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DISCIPLINED SPACES AND CENTRALIZER CLONE SEGMENTS

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ABSTRACT. Our main result implies that for any choice $1 \leq m \leq n \leq p$ of integers there exist finitary algebras A_1 and A_2 that generate the same variety, and such that the initial k-segments of their centralizer clones coincide exactly when $k \leq m$, are isomorphic exactly when $k \leq n$ and are elementarily equivalent exactly when $k \leq p$. The proof uses the existence and properties of disciplined topological spaces which we introduce and investigate here.

1. **Introduction.**

1.1 . In simplest terms, the centralizer clone Clo(A) of a universal algebra A is the category of all homomorphisms between finite powers $A^0 = \{\emptyset\}, A^1 = A, A^2, A^3, \ldots$ of A. For any integer $r \geq 1$, all homomorphisms between the powers $A^0, A^1, \ldots A^r$ form what is called the initial r-segment $\mathrm{Clo}_r(A)$ of Clo(A). It is clear that the equality of centralizer clone segments $\mathrm{Clo}_r(A_1)$ and $\mathrm{Clo}_r(A_2)$ of algebras A_1, A_2 always implies their isomorphism, and that their isomorphism always implies their elementary equivalence. The present paper demonstrates that, subject only to these obvious dependencies, the three upper boundaries of equality, isomorphism and elementary equivalence of centralizer clone segments of universal algebras may be chosen at will.

This result translates an earlier one [7] on continuous maps of metrizable spaces, and exploits the method used by the first author in her preprint [10] to separate equality from elementary equivalence for centralizer clone segments of universal algebras. These two sources, in turn, derive their essence from yet another first author's paper [9] on continuous maps of metrizable topological spaces. The original impetus for [10] came from Ralph McKenzie who suggested that, having completed [9], the first author also investigate centralizer clones of universal algebras.

Results of [7] on continuous maps of metrizable spaces can be 'translated' into our present results on homomorphisms of universal algebras only for metrizable spaces which are disciplined (see 2.2 below). We show that metrizable spaces constructed in [7] are disciplined and have other properties enabling the translation to proceed.

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A somewhat more general notion of a clone is needed for an accurate description of our results.

1.2 . Let $\omega = \{0, 1, 2, \ldots\}$ denote the set consisting of all finite ordinals $n = \{0, \ldots, n-1\}$.

A subcategory T of the category Set of all sets and maps is called a *clone on a non-empty set* X if the set obj T of all objects of T consists of all finite Cartesian powers $X^0 = \{\emptyset\}, X^1 = X, X^2 = X \times X, \ldots$ of X, the product projections $p_i^{(m)}: X^m \longrightarrow X$ with $i \in m$ are morphisms in T for any $m \in \omega$, and every $X^m \in$ obj T accompanied by all Cartesian product projections $p_i^{(m)}: X^m \longrightarrow X$ with $i \in m$ is a categorical product, in T, of m copies of X. This means that for any $m, n \in \omega$ and any m-tuple $f_i: X^n \longrightarrow X$ of maps in T, the map $f: X^n \longrightarrow X^m$ defined by

$$f(y) = \big(f_0(y), \ldots, f_{m-1}(y)\big) \text{ for all } y \in X^n$$

also belongs to T. In particular, for each $\psi: m \longrightarrow n$ with $m, n \in \omega$, the category T contains the map $p^{[\psi]}: X^n \longrightarrow X^m$ given for every $y = (y_0, \ldots, y_{n-1}) \in X^n$ by

$$p^{[\psi]}(y_0, \ldots, y_{n-1}) = (y_{\psi(0)}, \ldots, y_{\psi(m-1)}).$$

Set $\Gamma_T(m) = X^m$ for any $m \in \omega$, and $\Gamma_T(\psi) = p^{[\psi]}$ for any mapping $\psi: m \longrightarrow n$ with $m, n \in \omega$. Then $\Gamma_T: \mathrm{Ord}_\omega \longrightarrow T$ is a well-defined contravariant functor from the category Ord_ω of all finite ordinals and all mappings between them, and $\Gamma_T(\psi_j^{(n)}) = p_j^{(n)}$ for every map $\psi_j^{(n)}: 1 \longrightarrow n$ with $\psi_j^{(n)}(0) = j \in n$.

Let k be an abstract category and let H be an isomorphism of k onto a clone T on a non-empty set X. Then the composite $\alpha = H^{-1} \circ \Gamma_T : \mathrm{Ord}_\omega \longrightarrow k$ is a contravariant functor which endows k with the following additional structure:

(g) k has a uniquely determined object a such that obj $k = \{a^n \mid n \in \omega\}$, where a^n is an n-th power of a in the category k, and

(e) k has a specific enumeration of product projections $\pi_j^{(n)}: a^n \longrightarrow a$ by members of n.

Indeed, it suffices to set $a = \alpha(1)$, $a^n = \alpha(n)$ and $\pi_j^{(n)} = \alpha(\psi_j^{(n)})$ for $j \in n \in \omega$. Conversely, the information provided in (g) and (e) gives rise to such a contravariant functor $\alpha: \mathrm{Ord}_\omega \longrightarrow k$.

A category k for which (g) and (e) hold (or, equivalently, a category accompanied by such a functor $\alpha: \mathrm{Ord}_\omega \longrightarrow k$) will be called an *abstract clone* and $a \in$ obj k its *generating object*.

Therefore, up to trivial cases when α is not faithful, any abstract clone is the categorical dual of an algebraic theory as defined by Lawvere.

We say that abstract clones k and k' are *isomorphic*, and we write

$$k \simeq k',$$

if there exists an isofunctor Φ of k onto k' such that $\Phi \circ \alpha = \alpha'$ for the respective contravariant functors α and α' of Ord_ω into k and k'. In particular, any isomorphism preserves generating objects and projection enumeration.

1.3 . A pair (k, F) is called a *clone* whenever k is an abstract clone and $F\colon k \longrightarrow \mathrm{Set}$ is a faithful functor which preserves finite products.

We say that clones (k, F) and (k', F') are *isomorphic*, and we write

$$(k, F) \simeq (k', F'),$$

whenever the abstract clones k and k' are isomorphic. We say that the two clones are *equal*, and we write

$$(k, F) = (k', F'),$$

if there exists an isomorphism Φ of the abstract clone k onto the abstract clone k' (that is, an isofunctor Φ such that $\Phi \circ \alpha = \alpha'$) for which $F' \circ \Phi = F$.

1.4 . If $\mathcal{K}$ is a category with finite products, then every object a of $\mathcal{K}$ determines an abstract clone k, uniquely up to an isomorphism, as follows: for every $n \in \omega$ we select an n-th power a^n of a and an enumeration of its product projections, and then let k be the full subcategory of $\mathcal{K}$ generated by these powers.

If $(\mathcal{K}, \mathcal{U})$ is a concrete category such that $\mathcal{K}$ has finite products and the forgetful functor $U\colon \mathcal{K} \longrightarrow \mathrm{Set}$ preserves them (that is, if $(\mathcal{K}, U)$ has finite concrete products), then every object a of $\mathcal{K}$ with $U(a) \neq \emptyset$ determines, up to an equality of clones induced on the non-empty set $U(a)$, a clone (k, F), namely that consisting of an abstract clone k determined by a in $\mathcal{K}$ and the restriction F of U to k. Any such clone (k, F) will be denoted as $\mathrm{Clo}(a)$, or $\mathrm{Clo}(a, \mathcal{K})$ when $\mathcal{K}$ needs to be explicitly mentioned.

1.5 . Any nonvoid universal algebra A of a finitary type Σ determines two distinct clones on its underlying set X. As suggested by [6], for instance, these are:

(1) the clone $\mathrm{Clo}(A)$ formed by all homomorphisms between finite powers of A, called the *centralizer clone* of A, and

(2) the *operation clone* of A, which is the smallest clone on X containing all operations of A.

While the operation clone of A determines the centralizer clone $\mathrm{Clo}(A)$ uniquely, its abstract clone does not do this at all, as shown in Theorem 1.9 below.

1.6 . Let $r \geq 0$ be an integer. If k is an abstract clone generated by the object a, then its full subcategory generated by $\{a^j \mid j = 0, 1, \ldots, r\} \subset \mathrm{obj}\, k$ augmented by the enumeration of the product projections $\pi_i^{(j)}\colon a^j \longrightarrow a$ inherited from k is called the *r-segment* of k. A pair (k_r, F_r) is the r-segment of a clone (k, F) if k_r is the r-segment of k and F_r is the restriction of F to the full subcategory k_r of k. For any concrete category $(\mathcal{K}, U)$ with finite concrete products and any $a \in \mathrm{obj}\, \mathcal{K}$ with $U(a) \neq \emptyset$, the r-segment of a clone $\mathrm{Clo}(a)$ will be denoted by $\mathrm{Clo}_r(a)$.

We say that two r-segments k_r, k'_r of abstract clones k, k' are isomorphic and write

$$k_r \simeq k'_r$$

if there exists an isofunctor Φ_r of k_r onto k'_r such that $\Phi_r \circ \alpha_r = \alpha'_r$ where α_r, α'_r are restrictions of the respective contravariant functors α and α'.

If (k_r, F_r) and (k'_r, F'_r) are r-segments of clones, we say that they are *isomorphic* and write

$$(k_r, F_r) \simeq (k'_r, F'_r)$$

whenever $k_r \simeq k'_r$. Finally, the two clone r-segments (k_r, F_r) and (k'_r, F'_r) are *equal*, and we write

$$(k_r, F_r) = (k'_r, F'_r)$$

if there exists an isofunctor Φ_r of k_r onto k'_r such that $\Phi_r \circ \alpha_r = \alpha'_r$ and $F'_r \circ \Phi_r = F_r$.

1.7. In addition to their equality and isomorphism, we shall also investigate elementary equivalence of clones and clone segments. We say that clones or r-segments of clones (k, F) and (k', F') are *elementarily equivalent*, and write

$$(k, F) \approx (k', F'),$$

if the abstract clones or r-segments of abstract clones k and k' are elementarily equivalent (in symbols $k \approx k'$), that is, whenever k and k' satisfy the same formulas of the first order language of clone theory or of its fragment appropriate to r-segments. Paragraph 1.8 below introduces this language, as used by Taylor [8], and then in [9] and [7].

1.8. Any abstract clone k with a generating object a can be viewed as an ω-sorted universal algebra whose carrier X_n of the n-th sort is the set $k(a^n, a)$ of all k-morphisms $a^n \longrightarrow a$, and which has

(c) n distinct nullary operations $c_0^{(n)}, \ldots, c_{n-1}^{(n)}$ of each sort $n \in \omega$, and

(s) for any $m, n \in \omega$, an operation S_m^n of heterogeneous arity $(n, \overbrace{m, \ldots, m}^{n \text{ times}})$ whose values are of the sort m, that is,

$$S_m^n \colon X_n \times \overbrace{X_m \times \cdots \times X_m}^{n \text{ times}} \longrightarrow X_m.$$

Indeed, we need only interpret $c_0^{(n)}, \ldots, c_{n-1}^{(n)}$ as the product projections $\pi_0^{(n)}, \ldots, \pi_{n-1}^{(n)}$ of k and write

$$S_m^n(h, f_0, \ldots, f_{n-1}) = h \circ (f_0 \dot\times \cdots \dot\times f_{n-1})$$

in terms of the composition in k—where $f_0 \dot\times \cdots \dot\times f_{n-1}$ is the unique k-morphism f in $k(a^m, a^n)$ for which $\pi_i^{(n)} \circ f = f_i$ for all $i \in n$. These operations also satisfy all equations below, namely,

(RU) $S_n^n(h, c_0^{(n)}, \ldots, c_{n-1}^{(n)}) = h$ for every $n \in \omega$,

(LU) $S_m^n(c_i^{(n)}, f_0, \ldots, f_{n-1}) = f_i$ for all $m, n \in \omega$ and $i \in n$,

(AC) $S_m^p\big(h, S_m^n(g_0, f_0, \ldots, f_{n-1}), \ldots, S_m^n(g_{p-1}, f_0, \ldots, f_{n-1})\big) = S_m^n\big(S_n^p(h, g_0, \ldots, g_{p-1}), f_0, \ldots, f_{n-1}\big)$ for all $m, n, p \in \omega$.

Hence every abstract clone k determines a unique ω-sorted algebra whose operations are described in (c) and (s), and which satisfies equations (RU), (LU) and (AC). Conversely, any such ω-sorted algebra determines, up to an isomorphism, an abstract clone k.

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To see this, we choose an abstract generating object a for k and then formally require that $k(a^m, a^n)$ consist of all n-tuples of members of the carrier X_m of the sort m [with the unique 'empty' 0-tuple in case of $n = 0$]. If a 'composite' $g \circ f$ of $f = (f_0, \ldots, f_{n-1}) \in k(a^m, a^n)$ and $g = (g_0, \ldots, g_{p-1}) \in k(a^n, a^p)$ is defined by

$$g \circ f = \big(S^n_m(g_0, f_0, \ldots, f_{n-1}), \ldots, S^n_m(g_{p-1}, f_0, \ldots, f_{n-1})\big),$$

then the associativity of $\circ$ follows from (AC). The n-tuple $(c_0^{(n)}, \ldots, c_{n-1}^{(n)})$ is the unit in $k(a^n, a^n)$, and a^n is the n-th power of a with the projections $c_0^{(n)}, \ldots, c_{n-1}^{(n)} \in X_n = k(a^n, a)$ because of (RU) and (LU).

The first order language of clone theory is nothing else but the first order language of the ω-sorted algebras described above. Let $r \geq 1$ be an integer. Having restricted the sorts to $\{0, 1, \ldots, r\}$ and the operations to those $c_i^{(n)}$ and S^n_m for which $i \in n$ and $m, n \leq r$, we obtain a fragment of the first order language of clones which is just the first order language of clone r-segments.

Given an abstract clone or an r-segment of an abstract clone k, we shall call the above ω-sorted or $(r + 1)$-sorted algebra the *clone algebra* of k. The carrier of its n-th sort, that is, the set $k(a^n, a)$ of k-morphisms, will be denoted by $(k)_n$.

It is clear that two abstract clones or clone segments are isomorphic exactly when their corresponding clone algebras are isomorphic and, more generally, that product preserving functors that also preserve product projection enumeration correspond exactly to homomorphisms of the corresponding clone algebras.

1.9. MAIN THEOREM. *If $m, n, p \in \{1, 2, \ldots, \infty\}$ are such that $m \leq n \leq p$, then there exist finitary algebras A_1 and A_2 whose centralizer clones* $\mathrm{Clo}(A_1)$ *and* $\mathrm{Clo}(A_2)$ *satisfy*

$$m = \sup\{i \leq \omega \mid \mathrm{Clo}_i(A_1) = \mathrm{Clo}_i(A_2)\},$$
$$n = \sup\{i \leq \omega \mid \mathrm{Clo}_i(A_1) \simeq \mathrm{Clo}_i(A_2)\},$$
$$p = \sup\{i \leq \omega \mid \mathrm{Clo}_i(A_1) \approx \mathrm{Clo}_i(A_2)\}.$$

The algebras A_1, A_2 have isomorphic operation clones.

Next we outline the contents of the paper. In the second section, we assign a finitary algebra $\mathbb{A}(X)$, called a *trace* of X, to any metrizable topological space X. We also introduce the notion of a *disciplined* (metrizable) space X, for which we demonstrate a close connection between continuous maps $X^n \longrightarrow X$ and homomorphisms $\big(\mathbb{A}(X)\big)^n \longrightarrow \mathbb{A}(X)$. Then (see 2.9) we deduce that, for any two infinite disciplined spaces X_1, X_2,

$$\mathrm{Clo}_k(X_1) = \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) = \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

In the third section (see 3.5), we prove that

$$\mathrm{Clo}_k(X_1) \simeq \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \simeq \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big)$$

for any infinite disciplined spaces X_1, X_2 with no *absolute constants*—a clone notion introduced in 3.2. In the fourth section on elementary equivalence (see 4.1), we show that such spaces also satisfy

$$\mathrm{Clo}_k(X_1) \approx \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \approx \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

In [7], we presented a general construction method that produces metric spaces whose clones of continuous maps have certain useful properties, and then applied it to obtain a result similar to the present Theorem 1.9:

[7] for any choice of $m,n,p \in \{1,2,\ldots,\infty\}$ with $m \leq n \leq p$, there exist metric spaces X_1 and X_2 such that

$$\begin{aligned} m &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) = \mathrm{Clo}_i(X_2)\}, \\ n &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) \simeq \mathrm{Clo}_i(X_2)\}, \\ p &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) \approx \mathrm{Clo}_i(X_2)\}. \end{aligned}$$

In the fifth section, we briefly describe this method, and then supplement [7] by showing that all spaces it produces are disciplined, and almost none of them have absolute constants. Together with the results of Sections 2–4 mentioned earlier, this already shows that the algebras $A_1 = \mathbb{A}(X_1)$ and $A_2 = \mathbb{A}(X_2)$ satisfy the first statement of Theorem 1.9. Its remainder—the claim that the operation clones of A_1, A_2 are isomorphic—is proved in the sixth section, where we show that traces of any two infinite metrizable spaces satisfy precisely the same set of identities. This implies that operation clones of algebras $A_1 = A_1(m,n,p)$ and $A_2 = A_2(m,n,p)$ are isomorphic for all choices $m \leq n \leq p$ in $\{1,2,\ldots,\infty\}$, so that they determine the same abstract clone. Moreover, all these algebras will be constructed over the same underlying set. This means that an abstract (operation) clone can have representations on *the same set* which vary to the extent that their centralizer clones realize *every* triple $m \leq n \leq p$ in $\{1,2,\ldots,\infty\}$, in the sense described by Theorem 1.9.

Using [7] as a source of disciplined spaces without absolute constants, in the seventh section we apply the trace to 'translate' other results of [7] into results on finitary algebras. This section also describes alternate traces (unary and infinitary), and restates Theorem 1.9 for a finitary algebra and its reduct.

2. **Disciplined spaces and their algebraic traces.**

2.1 . To any metrizable topological space $X = (P,t)$ we now assign its *algebraic trace* $\mathbb{A}(X)$. This is a universal algebra $\mathbb{A}(X) = (Q, \{\cdot,\gamma\} \cup \{\pi_k \mid k \in \omega\})$ with a single binary operation $\cdot$ and countably many unary operations $\pi_0, \pi_1, \ldots$ and γ. The underlying set Q of $\mathbb{A}(X)$ is the disjoint union

$$Q = P \cup S \cup \{\lambda\},$$

where the set S consists of all one-to-one sequences $\{p_k \mid k \in \omega\}$ of members of P. The operations of $\mathbb{A}(X)$ are defined as follows:

$$q \cdot q' = \begin{cases} q & \text{if } q = q' \in P \\ \lambda & \text{otherwise;} \end{cases}$$

for every $k \in \omega$, we set

$$\pi_k(q) = \begin{cases} p_k & \text{if } q = \{p_k \mid k \in \omega\} \in S \\ q & \text{for every } q \in P \cup \{\lambda\}; \end{cases}$$

and finally, the operation γ which traces the topology t of X is given by

$$\gamma(q) = \begin{cases} p & \text{if } q \in S \text{ and } q \text{ converges to } p \text{ in } X \\ \lambda & \text{if } q \in S \text{ does not converge in } X \\ q & \text{for every } q \in P \cup \{\lambda\}. \end{cases}$$

It is clear that $\mathbb{A}(X)$ is isomorphic to $\mathbb{A}(X')$ whenever X is homeomorphic to X'. Conversely, if the traces $\mathbb{A}(X)$, $\mathbb{A}(X')$ are isomorphic then X is homeomorphic to X'. Indeed, if $\mathbb{A}(X)$ is a singleton then $X = X' = \emptyset$. Else $P \subset Q$ consists of all $q \in Q$ with $q \cdot q = q$ satisfying $q \cdot q' \neq q$ for every $q' \neq q$, and hence P is bijective to the underlying set P' of X', and X is homeomorphic to X' because of the remaining unary operations.

2.2. DEFINITION. Let $n \in \omega$. We say that a topological space $X = (P, t)$ is *n-disciplined* if every continuous map $f\colon X^n \longrightarrow X$ decomposes as a projection followed by a homeomorphism onto a closed subset of X or, more precisely, if for some set $M \subseteq n = \{0, \ldots, n-1\}$ and some homeomorhism $g\colon X^M \longrightarrow X$ of X^M onto a closed subset of X,

$$f = g \circ \pi^{(M)},$$

where $\pi^{(M)}\colon X^n \longrightarrow X^M$ is the projection of X^n onto X^M associated with the inclusion map $M \subseteq n$.

A space X is *disciplined* if it is n-disciplined for every $n \in \omega$.

REMARKS. (a) It is clear that every T_1-space is 0-disciplined.

(b) A space X is n-disciplined if and only if for every continuous map $f\colon X^n \longrightarrow X$ there exist an integer $k \geq 0$, a one-to-one mapping $\psi\colon k \longrightarrow n$, and a homeomorphism $g\colon X^k \longrightarrow X$ onto a closed subset of X such that $f = g \circ \pi^{[\psi]}$, where $\pi^{[\psi]}\colon X^n \longrightarrow X^k$ is the (surjective) projection given by $\pi^{[\psi]}(\varphi) = \varphi \circ \psi$ for every $\varphi \in X^n$. Any such decomposition $f = g \circ \pi^{[\psi]}$ will be called *standard*.

For any integer $n \geq 0$, let $\lambda^{(n)}\colon \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ be the constant map whose value is λ. Here is the central result of this section.

2.3. PROPOSITION. *Let $\mathbb{A}(X)$ be the algebraic trace of a disciplined metrizable space $X = (P, t)$, and let $n \geq 0$. Then*

(1) every continuous map $f\colon X^n \longrightarrow X$ has a unique extension to a homomorphism $E(f)\colon \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$, and $E(f)$ is constant if and only if f is constant,

(2) *if* $h\colon \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ *is a homomorphism, then either* $h = \lambda^{(n)}$, *or else* $h(P^n) \subseteq P$ *and the restriction of* h *to* P^n *maps* X^n *continuously into* X.

We recall that the algebraic trace $\mathbb{A}(X) = (Q, \{\cdot, \gamma\} \cup \{\pi_j \mid j \in \omega\})$ of a space $X = (P, t)$ is defined on the disjoint union $Q = P \cup S \cup \{\lambda\}$. For $k \geq 1$ and an arbitrary one-to-one map $g\colon P^k \longrightarrow P$ we define a map

$$h_g\colon Q^k \longrightarrow Q$$

as follows. For $q = (q_0, \ldots, q_{k-1}) \in Q^k$, we set

$$\text{(h1)} \qquad h_g(q_0, \ldots, q_{k-1}) = \begin{cases} \lambda & \text{if at least one } q_i = \lambda \\ g(q) & \text{if } q \in P^k. \end{cases}$$

The case still left undefined is that of a k-tuple $q \in (P \cup S)^k$ with $q_l \in S$ for at least one $l \in k$. For any such k-tuple $(q_0, \ldots, q_{k-1})$ and every $i \in k$ and $j \in \omega$ we first write

$$\tilde{q}_{j,i} = \begin{cases} q_i & \text{when } q_i \in P \\ \pi_j(q_i) & \text{when } q_i \in S, \end{cases}$$

and then assemble a sequence $\{\tilde{q}_j \mid j \in \omega\}$ with the members $\tilde{q}_j = (\tilde{q}_{j,0}, \ldots, \tilde{q}_{j,k-1}) \in P^k \subset Q^k$. This sequence is one-to-one because $q_l \in S$ for some $l \in k$. Since the mapping g is one-to-one,

$$\text{(h2)} \qquad h_g(q_0, \ldots, q_{k-1}) = \{g(\tilde{q}_j) \mid j \in \omega\}$$

is a one-to-one sequence of elements of P. Whence $h_g(q_0, \ldots, q_{k-1}) \in S$, and (h2) and (h1) define a mapping $h_g\colon Q^k \longrightarrow Q$ correctly and completely for any $k \geq 1$.

When $k = 0$, we have $P^0 = Q^0 = \{\emptyset\}$ and, for any $g\colon P^0 \longrightarrow P$, we define $h_g\colon Q^0 \longrightarrow Q$ to be the map with $h_g(\emptyset) = g(\emptyset)$.

Proposition 2.3 is proved in 2.4–2.9 below.

2.4. LEMMA. *Let* $k \geq 0$ *be an integer and let* $g\colon P^k \longrightarrow P$ *be a homeomorphism of* X^k *onto a closed subset of* X. *Then* $E(g) = h_g\colon \mathbb{A}(X)^k \longrightarrow \mathbb{A}(X)$ *is a homomorphism.*

PROOF. Write $h_g = h$. Since $\{p\}$ is a subalgebra of $\mathbb{A}(X)$ for every $p \in P$, we need consider only the case of $k \geq 1$. Let $q = (q_0, \ldots, q_{k-1})$ and $q' = (q'_0, \ldots, q'_{k-1})$ belong to Q^k.

a) First we show that the binary operation $\cdot$ is preserved by h.

a1) If $q \neq q'$, then $q_l \cdot q'_l = \lambda$ for some $i \in k$, so that $h(q \cdot q') = \lambda$. We show that $h(q) \cdot h(q') = \lambda$ as well. If $q_l \in S \cup \{\lambda\}$ for some $l \in k$, then $h(q) \in S \cup \{\lambda\}$ and hence $h(q) \cdot z = \lambda$ for all $z \in Q$. Since the operation $\cdot$ is commutative, in the only remaining case we have $q, q' \in P^k$. But h coincides with g on P^k and g is one-to-one, so that $h(q) \neq h(q')$ and $h(g) \cdot h(g') = \lambda$ follows.

a2) If $q = q'$, then either $q_l \in S \cup \{\lambda\}$ for some $l \in k$ and $h(q \cdot q') = \lambda = h(q) \cdot h(q')$ as in the previous case, or $q \in P^k$, and hence $q \cdot q = q$. But then $h(q \cdot q) = g(q)$ and $h(q) \cdot h(q) = g(q) \cdot g(q) = g(q)$.

b) Next we prove that h preserves π_j for each $j \in k$.

b1) If $q \in (P \cup \{\lambda\})^k$, then $\pi_j(q_i) = q_i$ for all $i \in k$, and hence $\pi_j(q) = q$. Also, $h(q) \in P \cup \{\lambda\}$ by (h1), and $\pi_j\big(h(q)\big) = h(q) = h\big(\pi_j(q)\big)$ follows.

b2) If $q_l = \lambda$ for some $l \in k$, then $\pi_j(q_l) = \lambda$, so that $h\big(\pi_j(q)\big) = \lambda = h(q) = \pi_j\big(h(q)\big)$, by (h1) again.

b3) In the remaining case of a $q \in (P \cup S)^k$ with $q_l \in S$ for some $l \in k$, the k-tuple $r = \big(\pi_j(q_0), \ldots, \pi_j(q_{k-1})\big)$ is such that $\pi_j(q_l) = q_l$ when $q_l \in P$, and $\pi_j(q_l) \in P$ is the j-th member of the sequence q_l when $q_l \in S$, so that $r \in P^k$ and $h(r) = g(r) = \pi_j\big(h(q)\big)$ by (h2).

c) Finally, we show that $h\big(\gamma(q_0), \ldots \gamma(q_{k-1})\big) = \gamma\big(h(q)\big)$ for all $q = (q_0, \ldots, q_{k-1}) \in Q^k$. This is clearly true for $q \in (P \cup \{\lambda\})^k$ because γ is the identity on $P \cup \{\lambda\}$ and $h(q) \in P \cup \{\lambda\}$ for any such q, and also for any k-tuple q with some $q_l = \lambda$, for then $h\big(\gamma(q_0), \ldots, \gamma(q_{k-1})\big) = \lambda = \gamma(\lambda) = \gamma\big(h(q)\big)$ by (h1).

In the remaining case, we have a $q \in (P \cup S)^k$ with $q_l \in S$ for some $l \in k$. As in the clause (h2) of the definition of h, let $\tilde{q} = \{\tilde{q}_j \mid j \in \omega\}$ be the one-to-one sequence of elements $\tilde{q}_j = (\tilde{q}_{j,0}, \ldots, \tilde{q}_{j,k-1})$ of P^k, such that $\tilde{q}_{j,i} = q_i$ when $q_i \in P$, and $\tilde{q}_{j,i}$ is the j-th member of q_i when $q_i \in S$. First, let us suppose that $\tilde{q}$ converges to a point $r = (r_0, \ldots, r_{k-1})$ in the space $X^k = (P, t)^k$. Then, for every $i \in k$, the sequence $\{\tilde{q}_{j,i} \mid j \in \omega\}$ converges to r_i in X, so that $\big(\gamma(q_0), \ldots, \gamma(q_{k-1})\big) = r$, and hence $h\big(\gamma(q_0), \ldots, \gamma(q_{k-1})\big) = h(r)$. On the other hand, $h(q) = \{g(\tilde{q}_j) \mid j \in \omega\} \in S$ by (h2), and the sequence $\{g(\tilde{q}_j) \mid j \in \omega\}$ converges to $g(r)$ because g is continuous. But $h(r) = g(r)$ since $r \in P^k$, and therefore $\gamma\big(h(q)\big) = h(r) = h\big(\gamma(q_0), \ldots, \gamma(q_{k-1})\big)$. Secondly, let us suppose that $\tilde{q}$ does not converge in X^k. Then there must be a sequence $\{\tilde{q}_{j,i} \mid j \in \omega\}$ which does not converge in X. Whence $\gamma\{\tilde{q}_{j,i} \mid j \in \omega\} = \lambda$. Simultaneously, since g is a homeomorphism of P^k onto a closed subset of (P, t), the sequence $\{g(\tilde{q}_j) \mid j \in \omega\} = h(q)$ cannot converge in (P, t) either. Therefore $\gamma\big(h(q)\big) = \lambda = h\big(\gamma(q_0), \ldots, \gamma(q_{k-1})\big)$. ∎

2.5. LEMMA. *Let $k \geq 0$ be an integer and let $h\colon \mathbb{A}(X)^k \longrightarrow \mathbb{A}(X)$ be a homomorphism. Then either h is constant with the value in $P \cup \{\lambda\}$, or else $h(\lambda, \ldots, \lambda) = \lambda$, $h(P^k) \subseteq P$, and the restriction of h to P^k is not constant.*

PROOF. (a) Since $z \cdot \lambda = \lambda$ for all $z \in Q$, for every $q \in Q^k$ we have $q \cdot (\lambda, \ldots, \lambda) = (\lambda, \ldots, \lambda)$, and hence also $h(q) \cdot h(\lambda, \ldots, \lambda) = h(\lambda, \ldots, \lambda)$. If $h(\lambda, \ldots, \lambda) \neq \lambda$, then $h(\lambda, \ldots, \lambda) \in P$ and hence $h(q) = h(\lambda, \ldots, \lambda)$ for all $q \in Q^k$. Therefore $h(\lambda, \ldots, \lambda) = \lambda$ for any h which is not a constant whose value belongs to P.

(b) Suppose that $p = (p_0, \ldots, p_{k-1}) \in P^k$ and $h(p) = \lambda$. We aim to show that h must be the constant with the value λ. Since $\gamma(z) = z$ exactly when $z \in P \cup \{\lambda\}$, the homomorphism h must map $(P \cup \{\lambda\})^k$ into $P \cup \{\lambda\}$. First we show that $h(p') = \lambda$ for any other $p' = (p'_0, \ldots, p'_{k-1}) \in P^k$. For $i \in k$, we set $q_i = p_i$ when $p'_i = p_i$ and, when $p'_i \neq p_i$, choose $q_i \in S$ with $\pi_0(q_i) = p_i$ and $\pi_1(q_i) = p'_i$. Then $\big(\pi_0(q_0), \ldots, \pi_0(q_{k-1})\big) = p$ and $\big(\pi_1(q_0), \ldots, \pi_1(q_{k-1})\big) = p'$. Whence $\pi_0\big(h(q_0, \ldots, q_{k-1})\big) = h\big(\pi_0(q_0), \ldots, \pi_0(q_{k-1})\big) = h(p) = \lambda$. But then $h(q_0, \ldots, q_{k-1}) = \lambda$ because $\pi_0^{-1}\{\lambda\} = \{\lambda\}$, and $h(p') = \pi_1\big(h(q_0, \ldots, q_{k-1})\big) = \pi_1(\lambda) = \lambda$ follows. Therefore $h(P^k) = \{\lambda\}$. Next we show that $h(q) = \lambda$ for every $q = (q_0, \ldots, q_{k-1}) \in (P \cup \{\lambda\})^k$. For any such q, there exist

$p = (p_0, \ldots, p_{k-1})$ and $p' = (p'_0, \ldots, p'_{k-1})$ in P^k with $q_i = p_i \cdot p'_i$ for every $i \in k$, and hence $h(q) = h(p) \cdot h(p') = \lambda \cdot \lambda = \lambda$. Finally, for any $q = (q_0, \ldots, q_{k-1}) \in Q^k$ we have $\big(\pi_0(q_0), \ldots, \pi_0(q_{k-1})\big) \in (P \cup \{\lambda\})^k$, so that $\pi_0\big(h(q)\big) = h\big(\pi_0(q_0), \ldots, \pi_0(q_{k-1})\big) = \lambda$. Since $\pi_0(z) = \lambda$ in $\mathbb{A}(X)$ only when $z = \lambda$, this proves that $h(q) = \lambda$. Whence h is the constant with the value λ.

If h is not constant, then (a) and (b) show that $h(\lambda, \ldots, \lambda) = \lambda$, $h(P^k) \subseteq P$, and that the restriction of h to P^k is not constant. ∎

2.6. LEMMA. *Let $k \geq 0$ be an integer and let $h_1, h_2 \colon \mathbb{A}(X)^k \longrightarrow \mathbb{A}(X)$ be homomorphisms such that $h_1(p) = h_2(p)$ for every $p \in P^k$. Then $h_1 = h_2$.*

PROOF. Since the claim is trivial when $k = 0$, let us assume that $k \geq 1$. By Lemma 2.5, the homomorphisms h_1, h_2 agree on $P^k \cup \{\lambda\}^k$.

(a) To show that $h_1(q) = h_2(q)$ for each $q = (q_0, \ldots, q_{k-1}) \in (P \cup \{\lambda\})^k$, we select $p = (p_0, \ldots, p_{k-1})$ and $p' = (p'_0, \ldots, p'_{k-1})$ in P^k so that $p_i \cdot p'_i = q_i$ for every $i \in k$, and conclude that $h_1(q) = h_1(p) \cdot h_1(p') = h_2(p) \cdot h_2(p') = h_2(q)$.

(b) Let $q = (q_0, \ldots, q_{k-1}) \in Q^k$. Then $\big(\pi_j(q_0), \ldots, \pi_j(q_{k-1})\big) \in (P \cup \{\lambda\})^k$ for every $j \in \omega$. The maps h_1, h_2 satisfy $\pi_j\big(h_1(q)\big) = h_1\big(\pi_j(q_0), \ldots, \pi_j(q_{k-1})\big)$ and $\pi_j\big(h_2(q)\big) = h_2\big(\pi_j(q_0), \ldots, \pi_j(q_{k-1})\big)$. But then $\pi_j\big(h_1(q)\big) = \pi_j\big(h_2(q)\big)$ for every $j \in \omega$ because h_1, h_2 agree on $(P \cup \{\lambda\})^k$, and $h_1(q) = h_2(q)$ follows. ∎

2.7. Now we are in a position to prove Proposition 2.3(1). Let $X = (P, t)$ be a disciplined metrizable space, and let $f \colon X^n \longrightarrow X$ be continuous with $n \geq 0$. Since X is disciplined, we have a standard decomposition $f = g \circ \pi^{[\psi]}$ of f, where $\pi^{[\psi]} \colon X^n \longrightarrow X^k$ for some $k \geq 0$ and an injective $\psi \colon k \longrightarrow n$, and $g \colon X^k \longrightarrow X$ is a homeomorphism of X^k onto a closed subset of X. Then the homomorphism $h_g \colon \mathbb{A}(X)^k \longrightarrow \mathbb{A}(X)$ from Lemma 2.4 extends g. Whence $E(f) = h_g \circ \pi^{[\psi]}$, where $\pi^{[\psi]}$ denotes the projection $\mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)^k$ associated with ψ, and $E(f)$ is a homomorphism extending f. Its uniqueness follows by Lemma 2.6. The homomorphism $E(f)$ is constant exactly when f is constant, by Lemma 2.5. This completes the proof of Proposition 2.3(1).

Proposition 2.3(2) will follow directly from Lemma 2.6 and the Lemma 2.8 below.

2.8. LEMMA. *Let $n \geq 0$ be an integer, and let $X = (P, t)$ be a metrizable space. Then any homomorphism $h \colon \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ other than $\lambda^{(n)}$ is an extension of a continuous map $f \colon X^n \longrightarrow X$.*

PROOF. If $h \neq \lambda^{(n)}$ then, by Lemma 2.5, either h is a constant whose value belongs to P, or else $h(\lambda, \ldots, \lambda) = \lambda$ and $h(P^n) \subseteq P$. Thus we need consider only the latter case for $n \geq 1$.

Suppose that the restriction $f \colon X^n \longrightarrow X$ of h to P^n is not continuous at some $r = (r_0, \ldots, r_{n-1}) \in P^n$. Then there exists a sequence $\{v_j \mid j \in \omega\}$ which converges to r in X^n and such that, for some $\epsilon > 0$, the distance from $f(v_j)$ to $f(r)$ is never less than ϵ. If we write $v_j = (v_{j,0}, \ldots v_{j,n-1})$ for each $j \in \omega$, then the sequence $v^i = \{v_{j,i} \mid j \in \omega\}$ converges to r_i for every $i \in n$. We need to produce a sequence $\{w_j \mid j \in \omega\}$ in P^n amenable to algebraic arguments, that is, a sequence whose coordinate sequences $\{w_{j,i} \mid j \in \omega\}$ are

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either one-to-one or constant for $i \in n$. We proceed as follows. Any sequence contains a subsequence that is either constant or one-to-one. There exists a subsequence $\{v_j^0 \mid j \in \omega\}$ of $\{v_j \mid j \in \omega\}$ whose first-coordinate sequence $\{v_{j,0}^0 \mid j \in \omega\}$ is one-to-one or constant. Then we find a subsequence $\{v_j^1 \mid j \in \omega\}$ of $\{v_j^0 \mid j \in \omega\}$ such that the sequence $\{v_{j,1}^1 \mid j \in \omega\}$ of its second coordinates is one-to-one or constant, and repeat this process until, having exhausted all n coordinates, we arrive at a subsequence $\{w_j \mid j \in \omega\}$ of $\{v_j \mid j \in \omega\}$. If $w_j = (w_{j,0}, \dots, w_{j,n-1})$ and $i \in n$, then the i-th coordinate sequence $w^i = \{w_{j,i} \mid j \in \omega\}$ is either constant with $w_{j,i} = r_i$ for all $j \in \omega$, or else it is one-to-one and converges to r_i.

For each $i \in n$, we choose $q_i \in Q$ as follows:

$$q_i = \begin{cases} w^i & \text{if } w^i \text{ is one-to-one} \\ r_i & \text{if } w^i \text{ is constant.} \end{cases}$$

Then $q_i \in S$ in the first instance, $q_i \in P$ in the second and, moreover, $\gamma(q_i) = r_i$ and $\pi_j(q_i) = w_{j,i}$ for all $i \in n$ and $j \in \omega$. Thus, in particular, $q = (q_0, \dots, q_{n-1}) \in (P \cup S)^n$ and $\big(\gamma(q_0), \dots, \gamma(q_{n-1})\big) = r$ in $\mathbb{A}(X)^n$. Also, for every $j \in \omega$,

$$\pi_j\big(h(q)\big) = h\big(\pi_j(q_0), \dots, \pi_j(q_{n-1})\big) = h(w_{j,0}, \dots, w_{j,n-1}) = f(w_j) \in P$$

because $h(P^n) \subseteq P$. From $\pi_j^{-1}(P) = P \cup S$ it follows that either $h(q) \in P$ or $h(q) \in S$. In the first case $h(q) = f(w_j)$ for all $j \in \omega$, while $h(q) = \{f(w_j) \mid j \in \omega\} \in S$ in the second. But then

$$\gamma\big(h(q)\big) = h\big(\gamma(q_0), \dots, \gamma(q_{n-1})\big) = h(r) = f(r),$$

and this is possible only when either $f(w_j) = f(r)$ for all $j \in \omega$, or $\{f(w_j) \mid j \in \omega\}$ is one-to-one and converges to $f(r)$. But this is impossible since $\{f(w_j) \mid j \in \omega\}$ is a subsequence of the sequence $\{f(v_j) \mid j \in \omega\}$ which does not converge to r. ∎

The proof of Proposition 2.3 is now complete.

2.9. PROPOSITION. *Let $k \geq 0$ be an integer and let X_1 and X_2 be disciplined metrizable spaces. Then*

$$\mathrm{Clo}_k(X_1) = \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) = \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

PROOF. This is an immediate consequence of Proposition 2.3. ∎

3. **Absolute constants in disciplined spaces.**

3.1 . Our Theorem 1.9 speaks about the equality $=$, isomorphism $\simeq$ and elementary equivalence $\approx$ of centralizer clone segments $\mathrm{Clo}_i(A_1)$ and $\mathrm{Clo}_i(A_2)$ of finitary algebras A_1 and A_2 which, as indicated earlier, are the algebraic traces of suitable disciplined metrizable spaces. The equality of centralizer clone segments of such algebras has already been treated by Proposition 2.9. Here we investigate their isomorphism, for which we prove a similar Proposition 3.5 and, in 3.8, also set down a basis for subsequent consideration of elementary equivalence.

We need the following concepts.

3.2. DEFINITION. (a) Let $F\colon k \longrightarrow \mathrm{Set}$ be a functor and let a be an object of the category k. We say that an element $x \in F(a)$ is an *absolute fixpoint* of a whenever, for any morphism $f \in k(a,a)$, either the mapping $F(f)$ is constant or $F(f)(x) = x$.

(b) Let $a \in \operatorname{obj} k$ generate a clone (k, F), and let $n \in \omega$. A morphism $c \in k(a^n, a)$ is called an *absolute constant* whenever $F(c)\colon F(a^n) \longrightarrow F(a)$ is a constant map whose value is an absolute fixpoint of a.

3.3. We recall that the first order language of clones is just the first order language of ω-sorted algebras with the operations $c_j^{(n)}$ and S_m^n with $j \in n$ and $m, n \in \omega$, as described in 1.8. The language has ω sorts of variables, with symbols $x^{(i)}, y^{(i)}, x_1^{(i)}, x_2^{(i)}, \ldots$ denoting the variables of the i-th sort for each $i \in \omega$. For any $r \geq 1$, the first order language of clone r-segments is the fragment of the first order language of clones which has variables $x^{(i)}$, $y^{(i)}, \ldots$ with $i = 0, 1, \ldots, r$ only, and operation symbols S_m^n and $c_j^{(n)}$ with $j \in n$ restricted to $m, n \leq r$.

Since in our case the variables of the i-th sort range through maps $Y^i \longrightarrow Y$, we simplify matters somewhat by treating all variables as maps. As is customary, we shall identify elements of Y with constant maps $Y \longrightarrow Y$, so that a variable $x^{(0)}$ will be replaced by a variable $x^{(1)}$ of the 1-st sort satisfying the predicate

$$\mathrm{p}(x^{(1)}) \equiv \forall y^{(1)} S_1^1(x^{(1)}, y^{(1)}) = x^{(1)}.$$

In what follows, we shall also use the predicates

$$\mathrm{afp}(x^{(1)}) \equiv \mathrm{p}(x^{(1)}) \wedge \forall y^{(1)}\big(\mathrm{p}(y^{(1)}) \vee S_1^1(y^{(1)}, x^{(1)}) = x^{(1)}\big) \text{ and}$$
$$\mathrm{ac}(z^{(n)}) \equiv \exists x^{(1)}\big(\mathrm{afp}(x^{(1)}) \wedge S_n^1(x^{(1)}, z^{(n)}) = z^{(n)}\big),$$

which describe absolute fixpoints and absolute constants in this language.

3.4. If $X = (P, t)$ is an infinite disciplined metrizable space then, by Lemma 2.5, any homomorphism $h\colon \mathbb{A}(X) \longrightarrow \mathbb{A}(X)$ with $h(\lambda) \neq \lambda$ is a constant map (whose value belongs to P). Whence λ is an absolute fixpoint of $\mathbb{A}(X)$, and the homomorphism $\lambda^{(n)}\colon \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ is an absolute constant in $\mathrm{Clo}\big(\mathbb{A}(X)\big)$ for every $n \geq 0$.

If X has no absolute fixpoints, then for every $p \in P$ there is a non-constant continuous map $f_p\colon X \longrightarrow X$ with $f_p(p) \neq p$. Hence the unique homomorphism $E(f_p)\colon \mathbb{A}(X) \longrightarrow \mathbb{A}(X)$ extending f_p—see Proposition 2.3(1)—is non-constant and such that $E(f_p)(s) \neq s$ for any $s \in S$ with $\pi_0(s) = p$. It follows that $\mathbb{A}(X)$ has no absolute fixpoints other than λ, that $\lambda^{(n)}$ is the only absolute constant $\mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$, and that $\mathrm{Clo}(X)$ has no absolute constants.

Hence, by 2.3, for every $n \in \omega$,

$$\mathrm{Clo}\big(\mathbb{A}(X)\big)_n = \{E(f) \mid f \in \mathrm{Clo}(X)_n\} \cup \{\lambda^{(n)}\},$$

and $\lambda^{(n)}$ is the unique absolute constant in $\mathrm{Clo}\big(\mathbb{A}(X)\big)_n$. Let

$$E\colon \mathrm{Clo}(X) \longrightarrow \mathrm{Clo}\big(\mathbb{A}(X)\big)$$

denote both the embedding of these clone algebras and also the functor preserving finite products and projection enumeration corresponding to it.

3.5. PROPOSITION. *Let $k \geq 1$ be an integer, and let X_1 and X_2 be infinite disciplined metrizable spaces without absolute fixpoints. Then*

$$\mathrm{Clo}_k(X_1) \simeq \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \simeq \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

This equivalence also holds for the respective full clones.

PROOF. In the diagram below, let $E^{(1)}$ and $E^{(2)}$ be restrictions of embeddings from 3.4 above to the respective clone segments.

$$\begin{array}{ccc} \mathrm{Clo}_k(X_1) & \xrightarrow{E^{(1)}} & \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \\ \Phi \big\downarrow & & \big\downarrow \Psi \\ \mathrm{Clo}_k(X_2) & \xrightarrow{E^{(2)}} & \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big) \end{array}$$

Suppose first that the functor Ψ in this diagram is an isomorphism. Then Ψ preserves elements satisfying the same first order formulas, and hence it must map absolute constants of its domain precisely onto absolute constants in its codomain. By 3.4, there is a unique isomorphism Φ completing the diagram above.

Conversely, let Φ be an isomorphism. We extend Φ to a product-preserving functor Ψ which sends the unique absolute constant $\lambda^{(n)}$ of $\mathrm{Clo}\big(\mathbb{A}(X_1)\big)_n$ to the unique absolute constant of $\mathrm{Clo}\big(\mathbb{A}(X_2)\big)_n$ for every $n \leq k$. Then the diagram above commutes. The fact that the bijection Ψ is a clone isomorphism or, equivalently, that it preserves all operations S^n_m with $m, n \leq k$, will follow from Lemma 3.8 below. The remainder is clear. ∎

3.6 . Let $K = (k, F)$ be a clone with the generating object a, and let $\psi: r \longrightarrow n$ be a map for some $r, n \in \omega$. As in 1.2 for clones of spaces, but now for any clone, we write $\pi^{[\psi]}: a^n \longrightarrow a^r$ to denote the k-morphism

$$\pi^{[\psi]} = \pi^{(n)}_{\psi(0)} \dot{\times} \cdots \dot{\times} \pi^{(n)}_{\psi(r-1)}.$$

Then, for any $h \in k(a^r, a)$,

$$h \circ \pi^{[\psi]} = S^r_n(h, \pi^{(n)}_{\psi(0)}, \ldots, \pi^{(n)}_{\psi(r-1)})$$

follows from the definition of clone algebra operations.

3.7 . Let $K = (k, F)$ be a clone with the generating object a. Since F is faithful and $a^n \neq a^m$ for distinct $n, m \in \omega$, any k-morphism $g \in k(a^n, a)$ is fully determined by $F(g)$. Hence there is no need to distinguish $F(g)$ from g, and we shall write g instead of $F(g)$ as well.

DEFINITION. Let $g \in k(a^n, a)$. We say that

$$g = h \circ \pi^{[\psi]}$$

is an *extremal decomposition* of g—in the clone K or in any of its m-segments with $m \geq n$—whenever $\psi: r \longrightarrow n$ is a one-to-one map, and for any decomposition $g = h_1 \circ \pi^{[\psi_1]}$

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with a one-to-one map $\psi_1 : s \longrightarrow n$ there exists a k-morphism $d \in k(a^s, a^r)$ such that $d \circ \pi^{[\psi_1]} = \pi^{[\psi]}$.

REMARKS. (a) Since ψ and ψ_1 in the definition above are both one-to-one, we have $r, s \leq n$. Thus the definition does apply not only to the full clone, but also to all its m-segments with $m \geq n$.

(b) Any factorizing morphism d is unique and has the form $d = \pi^{[\delta]}$ for some δ with $\psi_1 \circ \delta = \psi$.

(c) Since the map $\pi^{[\psi]}$ is surjective for any injective map ψ, an extremal decomposition is determined uniquely up to a commuting isomorphism.

OBSERVATION. If $g = h \circ \pi^{[\psi]} \in k(a^n, a)$ with $\psi : r \longrightarrow n$, and if $f_j \in k(a^m, a)$ for $j \in n$, then

$$\begin{aligned} S^n_m(g, f_0, \dots, f_{n-1}) &= g \circ (f_0 \dot{\times} \cdots \dot{\times} f_{n-1}) = h \circ (f_{\psi(0)} \dot{\times} \cdots \dot{\times} f_{\psi(r-1)}) \\ &= S^r_m(h, f_{\psi(0)}, \dots, f_{\psi(r-1)}). \end{aligned}$$

This follows immediately from $\pi^{[\psi]} = \pi^{(n)}_{\psi(0)} \dot{\times} \cdots \dot{\times} \pi^{(n)}_{\psi(r-1)}$.

3.8. The Lemma below summarizes the relationship of $\mathrm{Clo}\big(\mathbb{A}(X)\big)$ to $\mathrm{Clo}(X)$ in a form suitable for investigations of elementary equivalence.

Let $E : \mathrm{Clo}(X) \longrightarrow \mathrm{Clo}\big(\mathbb{A}(X)\big)$ be the embedding from 3.4. Proposition 2.3 shows that, for each $i \in n$, the homomorphism $E(\pi^{(n)}_i) \in \mathrm{Clo}\big(\mathbb{A}(X)\big)_n$ is the i-th projection, and that $E(\pi^{(n)}_i)$ uniquely extends the i-th projection $\pi^{(n)}_i \in \mathrm{Clo}(X)_n$. It follows that $E(\pi^{[\psi]})$ is the unique extension of $\pi^{[\psi]} = \pi^{(n)}_{\psi(0)} \dot{\times} \cdots \dot{\times} \pi^{(n)}_{\psi(r-1)}$ for any $\psi : r \longrightarrow n$ as well. This allows us to simplify the notation as follows: we write $\pi^{(n)}_i$ instead of $E(\pi^{(n)}_i) \in \mathrm{Clo}\big(\mathbb{A}(X)\big)$ and, more generally, $\pi^{[\psi]}$ instead of $E(\pi^{[\psi]}) \in \mathrm{Clo}\big(\mathbb{A}(X)\big)$.

LEMMA. *If $X = (P, t)$ is a disciplined metrizable space and $n \in \omega$, then* $\mathrm{Clo}(X)$ *and* $\mathrm{Clo}\big(\mathbb{A}(X)\big)$ *have the following properties:*

(1) $\mathrm{Clo}\big(\mathbb{A}(X)\big)_n = \{E(g') \mid g' \in \mathrm{Clo}(X)_n\} \cup \{\lambda^{(n)}\}$.

(2) For any $g' \in \mathrm{Clo}(X)_n$, any standard decomposition $g' = h' \circ \pi^{[\psi]}$ of g' is also an extremal decomposition of g'.

(3) Every $g = E(g') \in \mathrm{Clo}\big(\mathbb{A}(X)\big)_n \setminus \{\lambda^{(n)}\}$ has an extremal decomposition $g = E(h') \circ \pi^{[\psi]}$, where $g' = h' \circ \pi^{[\psi]}$ is an extremal decomposition of g'.

(4) Let $f = (f_0 \dot{\times} \cdots \dot{\times} f_{n-1}) : \mathbb{A}(X)^m \longrightarrow \mathbb{A}(X)^n$ and $g : \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ be in $\mathrm{Clo}\big(\mathbb{A}(X)\big)$. *Then either $g = \lambda^{(n)}$ and hence $g \circ f = \lambda^{(m)}$, or else g has an extremal decomposition $g = E(h') \circ \pi^{[\psi]}$ and*

$$g \circ f = \begin{cases} \lambda^{(m)} & \text{if } f_{\psi(i)} = \lambda^{(m)} \text{ for some } i \in r \\ E(h') \circ (f_{\psi(0)} \dot{\times} \cdots \dot{\times} f_{\psi(r-1)}) & \text{otherwise,} \end{cases}$$

with $f_{\psi(i)} = E(f'_{\psi(i)})$ for all $i \in r$.

These conclusions also hold for any initial k-segments of these clones when $k \geq 1$.

PROOF. Claim (1) is already contained in 3.4. To prove (2), let $g' = h' \circ \pi^{[\psi]}$ be a standard decomposition. To show that this is also an extremal decomposition, suppose

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that $g' = h_1 \circ \pi^{[\psi_1]}$ for some injective $\psi_1 : s \longrightarrow n$. Since the projections $\pi^{[\psi]}$ and $\pi^{[\psi_1]}$ are surjective and h' is a homeomorphism of X^r onto $h_1(X^s)$, the composite $d = (h')^{-1} \circ h_1$ exists and is continuous. But then $h' \circ d \circ \pi^{[\psi_1]} = h' \circ \pi^{[\psi]}$, and $d\pi^{[\psi_1]} = \pi^{[\psi]}$ follows because h' is injective.

Now we turn to (3). If $g \neq \lambda^{(n)}$, then $g = E(g')$ for a unique $g' : X^n \longrightarrow X$, by (1), and g' has an extremal decomposition $g' = h' \circ \pi^{[\psi]}$, by (2). From Lemma 2.6 it then follows that $E(\pi^{[\psi]}) = \pi^{[\psi]}$ and $g = E(h') \circ \pi^{[\psi]}$. To show that this is an extremal decomposition of g, suppose that $g = h_1 \circ \pi^{[\psi_1]}$ with an injective $\psi_1 : s \longrightarrow n$. Then $h_1 \neq \lambda^{(s)}$ because $g \neq \lambda^{(n)}$, so that $h_1 = E(h_1')$ for some $h_1' \in \mathrm{Clo}(X)_s$, by (1) again, and $g' = h_1' \circ \pi^{[\psi_1]}$ follows. Since the decomposition $g' = h' \circ \pi^{[\psi]}$ is extremal, we have $d \circ \pi^{[\psi_1]} = \pi^{[\psi]}$ for some $d \in \mathrm{Clo}(X)$, and hence $E(d) \circ \pi^{[\psi_1]} = \pi^{[\psi]}$ in $\mathrm{Clo}\big(\mathbb{A}(X)\big)$. Therefore $g = E(h') \circ \pi^{[\psi]}$ is an extremal decomposition of g, as claimed.

To prove (4), let $g \neq \lambda^{(n)}$. Then $g = E(g') = E(h') \circ \pi^{[\psi]}$ is an extremal decomposition of g in which h' is a homeomorphism of X^r onto a closed subset of X, by (3) and (2), and $g \circ f = E(h') \circ (f_{\psi(0)} \dot{\times} \cdots \dot{\times} f_{\psi(r-1)})$ by Observation 3.7. From the definition of the extension $E(h')$ of the homeomorphism h' it then follows that $g \circ f = \lambda^{(m)}$ whenever $f_{\psi(i)} = \lambda^{(m)}$ for at least one $i \in r$. If, on the other hand, $f_{\psi(i)} \neq \lambda^{(m)}$ for every $i \in r$, then $f_{\psi(i)} = E(f'_{\psi(i)})$ for all $i \in r$ by (1), and (4) follows. ∎

4. Elementary equivalence.

4.1. The purpose of this section is to verify the following claim about elementary equivalence of clones or their segments.

PROPOSITION. *For any two infinite disciplined metrizable spaces X_1 and X_2 with no absolute fixpoints, and for any $k \geq 1$,*

$$\mathrm{Clo}_k(X_1) \approx \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \approx \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

This equivalence also holds for the respective full clones.

Even though it is easy to show, using Lemma 3.8 and a natural uniform interpretation of sentences from the theory of $\mathrm{Clo}_k(X)$ in the theory of $\mathrm{Clo}_k\big(\mathbb{A}(X)\big)$, that $\mathrm{Clo}_k(X_1) \approx \mathrm{Clo}_k(X_2)$ follows from $\mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \approx \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big)$, sentences claiming the existence of absolute constants make a converse interpretation unwieldy, and we employ other means to prove the above Proposition. Specifically, we use the well-known fact that elementary equivalence is equivalent to isomorphism of prime limits (see [4] for mono-sorted algebras). We prove that an isomorphism of certain prime limits of $\mathrm{Clo}_k(X_1)$ and $\mathrm{Clo}_k(X_2)$ can be extended to an isomorphism of associated prime limits of $\mathrm{Clo}_k\big(\mathbb{A}(X_1)\big)$ and $\mathrm{Clo}_k\big(\mathbb{A}(X_2)\big)$ and, vice versa, that a restriction of an isomorphism of some prime limits of $\mathrm{Clo}_k\big(\mathbb{A}(X_1)\big)$ and $\mathrm{Clo}_k\big(\mathbb{A}(X_2)\big)$ is an isomorphism of the associated prime limits of $\mathrm{Clo}_k(X_1)$ and $\mathrm{Clo}_k(X_2)$.

4.2. In order to introduce an appropriate notation, we recall how ultrapowers of clone algebras are created.

Let K be a clone algebra as described in 1.8, and let K_n denote its carrier of the n-th sort. Given an ultrafilter $\mathcal{F}$ on a set I, for any $s, t \in K_n^I$ we define

$$s \sim_{\mathcal{F}} t \text{ iff } \mathrm{Eq}(s,t) = \{i \in I \mid s(i) = t(i)\} \in \mathcal{F}.$$

Then $\sim_{\mathcal{F}}$ is an equivalence. For any $s \in K_n^I$, the symbol $[s]_{\mathcal{F}}$ will denote the class of $\sim_{\mathcal{F}}$ containing s. If $\kappa_n^{(\mathcal{F})}$ is the natural map of K_n^I onto the ultrapower $K_n^{(\mathcal{F})} = K_n^I / \sim_{\mathcal{F}}$ with the kernel $\sim_{\mathcal{F}}$ then $\kappa_n^{(\mathcal{F})}(s) = [s]$, so that each $\sigma \in K_n^{(\mathcal{F})}$ may be replaced, as is customary, by any $s \in K_n^I$ with $[s]_{\mathcal{F}} = \sigma$. Whence $t \in [s]_{\mathcal{F}}$ exactly when $\mathrm{Eq}(s,t) \in \mathcal{F}$.

For any $w \in K_n$, we let $\bar{w} \in K_n^I$ stand for the constant sequence whose value is w. For each $j \in n$ we then write

$$(c_j^{(n)})^{(\mathcal{F})} = [\overline{c_j^{(n)}}]_{\mathcal{F}} \in K_n^{(\mathcal{F})},$$

where $c_j^{(n)}$ is the nullary operation from 1.8(c).

Let $(S_m^n)^I : K_n^I \times (K_m^I)^n \longrightarrow K_m^I$ be the componentwise extension of an operation $S_m^n : K_n \times K_m^n \longrightarrow K_m$ from 1.8(s), that is, let

$$\big((S_m^n)^I(s, t_0, \ldots, t_{n-1})\big)(i) = S_m^n\big(s(i), t_0(i), \ldots t_{n-1}(i)\big)$$

for every $i \in I$, any $s \in K_n$ and any $t_0, \ldots, t_{n-1} \in K_m$. If $s' \in [s]_{\mathcal{F}}$ and $t_j' \in [t_j]_{\mathcal{F}}$ for all $j \in n$, then

$$\mathrm{Eq}(s,s') \cap \bigcap\{\mathrm{Eq}(t_j, t_j') \mid j \in n\} \in \mathcal{F},$$

and hence also

$$\mathrm{Eq}\big((S_m^n)^I(s, t_0, \ldots, t_{n-1}), (S_m^n)^I(s', t_0', \ldots, t_{n-1}')\big) \in \mathcal{F}.$$

It follows that there is a well-defined and unique mapping

$$(S_m^n)^{(\mathcal{F})} : K_n^{(\mathcal{F})} \times (K_m^{(\mathcal{F})})^n \longrightarrow K_m^{(\mathcal{F})}$$

given for $\sigma \in K_n^{(\mathcal{F})}$ and $\tau_0, \ldots \tau_{n-1} \in K_m^{(\mathcal{F})}$ by

$$(S_m^n)^{(\mathcal{F})}(\sigma, \tau_0, \ldots, \tau_{n-1}) = [(S_m^n)^I(s, t_0, \ldots, t_{n-1})]_{\mathcal{F}}$$

with an arbitrary choice of $s \in \sigma$ and $t_j \in \tau_j$ for each $j \in n$.

Let $\chi : K \longrightarrow K^{(\mathcal{F})}$ be the diagonal map given by $\chi(k) = [\bar{k}]_{\mathcal{F}} \in K_n^{(\mathcal{F})}$ for all $k \in K_n$. Then χ is a one-to-one homomorphism of the clone algebra K into a heterogeneous algebra $K^{(\mathcal{F})}$ whose operations $(c_j^{(n)})^{(\mathcal{F})}$ and $(S_m^n)^{(\mathcal{F})}$ satisfy (RU), (LU) and (AC) from 1.8. Having identified every $k \in K$ with its image $\chi(k) \in K^{(\mathcal{F})}$, we thus obtain a clone algebra $K^{(\mathcal{F})}$ whose carrier of the n-th sort is $K_n^{(\mathcal{F})}$ and whose operations are $(c_j^{(n)})^{(\mathcal{F})} = c_j^{(n)}$ and $(S_m^n)^{(\mathcal{F})}$. The clone algebra $K^{(\mathcal{F})}$ will be called a *prime power* of K associated with $\mathcal{F}$.

Let K, L be clone algebras, and let $E : K \longrightarrow L$ be a clone homomorphism. The definition of its extension $E^{(\mathcal{F})} : K^{(\mathcal{F})} \longrightarrow L^{(\mathcal{F})}$ is evident, and it is clear that $E^{(\mathcal{F})}$ is an embedding whenever E is an embedding.

4.3. For any integer $r \geq 1$, let $\mathcal{F}_r$ be an ultrafilter on some set I_r and let $\mathbb{F} = \{\mathcal{F}_r \mid r \geq 1\}$. Set $K_{(0)} = K$, then inductively define $K_{(r)} = K_{(r-1)}^{(\mathcal{F}_r)}$ for each $r \geq 1$, and let $\chi_r: K_{(r-1)} \longrightarrow K_{(r)}$ denote the diagonal embedding arising at each step. If $0 \leq s < r$, then $\chi_r \circ \cdots \circ \chi_{s+1}$ is an injective homomorphism, and we may therefore assume that $K_{(s)}$ is a subalgebra of $K_{(r)}$ whenever $0 \leq s < r$. These composite embeddings, or rather inclusions, form a diagram whose colimit $\mathrm{PL}_{\mathbb{F}}(K)$ is an ω-sorted algebra isomorphic to the union of the ω-chain

$$K_{(0)} \xrightarrow{\chi_1} K_{(1)} \xrightarrow{\chi_2} K_{(2)} \longrightarrow \quad \cdots \quad ,$$

with colimit injections $\mu_s: K_{(s)} \longrightarrow \mathrm{PL}_{\mathbb{F}}(K)$. It is clear that the ω-sorted algebra $\mathrm{PL}_{\mathbb{F}}(K)$, called a *prime limit* of K over $\mathbb{F}$, satisfies (LU), (RU) and (AC) from 1.8 with the nullary operations $c_j^{(n)} \in K_{(0)}$ for all $j \in n$ and joint extensions $(S_m^n)^{(\mathbb{F})}$ of all $(S_m^n)^{(\mathcal{F}_r)}$ with $r \geq 1$.

For a clone homomorphism $E: K \longrightarrow L$, the definition of $\mathrm{PL}_{\mathbb{F}}(E): \mathrm{PL}_{\mathbb{F}}(K) \longrightarrow \mathrm{PL}_{\mathbb{F}}(L)$ is evident. It is clear that if E is an embedding then $\mathrm{PL}_{\mathbb{F}}(E)$ is also an embedding.

4.4. Let X be an infinite disciplined space with no absolute fixpoints, and let $\mathbb{F}$ be a sequence of ultrafilters. In the lemma below, we extend claims (1)–(4) of Lemma 3.8 about the clone algebras $\mathrm{Clo}(X)$ and $\mathrm{Clo}\big(\mathbb{A}(X)\big)$ to analogous claims about their prime limits $A = \mathrm{PL}_{\mathbb{F}}\big(\mathrm{Clo}(X)\big)$ and $A^* = \mathrm{PL}_{\mathbb{F}}\Big(\mathrm{Clo}\big(\mathbb{A}(X)\big)\Big)$.

The non-nullary clone operations will be denoted as S_n^m in A, and as $(S_n^m)^*$ in A^*, while $c_i^{(n)}$ will denote the clone constant corresponding to the product projection $\pi_i^{(n)}$ in $\mathrm{Clo}(X)$ and in $\mathrm{Clo}\big(\mathbb{A}(X)\big)$, in both A and A^*.

From 3.7 we recall that any existing extremal decomposition is determined uniquely up to a commuting isomorphism. We note that, in the lemma below, all decompositions are expressed by means of clone algebra operations, see Observation 3.7.

LEMMA. *Let X, $\mathbb{F}$, A, A^* be as above. Then:*

(1) There exists a one-to-one clone homomorphism $E: A \longrightarrow A^*$ such that the carrier A_n^* of the n-th sort of A^* is the disjoint union $A_n^* = \{E(g') \mid g' \in A_n\} \cup \{*^{(n)}\}$, and $*^{(n)}$ is the only absolute constant in A_n^*.

(2) Each $g' \in A_n$ has an extremal decomposition $g' = S_n^r(h', c_{\psi(0)}^{(n)}, \dots, c_{\psi(r-1)}^{(n)})$.

(3) $E: A \longrightarrow A^*$ preserves extremal decompositions, that is, if $g \in A_n^*$, $g = E(g')$ and $g' = S_n^r(h', c_{\psi(0)}^{(n)}, \dots, c_{\psi(r-1)}^{(n)})$ is an extremal decomposition of g' in A, then $g = (S_n^r)^*\big(E(h'), c_{\psi(0)}^{(n)}, \dots, c_{\psi(r-1)}^{(n)}\big)$ is an extremal decomposition of g in A^*.

(4) For $f_0, \dots, f_{n-1} \in A_m^*$ and $g \in A_n^*$, denote

$$C = (S_m^n)^*(g, f_0, \dots, f_{n-1}).$$

Then $C = *^{(m)}$ for $g = *^{(n)}$. Otherwise $g \in A_n^* \setminus \{*^{(n)}\}$ has an extremal decomposition $g = (S_n^r)^*(h, c_{\psi(0)}^{(n)}, \dots, c_{\psi(r-1)}^{(n)})$, by (1)–(3), and

$$C = \begin{cases} *^{(m)} & \text{if } f_{\psi(i)} = *^{(m)} \text{ for some } i \in r \\ (S_m^r)^*(h, f_{\psi(0)}, \dots, f_{\psi(r-1)}) & \text{if } f_{\psi(i)} \neq *^{(m)} \text{ for all } i \in r. \end{cases}$$

The same conclusions hold for prime limits $\mathrm{PL}_{\mathbb{F}}\big(\mathrm{Clo}_k(X)\big)$ *and* $\mathrm{PL}_{\mathbb{F}}\Big(\mathrm{Clo}_k\big(\mathbb{A}(X)\big)\Big)$ *of k-segments of the clone algebras.*

PROOF. Denote $A_{(0)} = \mathrm{Clo}(X)$, $A^*_{(0)} = \mathrm{Clo}\big(\mathbb{A}(X)\big)$, and let $E_{(0)}: A_{(0)} \longrightarrow A^*_{(0)}$ be as in 3.4. For every $s \geq 1$, write $A_{(s)} = (A_{(s-1)})^{(\mathcal{F}_s)}$, $A^*_{(s)} = (A^*_{(s-1)})^{(\mathcal{F}_s)}$ and $E_{(s)} = (E_{(s-1)})^{(\mathcal{F}_s)}$, where $\mathcal{F}_s \in \mathbb{F}$ is an ultrafilter on a set I_s.

From 3.4 and Lemma 3.8 it follows that $A_{(0)}$, $A^*_{(0)}$ and $E_{(0)}$ already satisfy (1), (2), (3) and (4). From the definition of a prime limit it follows that the lemma will be proved once we show that these four statements hold for $A_{(s)}$, $A^*_{(s)}$ and $E_{(s)}$ whenever they hold for $A_{(s-1)}$, $A^*_{(s-1)}$ and $E_{(s-1)}$.

Hence, let an ultrafilter $\mathcal{F}$ on a set I be given. We shall suppose that (1)–(4) hold for clone algebras $K, L = K^*$ and a clone-embedding $E: K \longrightarrow L$, and prove these statements for the ultrapowers $K^{(\mathcal{F})}$, $L^{(\mathcal{F})}$, and $E^{(\mathcal{F})}$. To simplify the notation, let us suppose that the embedding E is an inclusion, so that K is a subalgebra of L and each operation $(S^n_m)^*$ of L is a mere extension of the operation S^n_m of K.

To prove (1), assume that the carrier L_n of the n-th sort of L is the disjoint union $L_n = K_n \cup \{*^{(n)}\}$, in which $*^{(n)}$ is the only absolute constant. Clearly $K^{(\mathcal{F})}_n \subseteq L^{(\mathcal{F})}_n$. Furthermore, the sentence $\forall x^{(n)} \neg\, \mathrm{ac}(x^{(n)})$ holds in K_n, while its negation $\exists x^{(n)}\, \mathrm{ac}(x^{(n)})$ is satisfied in L_n by $*^{(n)}$ alone. For an arbitrary $f \in L^{(\mathcal{F})}_n$, the set I is the disjoint union of $J = \{i \in I \mid f(i) \in K_n\}$ and $J' = \{i \in I \mid f(i) = *^{(n)}\}$, so that exactly one of these sets belongs to $\mathcal{F}$. If $J \in \mathcal{F}$, then $f \in K^{(\mathcal{F})}_n$ and f is not an absolute constant. Else $J' \in \mathcal{F}$ and $f \sim_{\mathcal{F}} \overline{*^{(n)}}$, and f must be the absolute constant in $L^{(\mathcal{F})}_n$. Having identified diagonal members of $L^{(\mathcal{F})}_n$ with their values in L_n, we obtain (1) for $K^{(\mathcal{F})}_n$ and $L^{(\mathcal{F})}_n$.

For (2), select any $g \in K^{(\mathcal{F})}_n$. Then $J = \{i \in I \mid g(i) \in K_n\} \in \mathcal{F}$. Since $g(i) \neq *^{(n)}$ for all $i \in J$, every such $g(i)$ has an extremal decomposition $g(i) = S^{r_i}_n\big(h(i), c^{(n)}_{\psi_i(0)}, \dots, c^{(n)}_{\psi_i(r_i-1)}\big)$ in K_n. Each $\psi_i: r_i \longrightarrow n$ is injective, there are only finitely many of these maps, and hence there is a unique injective map $\psi: r \longrightarrow n$ for which $J_\psi = \{i \in J \mid \psi_i = \psi\} \in \mathcal{F}$. But then $g = S^r_n(h, c^{(n)}_{\psi(0)}, \dots, c^{(n)}_{\psi(r-1)})$ in $K^{(\mathcal{F})}$ because $h(i) \in K_n$ for every $i \in J$.

To see that this decomposition is extremal, let also $g = S^{r_1}_n(h_1, c^{(n)}_{\psi_1(0)}, \dots, c^{(n)}_{\psi_1(r_1-1)})$ with an injective map $\psi_1: r_1 \longrightarrow n$. Then there exists some $J_1 \in \mathcal{F}$ so that the left hand side in

$$S^r_n\big(h(i), c^{(n)}_{\psi(0)}, \dots, c^{(n)}_{\psi(r-1)}\big) = S^{r_1}_n\Big(h_1(i), c^{(n)}_{\psi_1(0)}, \dots, c^{(n)}_{\psi_1(r_1-1)}\Big)$$

is an extremal decomposition of $g(i)$ in K, and the equality itself holds for every $i \in J_1$. Since $J_1 \neq \emptyset$, there exists a unique $\delta: r \longrightarrow r_1$ with $\psi_1 \circ \delta = \psi$ and, because $J_1 \in \mathcal{F}$, the decomposition $g = S^r_n(h, c^{(n)}_{\psi(0)}, \dots, c^{(n)}_{\psi(r-1)})$ of g is extremal. This proves (2).

For (3), we need that an extremal decomposition $g = S^r_n(h, c^{(n)}_{\psi(0)}, \dots, c^{(n)}_{\psi(r-1)})$ in $K^{(\mathcal{F})}_n$ of any $g \in K^{(\mathcal{F})}_n$ is its extremal decomposition also in $L^{(\mathcal{F})}_n$. This follows easily from the fact that (4) holds for K and L.

To prove (4), let $f_0, \dots, f_{n-1} \in L^{(\mathcal{F})}_m$, $g \in L^{(\mathcal{F})}_n$, and let $C = (S^n_m)^*(g, f_0, \dots, f_{n-1})$ be the composite in $L^{(\mathcal{F})}$. If $g = *^{(n)}$, then $C = *^{(m)}$ because $*^{(n)}$ is a constant. Any

$g \in L_n^{(\mathcal{F})} \setminus \{*^{(n)}\}$ belongs to $K_n^{(\mathcal{F})}$, by (1), and hence has an extremal decomposition $g = S_n^r(h, c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)})$, by (2). If $f_{\psi(j)} = *^{(m)}$ for some $j \in r$, then $C = *^{(n)}$ follows from the definition of $(S_m^n)^*$ on the ultrapower $L^{(\mathcal{F})}$ and the fact that $*^{(m)}$ is constant. Otherwise also $f_{\psi(j)} \in K_m^{(\mathcal{F})}$ for all $j \in n$, by (1), and hence there exists some $J_2 \in \mathcal{F}$ such that $C(i) = S_m^n\big(g(i), f_0(i), \ldots, f_{n-1}(i)\big)$ in K for all $i \in J_2$. But then $C = S_m^n(g, f_0, \ldots, f_{n-1})$ in $K^{(\mathcal{F})}$, and this completes the proof of (4) for $K^{(\mathcal{F})}$ and $L^{(\mathcal{F})}$. ∎

4.5. LEMMA. *Let $E_A: A \longrightarrow A^*$ and $E_B: B \longrightarrow B^*$ be clone algebra embeddings satisfying (1)–(4) of Lemma 4.4. Then*

$$A \simeq B \text{ if and only if } A^* \simeq B^*.$$

PROOF. To simplify the notation, let us suppose again that the embeddings E_A, E_B are inclusions. We let $(S_m^n)_A, (S_m^n)_A^*, (S_m^n)_B, (S_m^n)_B^*$ stand for the non-nullary clone operations of A, A^*, B, B^* respectively, and $c_i^{(n)}$ for the nullary operations of each of these four algebras.

Let $\Phi: A \longrightarrow B$ be a clone algebra isomorphism. Then Φ maps each sort A_n of A bijectively onto the sort B_n of B in such a way that $\Phi(c_j^{(n)}) = c_j^{(n)}$ for every $j \in n$. Furthermore, if $g = (S_n^r)_A(h, c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)})$ is an extremal decomposition of some $g \in A_n$ then $\Phi(g) = (S_n^r)_B\big(\Phi(h), c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)}\big)$ is an extremal decomposition of $\Phi(g)$.

From (1) we recall that $A_n^* = A_n \cup \{*^{(n)}\}$ and $B_n^* = B_n \cup \{*^{(n)}\}$, and set

$$\Psi(g) = \begin{cases} \Phi(g) & \text{for all } g \in A_n \\ *^{(n)} & \text{for } g = *^{(n)}. \end{cases}$$

It is clear that Ψ maps each A_n^* bijectively onto B_n^*. To prove that Ψ is an isomorphism, we only need to show that

$$\text{(C)} \qquad \Psi\big((S_m^n)_A^*(g, f_0, \ldots, f_{n-1})\big) = (S_m^n)_B^*\big(\Psi(g), \Psi(f_0), \ldots, \Psi(f_{n-1})\big)$$

for any choice of $f_0, \ldots, f_{n-1} \in A_m^*$ and $g \in A_n^*$.

First suppose that $g = *^{(n)}$.

Here $(S_m^n)_A^*(g, f_0, \ldots, f_{n-1}) = *^{(m)}$ and $(S_m^n)_B^*\big(\Psi(g), \Psi(f_0), \ldots, \Psi(f_{n-1})\big) = *^{(m)}$, from (4) applied to both A^* and B^*. But then (C) follows because $\Psi(*^{(m)}) = *^{(m)}$.

Let $g \neq *^{(n)}$ next.

Then g has an extremal decomposition $g = (S_n^r)_A(h, c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)})$ with $h \neq *^{(r)}$, and hence $\Psi(g) = (S_n^r)_B\big(\Phi(h), c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)}\big)$ is an extremal decomposition of $\Psi(g)$. If $f_{\psi(i)} = *^{(m)}$ for some $i \in r$, then $\Psi(f_{\psi(i)}) = *^{(m)}$, and (C) follows from an application of (4) to both A^* and B^*. Otherwise $f_{\psi(i)} \in A_n$ for all $i \in r$ by (1), and (C) obtains from (4) because Φ is an isomorphism of A onto B. Therefore $\Psi: A^* \longrightarrow B^*$ is an isomorphism, as claimed.

Conversely, assume $\Psi: A^* \longrightarrow B^*$ to be an isomorphism of these clone algebras. If $f \in A_n$ then $\neg\,\mathrm{ac}(f)$ by (1), and hence $\neg\,\mathrm{ac}\big(\Psi(f)\big)$ because $\neg\,\mathrm{ac}(x^{(n)})$ is a first order formula

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and Ψ is an isomorphism. But then $\Psi(f) \in B_n$ by (1) again and, therefore, the restriction Φ of Ψ to A maps A isomorphically onto B. ∎

4.6. Now we are ready to complete the proof of Proposition in 4.1, which claims that for any two infinite disciplined metrizable spaces X_1, X_2 with no absolute fixpoints and any $k \in \{1\ldots,\infty\}$,

$$\mathrm{Clo}_k(X_1) \approx \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \approx \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

But this follows immediately from 4.4–5 and the fact that, similarly to monosorted algebras (see [4] for instance), heterogeneous algebras of the same type are elementarily equivalent if and only if they have isomorphic prime limits.

4.7. In 1.9, we noted the following result.

THEOREM [7]. *If $m, n, p \in \{1, 2, \ldots, \omega\}$ satisfy $m \leq n \leq p$, then there exist infinite metrizable spaces X_1 and X_2 such that*

$$\begin{aligned} m &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) = \mathrm{Clo}_i(X_2)\}, \\ n &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) \simeq \mathrm{Clo}_i(X_2)\}, \\ p &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) \approx \mathrm{Clo}_i(X_2)\}. \end{aligned}$$ ∎

Thus to demonstrate the claim of Theorem 1.9 about the three clone segments, in view of Propositions 2.9, 3.5 and 4.1 we need only show that the spaces X_1 and X_2 of Theorem [7] may be chosen to be disciplined and without absolute fixpoints. This we do in the next section.

5. **Disciplined spaces with no absolute fixpoints abound.**

5.1. Let $\Sigma = \bigcup_{n=0}^{\infty} \Sigma_n$ be a finitary type of (mono-sorted) universal algebras in which Σ_n denotes the set of all n-ary operation symbols. For $\sigma \in \Sigma$, we write $\operatorname{ar}\sigma = n$ whenever $\sigma \in \Sigma_n$, and always assume that Σ_0 is infinite.

Let $\mathbb{P} = (P, \{p_\sigma \mid \sigma \in \Sigma\})$ be an absolutely free Σ-algebra over the empty set of generators. It is well-known that $P = \bigcup_{k=0}^{\infty} P_k$, with

$$P_0 = \{p_\sigma \mid \sigma \in \Sigma_0\} \text{ and } P_{k+1} = P_k \cup \bigcup \{p_\sigma(P_k^{\operatorname{ar}\sigma}) \mid \sigma \in \Sigma \setminus \Sigma_0\},$$

where the unions are disjoint and every p_σ with $\sigma \in \Sigma \setminus \Sigma_0$ is one-to-one.

5.2. For any integer $m \geq 1$, let $\mathbb{T}(x_0, \ldots, x_{m-1})$ denote the term algebra of type Σ over the set $\{x_0, \ldots, x_{m-1}\}$. Then $\mathbb{T}(x_0, \ldots, x_{m-1})$ is an absolutely free Σ-algebra freely generated by the set $\{x_0, \ldots, x_{m-1}\}$, see [4] or [6].

We aim to define a mapping $\varrho^{(m)}$ of $\mathbb{T}(x_0, \ldots, x_{m-1})$ into the set P^{P^m} of all maps $P^m \longrightarrow P$, where P is the underlying set of the $\emptyset$-generated absolutely free Σ-algebra $\mathbb{P}$ from 5.1. We write $\pi_i^{(m)}$ to denote the i-th product projection $P^m \longrightarrow P$ and, for each $\sigma \in \Sigma_0$, we let $\sigma^{(m)}$ stand for the constant map $P^m \longrightarrow P$ with the value $p_\sigma \in P_0$.

We define a mapping $\varrho^{(m)}$ recursively as follows.

$$\varrho^{(m)}(x_i) = \pi_i^{(m)} \text{ for } i \in m,$$
$$\varrho^{(m)}(\sigma) = \sigma^{(m)} \text{ for } \sigma \in \Sigma_0, \text{ and}$$
$$\varrho^{(m)}\big(\sigma(t_0,\dots,t_{n-1})\big) = p_\sigma \circ \big(\varrho^{(m)}(t_0)\dot{\times}\cdots\dot{\times}\varrho^{(m)}(t_{n-1})\big) \text{ for } \sigma \in \Sigma_n \text{ with } n \geq 1.$$

It is easily verified that $\varrho^{(m)}\colon \mathbb{T}(x_0,\dots,x_{m-1}) \longrightarrow P^{P^m}$ is one-to-one.

For any given $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, let $\mathbb{T}_\Omega(x_0,\dots,x_{m-1})$ consist of all those members of $\mathbb{T}(x_0,\dots,x_{m-1})$ which have a subterm

$$\sigma'(x_{i_0},\dots,x_{i_{\operatorname{ar}\sigma'-1}}) \text{ with } \sigma' \in \Omega \text{ and distinct } x_{i_0},\dots,x_{i_{\operatorname{ar}\sigma'-1}} \in \{x_0,\dots,x_{m-1}\}.$$

Denote $H^{(m)} = \varrho^{(m)}\big(\mathbb{T}(x_0,\dots,x_{m-1})\big)$ and $H_\Omega^{(m)} = \varrho^{(m)}\big(\mathbb{T}_\Omega(x_0,\dots,x_{m-1})\big)$.

Now we are ready to state the main result of [7].

M [7]. *If* $\operatorname{card}\Sigma_0 \geq 2^{\aleph_0} + \operatorname{card}(\Sigma\setminus\Sigma_0)$, *then for every* $\Omega \subseteq \Sigma\setminus(\Sigma_0\cup\Sigma_1)$ *there exists a metric* τ_Ω *on the set* P *so that, for the metric space* $X_\Omega = (P,\tau_\Omega)$ *and any integer* $m \geq 1$, *a mapping* $f\colon X_\Omega^m \longrightarrow X_\Omega$ *is continuous exactly when* $f \in H^{(m)} \setminus H_\Omega^{(m)}$. ∎

In this section we prove that the spaces X_Ω constructed in [7] are disciplined, and that they have no absolute fixpoints when $\Sigma \setminus \Sigma_0 \neq \emptyset$. To do this, we need to use some of their structural properties that were stated only implicitly in [7] or had only an auxiliary rôle there.

5.3. As in 3.4 of [7], we write $P_0 = \{p_\sigma \mid \sigma \in \Sigma_0\}$, $B_\sigma = p_\sigma(P^n)$ for $\sigma \in \Sigma_n$ with $n \geq 1$, and $B = \bigcup\{B_\sigma \mid \sigma \in \Sigma \setminus \Sigma_0\} = P \setminus P_0$.

In 3.5 of [7] it is shown that (P,τ_Ω) is a metric space of diameter 1 such that

(α) B is a closed subset of (P,τ_Ω),

(β) p_σ is an isometry $(P^n,\tau_\Omega^n) \longrightarrow (B_\sigma,\tau_\Omega\restriction B_\sigma)$ when $\sigma \in \Sigma_n \setminus \Omega$,

(γ) $\tau_\Omega(B_\sigma,B_{\sigma'}) = 1$ whenever $\sigma,\sigma' \in \Sigma\setminus\Sigma_0$ are distinct,

(δ) p_σ is an isometry $(P^n,\tilde{\tau}_\Omega^n) \longrightarrow (B_\sigma,\tau_\Omega\restriction B_\sigma)$ when $\sigma \in \Sigma_n \cap \Omega$,

where $\tilde{\tau}_\Omega^n$ in (δ) is an auxiliary metric which we describe in 5.4 below.

From (α) and (γ) it follows that every set B_σ with $\sigma \in \Sigma\setminus\Sigma_0$ is closed in X_Ω, and (β) implies that p_σ is a homeomorphism of X_Ω^n onto the closed subset B_σ for any $\sigma \in \Sigma_n \setminus \Omega$.

5.4. For P as in 5.1 and B as in 5.3, let us write

$$P^n[i,j] = \{(z_0,\dots,z_{n-1}) \in P^n \mid z_i = z_j\} \text{ for distinct } i,j \in n,$$
$$P^n[i,c] = \{(z_0,\dots,z_{n-1}) \in P^n \mid z_i = c\} \text{ for } i \in n \text{ and } c \in P,$$
$$P^n[i,B] = \{(z_0,\dots,z_{n-1}) \in P^n \mid z_i \in B\} \text{ for } i \in n.$$

and call these sets *small subsets* of X_Ω^n. It is clear that $P^n[i,j]$ and $P^n[i,c]$ are closed subsets of X_Ω^n, and that $P^n[i,B]$ is a closed subset of X_Ω^n because of 5.3(α).

We need to show that, on any small subset of X_Ω^n, the product metric τ_Ω^n induces the same topology as an auxiliary metric $\tilde{\tau}_\Omega^n$ from [7]. The latter metric is given for $n \geq 2$ in 3.4 of [7] by

$$\tilde{\tau}_\Omega^n(x,y) = \min\{1, \tau_\Omega^n(x,y) + |g(x) - g(y)|\},$$

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for a real-valued function $g: P^n \longrightarrow [0,1]$ and a point $a^{(n)} = (a_0^{(n)}, \ldots, a_{n-1}^{(n)}) \in P_0^n$ satisfying $a_i^{(n)} \neq a_j^{(n)}$ whenever $i,j \in n$ are distinct.

The function g, the point $a^{(n)}$, and certain small subsets of X_Ω^n are related as follows:

(1) g is continuous on $X_\Omega^n \setminus \{a^{(n)}\}$,

(2) $g(x) = 0$ for any $x \in \bigcup\{P^n[i, a_i^{(n)}] \mid i \in n\}$, and also for any x outside a neighbourhood of $a^{(n)}$ which is disjoint with the sets $\bigcup\{P^n[i,j] \mid i \neq j, i,j \in n\}$ and $\bigcup\{P^n[i,B] \mid i \in n\}$,

(3) g is discontinuous at $a^{(n)}$.

Therefore the restriction of g to any small subset of X_Ω^n—including the sets $P^n[i, a_i^{(n)}]$—is continuous, so that $\tilde{\tau}_\Omega^n$ and τ_Ω^n induce the same topology on every small subset of X_Ω^n. This implies that the identity map of $X_\Omega^n = (P^n, \tau_\Omega^n)$ onto $(P^n, \tilde{\tau}_\Omega^n)$ sends each small subset homeomorphically onto a closed subset of $(P^n, \tilde{\tau}_\Omega^n)$. Since each B_σ with $\sigma \in \Sigma \setminus \Sigma_0$ is closed, 5.3(δ) implies that, for any $\sigma \in \Sigma_n \cap \Omega$, the restriction of p_σ to any small subset of X_Ω^n is a homeomorphism onto a closed subset of X_Ω.

5.5 . Here is a summary of earlier observations which will be needed in Proposition 5.6 and in Section 7:

(1) if $\sigma \in \Sigma_n \setminus \Omega$ and $n \geq 1$, then p_σ is a homeomorphism of X_Ω^n onto a closed subset of X_Ω,

(2) if $\sigma \in \Sigma_n \cap \Omega$, then p_σ is not continuous, but its restriction to any small subset of X_Ω^n is a homeomorphism onto a closed subset of X_Ω,

(3) the topology of $(P^n, \tilde{\tau}_\Omega^n)$ is finer than the topology of (P^n, τ_Ω^n) for every $n \geq 2$.

5.6. PROPOSITION. *The space X_Ω is disciplined for every $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$.*

PROOF. Write $X = X_\Omega$. Since X is metrizable, we need only show that, for any $m \geq 1$, every non-constant continuous map $f: X^m \longrightarrow X$ is of the form $f = g \circ \pi^{(M)}$, where $\pi^{(M)}: X^m \longrightarrow X^M$ is the projection associated with some $M \subseteq m$, and g is a homeomorphism of X^M onto a closed subset of X.

Let $m \geq 1$ and let $f: X^m \longrightarrow X$ be non-constant and continuous. According to 5.2, there is a unique term $t_f \in \mathbb{T}(x_0, \ldots, x_{m-1}) \setminus \mathbb{T}_\Omega(x_0, \ldots, x_{m-1})$ with $\varrho^{(m)}(t_f) = f$.

We use the term recursivity of the absolutely free Σ-algebra $\mathbb{T}(x_0, \ldots, x_{m-1})$ as follows.

If $t_f = x_i$ with $i \in m$, then $f = \pi_i^{(m)}$ is the i-th product projection $X^m \longrightarrow X$, and hence it has a required decomposition, namely $f = \mathrm{id}_X \circ \pi_i^{(m)}$. Otherwise $t_f = \sigma(t_0, \ldots, t_{n-1})$ for some $\sigma \in \Sigma$ and $t_0, \ldots, t_{n-1} \in \mathbb{T}(x_0, \ldots, x_{m-1}) \setminus \mathbb{T}_\Omega(x_0, \ldots, x_{m-1})$. Since f is non-constant, we must have $\sigma \in \Sigma_n$ with $n \geq 1$, and $f = p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{n-1})$, where $f_j = \varrho^{(m)}(t_j)$ for all $j \in n$ and at least one of the maps $f_j: X^m \longrightarrow X$ is non-constant. Assuming that

for every $j \in n$, there is some $M_j \subseteq m$ and some homeomorphism g_j of X^{M_j} onto a closed subset of X such that $f_j = g_j \circ \pi^{(M_j)}$,

(where we write $M_j = \emptyset$, $\pi^{(M_j)}: X^m \longrightarrow X^\emptyset$ and $g_j: X^\emptyset \longrightarrow X$ when f_j is a constant map), we shall construct a required decomposition $f = g \circ \pi^{(M)}$.

First of all, we note that $\bar{g} = \prod\{g_j \mid j \in n\}$ is a homeomorphism of $\prod\{X^{M_j} \mid j \in n\}$ onto a closed subset of X^n. Set $M = \bigcup\{M_j \mid j \in n\}$, and let $\psi_j: M_j \longrightarrow M$ denote the

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inclusion map. For any $\varphi \in X^M$ write $h(\varphi) = (\varphi \circ \psi_0, \ldots, \varphi \circ \psi_{n-1})$. Then

$$h: X^M \longrightarrow \prod\{X^{M_j} \mid j \in n\}$$

is a homeomorphism of X^M onto a closed subset of $\prod\{X^{M_j} \mid j \in n\}$.

If $\Delta: X^m \longrightarrow (X^m)^n$ is the 'diagonal map' defined by $\Delta(\varphi) = (\varphi, \ldots, \varphi)$ for all $\varphi \in X^m$, and if $\pi = \prod\{\pi^{(M_j)} \mid j \in n\}$, then $\pi \circ \Delta = h \circ \pi^{(M)}$. Since $\tilde{f} = f_0 \dot{\times} \cdots \times f_{n-1}$ has a decomposition

$$X^m \xrightarrow{\Delta} (X^m)^n \xrightarrow{\pi} \prod\{X^{M_j} \mid j \in n\} \xrightarrow{\bar{g}} X^n,$$

it follows that $\tilde{f} = (\bar{g} \circ h) \circ \pi^{(M)}$, where $\bar{g} \circ h$ is a homeomorphism of X^M onto a closed subset of X^n.

For $n \geq 1$ and $\sigma \in \Sigma_n \setminus \Omega$, the mapping p_σ is a homeomorphism of X^n onto a closed subset of X, by 5.5(1), so that the composite $g = p_\sigma \circ \bar{g} \circ h$ is a homeomorphism of X^M onto a closed subset of X again. Therefore $f = p_\sigma \circ \tilde{f} = g \circ \pi^{(M)}$ has a required decomposition.

Let $\sigma \in \Sigma \cap \Omega$. Then $\sigma \in \Sigma_n \cap \Omega$ for some $n \geq 2$, and we show that $\tilde{f}(X^m)$ is contained in a small subset of X^n. Indeed, if at least one of the component maps f_j of $\tilde{f} = f_0 \dot{\times} \cdots \dot{\times} f_{n-1}$ is the constant with the value c, then $\tilde{f}(X^m) \subseteq P^n[j, c]$. Assume that no map f_j with $j \in n$ is constant. If one of them, say f_{j_0}, is not a product projection, then $f_{j_0} = p_{\sigma'} \circ h$ for some $h: X^m \longrightarrow X^{\operatorname{ar} \sigma'}$ and $\sigma' \in \Sigma \setminus \Sigma_0$, and hence $f(P^m) \subseteq P^n[j_0, B]$. In the remaining case, every f_j is a projection $\pi_{i_j}^{(m)}: X^m \longrightarrow X$, so that $\tilde{f}(x_0, \cdots, x_{m-1}) = (x_{i_0}, \ldots, x_{i_{n-1}})$. Should $x_{i_0}, \ldots, x_{i_{n-1}}$ be pairwise distinct, then $\sigma(x_{i_0}, \ldots, x_{i_{n-1}}) \in \mathbb{T}_\Omega(x_0, \ldots, x_{m-1})$ and hence f would not be continuous. Therefore $i_r = i_s$ for some distinct $r, s \in n$, and hence $f(P^m) \subseteq P^n[i_r, i_s]$. Altogether, $\tilde{f}(P^m)$ is always contained in a small subset of X^n. Since $\tilde{f}(P^m)$ is closed in X^n, it is also closed in any small subset containing it. By 5.5(2), the restriction of p_σ to $\tilde{f}(P^m)$ maps the set $\tilde{f}(P^m) = \bar{g}h(X^M)$ homeomorphically onto a closed subset of X, so that $g = p_\sigma \circ \bar{g} \circ h$ is a homeomorphism of X^M onto a closed subset of X and $f = p_\sigma \circ \tilde{f} = g \circ \pi^{(M)}$ also in this case.

Therefore every continuous non-constant $f: X^m \longrightarrow X$ with $m \geq 1$ has a required decomposition. ∎

5.7. LEMMA. *If $\Sigma \setminus \Sigma_0 \neq \emptyset$ and $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ is arbitrary, then there exists a continuous map $s: X_\Omega \longrightarrow X_\Omega$ such that $s \circ f \neq f$ for every $f: X_\Omega^k \longrightarrow X_\Omega$ with $k \geq 0$. In particular, the space X_Ω has no absolute fixpoints.*

PROOF. If $\sigma \in \Sigma_n$ and $n \geq 1$, then either $n = 1$ and $\sigma \notin \Omega$, or else $n \geq 2$ and the diagonal of X_Ω^n is a closed subset of the small subset $P^n[0, 1] \subset X_\Omega^n$. Hence $s(x) = p_\sigma(x, \ldots, x)$ is a homeomorphism of X_Ω onto a closed subset of B_σ for any $\sigma \in \Sigma \setminus \Sigma_0$. If $s(a) = a$ for some $a \in P$ then $\sigma(a, \ldots, a) = a$ in the absolutely free $\emptyset$-generated Σ-algebra $\mathbb{P}$, which is impossible. Therefore $s \circ f \neq f$ for every continuous $f: X_\Omega^k \longrightarrow X_\Omega$ with $k \geq 0$. ∎

5.8. COROLLARY. *If $\Sigma \setminus \Sigma_0 \neq \emptyset$ and $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, then X_Ω is a disciplined space without absolute fixpoints.* ∎

6. **Operation clones of traces.**

6.1. To complete the proof of Theorem 1.9, we recall that the trace $\mathbb{A}(X) = (Q, Z \cup \{\cdot\})$ of any infinite metrizable space $X = (P, t)$ has a binary operation $\cdot$ and a countable set $Z = \{\gamma\} \cup \{\pi_j \mid j \in \omega\}$ of unary operations on the set $Q = S \cup P \cup \{\lambda\}$, in which S is the collection of all one-to-one sequences in P. One can easily verify that the binary operation $\cdot$ and the set Z of the unary operations satisfy these identities:

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot y = y \cdot x$
(2) $\sigma(x \cdot y) = x \cdot y$ for all $\sigma \in Z$,
(3) $\rho\sigma(x) = \sigma(x)$ for all $\rho, \sigma \in Z$,
(4) $\sigma(x) \cdot \sigma(x) = \sigma(x)$ for all $\sigma \in Z$,
(5) $\pi_0(x) \cdot \pi_1(x) \cdot y = x \cdot y$,
(6) $\pi_j(x) \cdot \pi_{j'}(x) \cdot t(x) = \pi_0(x) \cdot \pi_1(x)$ for all $j \neq j'$ in ω, provided $t(x) = x$, or $t(x) = \sigma(x)$ for some $\sigma \in Z$, or $t(x)$ is missing altogether.

6.2. PROPOSITION. *Let $X = (P, t)$ be an infinite metrizable space with no isolated points. Then an identity holds in $\mathbb{A}(X)$ if and only if it is a consequence of the identities described in (1)–(6) above.*

PROOF. For $n, n' \geq 1$, let $t(x_1, \ldots, x_n)$ and $t'(x'_1, \ldots, x'_{n'})$ be terms in the basic operations of $\mathbb{A}(X)$ over the variables indicated. We need to show that any identity

$$t(x_1, \ldots, x_n) = t'(x'_1, \ldots, x'_{n'}) \tag{I}$$

either fails to hold in $\mathbb{A}(X)$ or else follows from (1)–(6).

The following easily established properties of $\mathbb{A}(X)$ will be used:

(a) if $t(x_1, \ldots, x_m)$ is a term and $p \in P$, then $t(p, \ldots, p) = p$,
(b) if $q, q' \in Q$ and $q \cdot q' \neq \lambda$, then $q = q' = q \cdot q' \in P$,
(c) if $t(x_1, \ldots, x_n)$ is a term and $(q_1, \ldots, q_n) \in Q^n$ with $q_i = \lambda$ for some $i = 1, \ldots, n$, then $t(q_1, \ldots, q_n) = \lambda$.

We claim that an identity (I) is satisfied in $\mathbb{A}(X)$ only when it is regular, that is, when it has the form

$$t(x_1, \ldots, x_n) = t'(x_1, \ldots, x_n). \tag{R}$$

If not, then, say, $x_1 \notin \{x'_1, \ldots, x'_{n'}\}$. Select any $p \neq p'$ in P and substitute p for x_1 and p' for all other variables occurring in (I). Then $t'(p', \ldots, p') = p'$ and $t(p, p', \ldots p') \in \{p, \lambda\}$, by (a), (b) and (c). Since $p' \notin \{p, \lambda\}$, such an identity fails to hold in any $\mathbb{A}(X)$.

Before discussing reasons why (R) holds or fails in $\mathbb{A}(X)$, we note that any term t in which $\cdot$ does not occur is either a variable or $t(x) = \sigma(x)$ holds in $\mathbb{A}(X)$ for some $\sigma \in Z$, by (3).

CASE 1. Suppose that $\cdot$ occurs on neither side of (R). Since members of Z act differently from one another and from the identity map on the trace $\mathbb{A}(X)$ of any infinite X, the identity (R) holds in $\mathbb{A}(X)$ exactly when it follows from (1)–(6).

We turn to the remaining possibilities now.

Suppose that $\cdot$ does occur in $t(x_1,\ldots,x_n)$. Then (1)–(3) imply that $t = t_1\cdot\ldots\cdot t_r$ where each t_i is either one of the variables of t, or $t_i(x) = \sigma(x)$ holds in $\mathbb{A}(X)$ for some $\sigma \in Z$. Having applied (5) twice to each t_i which is a variable x, we find that $t_i(x) = \pi_0(x)\cdot\pi_1(x)$ holds in $\mathbb{A}(X)$, and may henceforth assume that every t_i has the form $t_i(x) = \sigma(x)$ for some $\sigma \in Z$. Then, using (4) to eliminate all repetitions, we may also assume that $t_i \neq t_j$ whenever $i \neq j$. Having applied (6), we also conclude that each variable x needs to occur in t at most twice, and that $t(x) = t_x(x)$ if t is unary or else $t(x) = t_x(x)\cdot T_x$—where the variables of T_x are the variables of t other than x, and $t_x(x)$ is one of the unary terms

(U) $$\pi_0(x)\cdot\pi_1(x), \gamma(x), \text{ or } \pi_j(x)\cdot\gamma(x), \pi_j(x) \text{ with } j \in \omega$$

in either case.

In particular, for any unary term t in which $\cdot$ does occur, in any $\mathbb{A}(X)$ we have either $t(x) = \pi_0(x)\cdot\pi_1(x)$ or $t(x) = \pi_j(x)\cdot\gamma(x)$ for some $j \in \omega$.

For a future use we note that, since an infinite disciplined metrizable space has no isolated points, for any $p \in P$ there exists a one-to-one sequence s converging to p. In particular, there are sequences $s, s_j \in S$ converging to p such that $\pi_j(s) \neq p$ for all $j \in \omega$, and $\pi_k(s_j) = p$ exactly when $k = j \in \omega$. In algebraic terms, this respectively means that $\pi_j(s)\cdot\gamma(s) = \lambda$ for all $j \in \omega$, and that $\pi_k(s_j)\cdot\gamma(s) \neq \lambda$ if and only if $k = j \in \omega$.

CASE 2. Suppose that $\cdot$ occurs in t but not in t'. Thus $t'(x) = x$ or $t'(x) = \sigma(x)$ for some $\sigma \in Z$. We claim that (R) fails to hold in any $\mathbb{A}(X)$. In the first case, $t(s) \in P\cup\{\lambda\} \not\ni s = t'(s)$ for any $s \in S$. In the second, we choose a convergent sequence $s \in S$ so that $\pi_j(s) \neq \gamma(s)$ for all $j \in \omega$. Then $t(s) = \lambda \notin P$ and $t'(s) = \sigma(s) \in P$ for every $\sigma \in Z$. Whence (R) fails to hold in $\mathbb{A}(X)$ in either case.

CASE 3. Suppose that $\cdot$ occurs in both t and t'. Let the identity (R) hold in $\mathbb{A}(X)$, and let x be one of its variables. Then $t = t_x(x)\cdot T_x$ and $t' = t'_x(x)\cdot T'_x$—where x occurs neither in T_x nor in T'_x and $t_x(x)$, $t'_x(x)$ are amongst the unary terms listed in (U).

(3a) Suppose that $t_x(x) = \sigma(x)$ for some $\sigma \in Z$. To show that $t_x(x) = t'_x(x)$, we select an $s \in S$ so that $\gamma(s) \in P$ and $\gamma(s) \neq \pi_j(s)$ for all $j \in \omega$, and then substitute s for x and $\sigma(s)$ for each of the remaining variables in (R). Then $\sigma(s) \in P$ and, if T_x and T'_x are present, also $T_x = T'_x = \sigma(s)$, by (a). If (R) is a unary identity, then $\sigma(s) = t'_x(s)$. If not, then $\sigma(s) = t = t'_x(s)\cdot\sigma(s)$ by (4), and hence $\sigma(s) = t'_x(s)$ follows again, this time from (b). Since s is one-to-one and $\gamma(s) \neq \pi_j(s)$, we have $\pi_0(s)\cdot\pi_1(s) = \lambda = \pi_j(s)\cdot\gamma(s)$ for all $j \in \omega$, and also $\sigma'(s) \neq \sigma(s)$ in P whenever $\sigma, \sigma' \in Z$ are distinct. Since $t'_x(x)$ is one of the terms listed in (U), the only remaining possibility is that $t'_x(x) = \sigma(x)$. But then $t_x(x) = t'_x(x)$, as claimed.

(3b) To show that $t_x(x) = t'_x(x)$ in all cases, suppose now that $t_x(x) = \pi_j(x)\cdot\gamma(x)$ for some $j \in \omega$. Select $s_j \in S$ so that $\gamma(s_j) = \pi_j(s_j)$, and substitute s_j for x and $\pi_j(s_j) \in P$ for all other variables in (R). Then $t_x(s_j) = \pi_j(s_j) \in P$, and $T_x = T'_x = \pi_j(s_j)$—if these terms are present. Then $\pi_j(s_j) = t'_x(s_j)$ follows as in (3a), from where we also already know that $t'_x(x)$ cannot be $\sigma(x)$ for any $\sigma \in Z$. If $t'_x(x) = \pi_0(x)\cdot\pi_1(x)$, then $t'_x(s_j) = \lambda \notin P$ because s_j is one-to-one, and (R) fails. The remaining possibility given by (U) is that

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$t'_x(x) = \pi_k(x) \cdot \gamma(x)$ for some $k \in \omega$. But $t'_x(s_j) = \pi_k(s_j) \cdot \gamma(s_j) = \pi_k(s_j) \cdot \pi_j(s_j) \in P$ only when $k = j$ because of (b) and the fact that s_j is one-to-one. Therefore $t_x(x) = t'_x(x)$ whenever neither term is equal to $\pi_0(x) \cdot \pi_1(x)$ in $\mathbb{A}(X)$. This concludes the proof that $t_x(x) = t'_x(x)$ follows for every variable x of any identity (R) which holds in $\mathbb{A}(X)$.

Altogether, any identity (I) either follows from (1)–(6) or else fails to hold in every $\mathbb{A}(X)$. ■

The proof of Theorem 1.9 is now complete.

7. **Consequences and modifications.**

7.1 . Let $(\mathcal{K}, \mathrm{U})$ be a concrete category with finite products preserved by its forgetful functor U. Then the clone $\mathrm{Clo}(a)$ exists for every object a of $\mathcal{K}$ with non-empty underlying set. If b is another such object, then the suprema

$$i(a,b) = \sup\{k+1 \mid \mathrm{Clo}_k(a) \simeq \mathrm{Clo}_k(b)\} \text{ and}$$
$$e(a,b) = \sup\{k+1 \mid \mathrm{Clo}_k(a) \approx \mathrm{Clo}_k(b)\}$$

exist and their values belong to the set $\{1, 2, \ldots, \infty\}$. It is clear that the two resulting functions satisfy

(1) $\varphi(a,a) = \infty$,
(2) $\varphi(a,b) = \varphi(b,a)$, and
(3) $\varphi(b,c) \geq \min\{\varphi(a,b), \varphi(a,c)\}$.

This leads to a natural question about the representability of φ by i or e over some category $\mathcal{K}$ of universal algebras. The question is addressed in Theorems 7.2–3 below. These follow from Theorems 1 and 2 of [7]—now valid for infinite disciplined metrizable spaces without absolute fixpoints—by means of Propositions 3.5 and 4.1.

REMARK. For the reciprocal $\rho = 1/\varphi$ of a function φ satisfying (1)–(3) we have

$$\rho(a,a) = 0,\ \rho(a,b) = \rho(b,a) \text{ and } \rho(b,c) \leq \max\{\rho(a,b), \rho(a,c)\},$$

which are the axioms of a non-archimedean pseudometric. Every metrizable 0-dimensional topological space can be metrized by a non-archimedean metric ρ with values in the set of integer reciprocals, see [2]. Theorem 7.2 below implies that any such pseudometric space can be represented by the reciprocal $1/i$ of our function i on a suitable collection of universal algebras.

7.2. THEOREM. *Let C be a set, and let $\varphi: C \times C \longrightarrow \{1, 2, \ldots, \infty\}$ be a function satisfying 7.1(1)–(3). Then there exists a system $\{A_c \mid c \in C\}$ of finitary algebras with pairwise isomorphic clones of operations, and such that $i(A_a, A_b) = \varphi(a,b)$ for every $(a,b) \in C \times C$. Moreover, if $\varphi(a,b) > 1$ for all $(a,b) \in C$, then all algebras A_c can be constructed over the same underlying set.* ■

Since there are only countably many formulas in the first order language of clones and their k-segments, a result by Erdős and Radó [3] implies that an analogue of Theorem 7.2 for elementary equivalence cannot hold when card $C > 2^{\aleph_0}$. In Theorem 7.3 below we present a partial positive result that leaves open the remaining case of $\aleph_0 <$ card $C \leq 2^{\aleph_0}$.

7.3. THEOREM. *Let C be a countable set, and let $\varphi\colon C \times C \longrightarrow \{1, 2, \ldots, \infty\}$ be a function satisfying 7.1(1)–(3). Then there is a system $\{B_c \mid c \in C\}$ of finitary algebras with pairwise isomorphic clones of operations, and such that $e(B_a, B_b) = \varphi(a, b)$ for every $(a, b) \in C \times C$. Moreover, if $\varphi(a, b) > 1$ for all $(a, b) \in C$, then all algebras B_c can be constructed over the same underlying set.* ∎

7.4. CONCLUDING REMARKS. Other conclusions can be easily obtained by various manipulations of the algebraic trace $\mathbb{A}(X)$.

(A) Theorems 7.2, 7.3 and 1.9—without their claims about operation clone isomorphism—hold also for algebras of other similarity types.

(A1) Countably many unary operations suffice. To see this, for any infinite disciplined metrizable space $X = (P, t)$ with no absolute fixpoints, we simply replace the trace $\mathbb{A}(X) = (Q, Z \cup \{\cdot\})$ by an algebra $\mathbb{B}(X) = (Q \times Q, \{\sigma_2 \mid \sigma \in Z\} \cup \{\beta, P_0, P_1\})$ in which, for all $(x_0, x_1) \in Q \times Q$, we set $\sigma_2(x_0, x_1) = \big(\sigma(x_0), \sigma(x_1)\big)$ when $\sigma \in Z$, $\beta(x_0, x_1) = (x_0 \cdot x_1, x_0 \cdot x_1)$ and $P_i(x_0, x_1) = (x_i, x_i)$ for $i \in 2$. Then $\mathbb{B}(X)$ is a unary algebra such that $g\colon \mathbb{B}(X)^n \longrightarrow \mathbb{B}(X)$ is a homomorphism exactly when $g = f \times f$ for some homomorphism $f\colon \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$.

(A2) A single ω-ary operation suffices as well. For any infinite disciplined metrizable space $X = (P, t)$ with no absolute fixpoints, we replace $\mathbb{A}(X)$ by an algebra $\mathbb{C}(X) = (R, \Gamma)$ of type $\{\omega\}$ in which $R = P \cup \{\lambda\}$ and, for any $s \in R^\omega$,

$$\Gamma(s) = \begin{cases} p & \text{if } s \in P^\omega \text{ converges to } p \\ \lambda & \text{in all other cases.} \end{cases}$$

It is clear that, for $n \geq 0$, any homeomorphism h of X^n onto a closed subset of X extends to a homomorphism $E(h)\colon \mathbb{C}(X)^n \longrightarrow \mathbb{C}(X)$, and that $E(h)$ is constant whenever h is. Proposition 2.3 is, in fact, somewhat easier to prove for $\mathbb{C}(X)$ than for $\mathbb{A}(X)$, and the remainder is straightforward.

(A3) In a forthcoming paper, it will be shown that three unary operations are also sufficient. The proof uses a modification of a type-reducing construction from [5].

(B) Theorem 1.9 and its unary forms hold also for an algebra and its reduct—in place of two algebras with isomorphic operation clones.

The spaces X_1 and X_2 used to prove Theorem 1 in [7], of which Theorem 1.9 is an algebraic translation by means of their traces $\mathbb{A}(X_i)$, have the same underlying set and satisfy $\mathrm{Clo}(X_1) \subset \mathrm{Clo}(X_2)$, see also 5.5(3). The algebras $\mathbb{A}(X_1)$ and $\mathbb{A}(X_2)$ inherit these properties. Hence $\mathbb{A}(X_1)$ can be expanded to an algebra $\mathbb{D}(X_1)$ by adding a new unary operation δ_1 defined by $\delta_1(x) = \gamma_2(x)$, where γ_2 was the unary 'convergence' operation of $\mathbb{A}(X_2)$. Then $\mathrm{Clo}\big(\mathbb{D}(X_1)\big) = \mathrm{Clo}\big(\mathbb{A}(X_1)\big)$ follows from Proposition 2.3 and the fact that $\mathrm{Clo}(X_1) \subset \mathrm{Clo}(X_2)$. The reduct $\mathbb{D}(X_2) = \mathbb{A}(X_2)$ is then obtained from $\mathbb{D}(X_1)$ by removing the original 'convergence' operation γ_1 of $\mathbb{A}(X_1)$.

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