}{2} \log^2 \epsilon - (2\gamma_1^2 + \beta) \log \epsilon + (\gamma_1^3 - 2\gamma_1 \gamma_2 + \gamma_1 \beta + \nu - \psi)$$

for (IIa) and

$$(5.11) \quad -\delta \log(\epsilon) + (\gamma_1 \delta + \nu')$$

¹⁶the Pochhammer symbol $\binom{\frac{1}{2}}{k} = |(-\frac{1}{2})_k|$.

for (IIb). The meaning of the assorted Greek letters is

$$\begin{aligned} \nu &:= \frac{1}{\pi} \sum' \frac{\left(\frac{1}{2}\right)_{k_1} \left(\frac{1}{2}\right)_{k_2} \left(\frac{1}{2}\right)_a \left(\frac{1}{2}\right)_b}{(b-a+\frac{1}{2})(b+k_1+\frac{1}{2})(a+k_1)(a+k_2)}, \\ \nu' &:= \frac{1}{\pi} \sum' \frac{\left(\frac{1}{2}\right)_{k_1} \left(\frac{1}{2}\right)_{k_2} \left(\frac{1}{2}\right)_a \left(\frac{1}{2}\right)_b}{(b-a+\frac{1}{2})(b+k_1+\frac{1}{2})(b+k_2+\frac{1}{2})(a+k_1)}, \\ \beta &:= \frac{1}{\pi} \sum' \frac{\left(\frac{1}{2}\right)_a \left(\frac{1}{2}\right)_b}{(b+\frac{1}{2})a(a+b+\frac{1}{2})}, \quad \delta := \frac{1}{\pi} \sum' \frac{\left(\frac{1}{2}\right)_a \left(\frac{1}{2}\right)_b}{(b+\frac{1}{2})^2(a+b+\frac{1}{2})}, \\ \psi &:= \frac{1}{\pi} \sum' \frac{\left(\frac{1}{2}\right)_a \left(\frac{1}{2}\right)_b}{(b+\frac{1}{2})a^2(a+b+\frac{1}{2})}, \end{aligned}$$

where \sum' denotes summation over all 2, 3, or 4-tuples of non-negative integers for which the denominator is nonzero.

Now it is easy to prove that $\beta + \delta = \gamma_1 \tilde{\gamma}_1 + \tilde{\gamma}_2 = 2\gamma_1^2 + \gamma_2$. Adding (5.9), (5.10), and (5.11), replacing $\log \epsilon$ by $\log s + 4\gamma_1$, and using (A.1)–(A.3) gives

$$-\frac{1}{6} \log^3 s + 2\zeta(2) \log s + \left(6\zeta(3) - 2\gamma_1 \zeta(2) - \frac{8}{3} \gamma_1^3 + \nu + \nu' - \psi\right).$$

The following lemma will be proved in Section 5.5.

Lemma 5.5 $\nu + \nu' - \psi = \frac{8}{3} \gamma_1^3 + 2\gamma_1 \zeta(2) - 12\zeta(3)$.

Reinstating the factor of $\frac{2^3}{(2\pi i)^3}$, we have the following theorem.

Theorem 5.6 For $d = 3$, the canonically normalized $\tilde{\Pi}_d^{\text{nilp}}(s)$ is given by

$$-\frac{4}{3} \ell^3(s) + 16 \frac{\zeta(2)}{(2\pi i)^2} \ell(s) - \underbrace{48 \frac{\zeta(3)}{(2\pi i)^3}}_{\tilde{a}_{00}}.$$

We conclude that the extension class $\xi = \tilde{a}_{00} \in \mathbb{C}/\mathbb{Q}$ satisfies Conjecture 2.3.

5.5 The G_2 -VHS \mathcal{V}_6

The comparable simplifications on (5.2) for $d = 6$ lead to $(\frac{2^6}{(2\pi i)^6}$ times)

$$\int_{[0,1]^{\times 6}} \left(\oint_{|t|=\epsilon} \frac{\sqrt{X_5 X_6} \sqrt{1-t}}{\sqrt{F_6(X_1, \dots, X_6, t)}} \frac{dt}{2\pi i t} \right) dX_1 \cdots dX_6,$$

where

$$F(X_1, \dots, X_6, t) = X_1 X_2 \prod_{i=1}^6 (1 - X_i) \prod_{j=1,3,5} \{(1 - X_j \cdots X_6)t + X_j \cdots X_6\}.$$

The region of integration breaks, as before, into (I) $X_1 \cdots X_6 > \eta$, (IIa,b) $X_3 \cdots X_6 > \eta > X_1 \cdots X_6$, (IIIa,b) $X_5 X_6 > \eta > X_3 \cdots X_6$, and (IVa,b) $\eta > X_5 X_6$. Working modulo

$\mathcal{O}(\epsilon \log^6 \epsilon)$, the region (I) integral again is just (5.5), and yields

$$\begin{aligned} & \frac{1}{720} \log^6 \epsilon - \frac{\gamma_1}{20} \log^5 \epsilon + \left\{ -\frac{\gamma_2}{4} + \frac{5\gamma_1^2}{8} \right\} \log^4 \epsilon + \left\{ -\gamma_3 + 5\gamma_1\gamma_2 - \frac{10}{3}\gamma_1^3 \right\} \log^3 \epsilon \\ & + \left\{ -3\gamma_4 + 15\gamma_1\gamma_3 + \frac{15}{2}\gamma_2^2 - 30\gamma_1^2\gamma_2 + \frac{15}{2}\gamma_1^4 \right\} \log^2 \epsilon \\ & + \left\{ -6\gamma_5 + 30\gamma_1\gamma_4 + 30\gamma_2\gamma_3 - 60\gamma_1\gamma_2^2 - 60\gamma_1^2\gamma_3 + 60\gamma_1^3\gamma_2 - 6\gamma_1^5 \right\} \log \epsilon \\ & + \left\{ -6\gamma_6 + 30\gamma_1\gamma_5 + 30\gamma_2\gamma_4 + 15\gamma_3^2 - 20\gamma_2^3 - 120\gamma_1\gamma_2\gamma_3 - 60\gamma_1^2\gamma_4 \right. \\ & \quad \left. + 90\gamma_1^2\gamma_2^2 + 60\gamma_1^3\gamma_3 - 30\gamma_1^4\gamma_2 + \gamma_1^6 \right\}. \end{aligned}$$

Notice (in view of (A.4)) the $-36\zeta(5)$ in the coefficient of $\log \epsilon$.

Unfortunately, other regions produce some series we are (at present) unable to evaluate; for instance, the coefficient of $\log \epsilon$ in the (IIb) integral contains the term

$$\frac{1}{\pi} \sum' \frac{\left(\frac{1}{2}\right)_{k_1} \left(\frac{1}{2}\right)_{k_2} \left(\frac{1}{2}\right)_{k_3} \left(\frac{1}{2}\right)_{k_4} \left(\frac{1}{2}\right)_{a_1} \left(\frac{1}{2}\right)_{a_2} \left(\frac{1}{2}\right)_{a_3}}{\left(a_3 - a_2 - a_1 + \frac{1}{2}\right) \left(k_1 + a_3 + \frac{1}{2}\right) \times \left(k_2 + a_3 + \frac{1}{2}\right) \left(k_3 - k_2 - a_2\right) \left(k_4 - k_3\right) \left(k_4 + a_1\right)}.$$

So we will limit ourselves here to evaluating only the first four terms of $\widetilde{\Pi}_6^{\text{nilp}}$. Adding the contributions from (IIa,b) to the first line of the region (I) result, gives

$$\begin{aligned} & \frac{1}{6!} \log^6 \epsilon - \frac{7\gamma_1}{5!} \log^5 \epsilon + \left\{ \frac{49}{48}\gamma_1^2 - \frac{5}{24}\zeta(2) \right\} \log^4(\epsilon) \\ & + \left\{ -\frac{109}{12}\gamma_1^3 + \frac{37}{6}\gamma_1\zeta(2) - 2\zeta(3) - \frac{\nu + \nu' - \psi}{6} \right\} \log^3 \epsilon. \end{aligned}$$

Normalizing this and multiplying by $\frac{1}{(2\pi i)^6}$, modulo $\mathcal{O}(\log^2 s)$ we have

$$\begin{aligned} & \frac{1}{2^6} \widetilde{\Pi}_6^{\text{nilp}}(s) \equiv \\ & \frac{1}{6!} \ell^6(s) - \underbrace{\frac{5\zeta(2)}{24(2\pi i)^2} \ell^4(s)}_{\widetilde{a}_{40}/2^6} + \underbrace{\frac{1}{(2\pi i)^3} \left(\frac{4}{9}\gamma_1^3 + \frac{1}{3}\gamma_1\zeta(2) - 2\zeta(3) - \frac{\nu + \nu' - \psi}{6} \right)}_{\widetilde{a}_{30}/2^6} \ell^3(s). \end{aligned}$$

Referring to the end of Section 3, Proposition 5.4 now implies that $\widetilde{a}_{40}, \widetilde{a}_{30} \in \mathbb{Q}$. For \widetilde{a}_{40} , this is clearly the case, but \widetilde{a}_{30} belongs to $i\mathbb{R}$ and so must be zero, proving Lemma 5.5. This is particularly striking, as our knowledge that \widetilde{a}_{30} is rational depends on Proposition 5.1(ii). So it is thanks to G_2 that we can evaluate $\nu + \nu' - \psi$, and with it, the $d = 3$ LMHS.

For $d = 6$, we expect Conjecture 2.4 to remain true:

Conjecture 5.7 *The canonically normalized $\widetilde{\Pi}_6^{\text{nilp}}(s)$ is given by*

$$\frac{4}{45} \ell^6(s) + \frac{5}{9} \ell^4(s) + q_2 \ell^2(s) + q_1 \frac{\zeta(5)}{(2\pi i)^5} \ell(s) + q_0,$$

where $q_0, q_1, q_2 \in \mathbb{Q}$.

Appendix A Some Hypergeometric Special Values

The vanishing cycle periods $\int_{\mu_t} \omega_t$ (for each $d \geq 1$) in Section 5.2 are close cousins of the hypergeometric functions

$${}_{d+1}F_d \left(\begin{matrix} \frac{1}{2}, \dots, \frac{1}{2} \\ 1, \dots, 1 \end{matrix} \middle| t \right) = \frac{2^{d-1}}{(2\pi i)^d} \int_{0 \leq x_d \leq \dots \leq x_1 \leq 1} \frac{dx_1 \wedge \dots \wedge dx_d}{\sqrt{(\prod_{i=1}^d x_i)(x_1 - 1)(\prod_{j=1}^{d-1} (x_{j+1} - x_j))(1 - tx_d)}}$$

with the main discrepancy arising from the alternation between x_i and $(1 - x_j)$ (rather than simply having $\prod x_i$) under the radical in our setup. So it is not surprising that hypergeometric special values such as (5.6)–(5.7) appear in the coefficients of powers of $\log t$ in $\int_{\tau_t} \omega_t$. In this appendix, we shall explain how to derive expressions for these constants in terms of Riemann zeta values. Writing (by abuse of notation) $\zeta(1) := \log 4$, we have the following for (5.6):

(A.1) $\gamma_1 = \zeta(1),$

(A.2) $\gamma_2 = \zeta(2) - \frac{1}{2}\zeta(1)^2,$

(A.3) $\gamma_3 = 2\zeta(3) - \zeta(2)\zeta(1) + \frac{1}{6}\zeta(1)^3,$

(A.4) $\gamma_4 = \frac{9}{4}\zeta(4) - 2\zeta(3)\zeta(1) + \frac{1}{2}\zeta(2)\zeta(1)^2 - \frac{1}{24}\zeta(1)^4,$
 $\gamma_5 = 6\zeta(5) - \frac{9}{4}\zeta(4)\zeta(1) - 2\zeta(3)\zeta(2) + \zeta(3)\zeta(1)^2 - \frac{1}{6}\zeta(2)\zeta(1)^3$
 $\quad \quad \quad + \frac{1}{120}\zeta(1)^5,$
 $\gamma_6 = \frac{79}{16}\zeta(6) - 6\zeta(5)\zeta(1) + \frac{9}{8}\zeta(4)\zeta(1)^2 - 2\zeta(3)^2 + 2\zeta(3)\zeta(2)\zeta(1)$
 $\quad \quad \quad - \frac{1}{3}\zeta(3)\zeta(1)^3 + \frac{1}{24}\zeta(2)\zeta(1)^4 - \frac{1}{720}\zeta(1)^6.$

We also record some values of (5.7):¹⁷

(A.5) $\tilde{\gamma}_0 = 1, \tilde{\gamma}_1 = \zeta(1), \tilde{\gamma}_2 = \zeta(2) + \frac{1}{2}\zeta(1)^2, \tilde{\gamma}_3 = 2\zeta(3) + \zeta(2)\zeta(1) + \frac{1}{6}\zeta(1)^3.$

The values $\gamma_4, \gamma_5, \gamma_6, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ were computed using Mathematica [MATH], though the method used below for $\gamma_1, \gamma_2, \gamma_3$ would also suffice.

We shall proceed by expressing the associated hypergeometric function

$$h_n(t) = \sum_{k=1}^{\infty} t^k \binom{-1/2}{k} k^{-n} = (-t/2) {}_{n+2}F_{n+1} \left(\begin{matrix} 1, \dots, 1, \frac{3}{2} \\ 2, \dots, 2 \end{matrix} \middle| -t \right)$$

in terms of polylogarithms for $n \leq 3$. Since we are interested in $\gamma_n = h_n(-1)$, we introduce the auxiliary function $f_n(u) = h_n(u^2 - 1)$. Manipulation of power series shows that

$$\frac{dh_n}{dt} = (1/t)h_{n-1}(t), \quad h_n(0) = 0,$$

¹⁷These suggest a delightful relation to the γ_n , which, in fact, fails for $n \geq 4$.

and therefore

$$\frac{df_n}{du} = \frac{2u}{u^2 - 1} f_{n-1}(u) = \left(\frac{1}{u - 1} + \frac{1}{u + 1} \right) f_{n-1}(u).$$

In particular, since $f_n(1) = h_n(0) = 0$, we have an iterated integral formula

$$f_n(u) = \int_1^u \frac{df_n}{dv} dv = \int_1^u \left(\frac{1}{v - 1} + \frac{1}{v + 1} \right) f_{n-1}(v) dv,$$

which starts with

$$f_0(u) = h_0(u^2 - 1) = 1/u - 1$$

by the binomial series.

Lemma A.1 $f_1(u) = -2(\log(u + 1) - \log(2)).$

Proof By the above

$$f_1(u) = \int_1^u \frac{2v}{v^2 - 1} (1/v - 1) dv = -2 \int_1^u \frac{dv}{1 + v},$$

which gives the result. ■

It follows that $\gamma_1(= f_1(0)) = \log(4)$, as desired.

At this point, we observe that the differential of $f_1(u)$ is $-2/(u + 1)$, so we are really calculating a series of iterated integrals on $\mathbb{P}^1 \setminus \{1, -1, \infty\}$. We therefore make the change of variables $2w = v + 1$ to obtain

$$f_n(2u - 1) = \int_1^u \left(\frac{1}{w - 1} + \frac{1}{w} \right) f_{n-1}(2w - 1) dw.$$

Integration either by hand or with Mathematica gives the following lemma.

Lemma A.2

$$\begin{aligned} f_2(2w - 1) &= 2 \operatorname{Li}_2(1 - w) - \log^2(w) \\ f_3(2w - 1) &= 2 \operatorname{Li}_3(1 - w) - 2 \operatorname{Li}_3(w) + 2\zeta(3) \\ &\quad + 2\zeta(2) \log(w) - \log(1 - w) \log^2(w) - (1/3) \log^3(w). \end{aligned}$$

To integrate $f_1(u)$ by hand to obtain $f_2(u)$, we observe that

$$f_1(u) = 2 \sum_{\ell=1}^{\infty} \frac{(u - 1)^\ell (-1)^\ell}{2^\ell \ell}$$

at $u = 1$. This allows us to compute the integral of $f_1(u)/(u - 1)$ in terms of Li_2 . The integral of $f_1(u)/(u + 1)$ is elementary. To determine $f_3(u)$ we must integrate both $f_2(v)/(v - 1)$ and $f_2(v)/(v + 1)$. To this end, we use the following identity that allows us to interchange $\operatorname{Li}_2(w)$ and $\operatorname{Li}_2(1 - w)$.

Lemma A.3 We have that

$$\operatorname{Li}_2(w) + \operatorname{Li}_2(1 - w) = \zeta(2) - \log(w) \log(1 - w)$$

and hence $2 \operatorname{Li}_2(1/2) = \zeta(2) - \frac{1}{4} \zeta^2(1).$

Proof Differentiate both sides with help of the formulas:

$$\frac{d}{dw} \operatorname{Li}_2(w) = -\frac{\log(1-w)}{w}, \quad \frac{d}{dw} \operatorname{Li}_2(1-w) = \frac{\log(w)}{1-w}.$$

The constant of integration $\zeta(2)$ is obtained by taking the limit at $w \rightarrow 1$. ■

The two remaining integrals needed to calculate f_3 are given by the following lemma.

Lemma A.4

$$\int \frac{\log^2(w)}{w-1} = \log(1-w) \log^2(w) + 2 \log(w) \operatorname{Li}_2(w) - 2 \operatorname{Li}_3(w)$$

$$\int \frac{\log(w) \log(1-w)}{w} dw = \operatorname{Li}_3(w) - \operatorname{Li}_2(w) \log(w).$$

Proof Differentiate both sides of each equation using

$$\frac{d}{dw} \operatorname{Li}_3(w) = \frac{\operatorname{Li}_2(w)}{w}, \quad \frac{d}{dw} \operatorname{Li}_2(w) = -\frac{\log(1-w)}{w}. \quad \blacksquare$$

Remark A.5 Based on empirical evidence gathered using Mathematica, it appears that

$$\gamma_d = \sum_P c_P \zeta(p_1) \cdots \zeta(p_r),$$

where the sum runs over all partitions P of d into a sum of positive integers, and the coefficients c_P are determined as follows:

- The coefficient of $\zeta(d)$ in γ_d is

$$c_d = \frac{2^d - 2}{d}, \quad d > 1.$$

- The coefficient of $\zeta(1)^d$ in γ_d is $\frac{(-1)^{d-1}}{d!}$.
- If $P = (p_1, \dots, p_r)$ is a partition of d with all $p_j > 1$, then

$$c_P = (-1)^{r-1} c_{p_1} \cdots c_{p_r}$$

assuming all p_j 's are distinct. More generally, if P contains elements with multiplicity $m_1, \dots, m_k > 1$, then

$$c_P = \frac{(-1)^{r-1}}{m_1! \cdots m_k!} c_{p_1} \cdots c_{p_r}$$

For example, $\zeta(a)\zeta(b)^2$ appears with coefficient $\frac{1}{2}c_a c_b^2$, while $\zeta(a)^3$ appears with coefficient $\frac{1}{6}c_a^3$.

- The coefficient of $\zeta(p_1) \cdots \zeta(p_r) \zeta(1)^c$ is equal to $\frac{(-1)^c}{c!}$ times the coefficient of $\zeta(p_1) \cdots \zeta(p_r)$. This assumes all $p_j > 1$.

For example, using γ_2, γ_3 we compute that

$$\gamma_4 = \frac{7}{2}\zeta(4) - 2\zeta(3)\zeta(1) - \frac{1}{2}\zeta(2)^2 + \frac{1}{2}\zeta(2)\zeta(1)^2 - \frac{1}{24}\zeta(1)^4,$$

where $\frac{7}{2}\zeta(4) - \frac{1}{2}\zeta(2)^2 = \frac{9}{4}\zeta(4)$. Likewise, to calculate the coefficient of π^6 in γ_6 , we consider

$$\frac{31}{3}\zeta(6) - \frac{7}{2}\zeta(4)\zeta(2) + \frac{1}{6}\zeta(2)^3 = \frac{79}{16}\zeta(6).$$

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