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CYCLIC ORDERS AND GRAPHS OF GROUPS

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Abstract We examine a cyclic order on the directed edges of a tree whose vertices have cyclically ordered links. We use it to show that a graph of groups with left-cyclically ordered vertex groups and convex left-ordered edge groups is left-cyclically orderable.

Keywords: cyclic ordering; graph of groups

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1. Introduction

Dicks and Sunic gave an elegant way of totally ordering the vertex set of a directed tree [9]. They applied this to give a simple proof of Vinogradov's result that free groups, and more generally, free products of left-orderable groups are left-orderable. The purpose of this text is to describe a cyclically ordered counterpart.

Our basic observation is that:

Lemma 1.1. *Let $T = (V, E)$ be a tree. Suppose there is a cyclic order on link(v) for each $v \in V$. Then there is an induced cyclic order on the directed edges of T.*

Using this natural cyclic order, we examine graphs of groups and obtain:

Theorem 1.2. *Let G split as a graph of groups with left-cyclically ordered vertex groups and convex left-ordered edge groups. Then G is left-cyclically ordered in a manner compatible with its vertex and edge groups.*

This generalizes the result of Baik and Samperton that free products of left-cyclically ordered groups are left-cyclically ordered [2]. Another recent study probing more deeply than our own, was given by Clay and Ghaswala who characterized when an amalgam of cyclically ordered groups is cyclically ordered [5]. The approach of Clay and Ghaswala specializes to give a proof of the amalgamated product case of our result. Moreover, it was pointed out to us that Calegari suggested a similar approach to cyclically ordering the boundary of the Bass–Serre tree of an amalgamated product [4, Ex 2.116], from which one can sometimes deduce a cyclic ordering on the amalgam after some additional care and hypotheses.

There has been increased activity in the study of cyclically ordered groups, which are a bit more general than ordered groups. Some perspective on the relationship between them is given by the intriguing characterization that G is left-ordered if and only if $G\times\mathbb{Z}_n$ is left cyclically ordered for each n [1]. Finally, we refer to [10] and [3] for surveys on cyclically ordered groups.

2. Cyclic orders

Definition 2.1. *(Cyclic order) A cyclic order on a set A is a function* $\Theta : A\times A\times A \to \{-1,0,1\}$ *satisfying the following conditions:*

- ***Non-degeneracy****: $\Theta(x,y,z)=\pm 1$ if and only if x,y,z are pairwise distinct.*
- ***Cyclicity****: If $\Theta(x,y,z)=1$, then $\Theta(z,x,y)=1$.*
- ***Asymmetry****: $\Theta(x,y,z)=-\Theta(y,x,z)$.*
- ***Transitivity****: If $\Theta(x,y,z)=1$ and $\Theta(x,z,w)=1$, then $\Theta(x,y,w)=1$.*

We write $[x,y,z]$ whenever $\Theta(x,y,z)=1$.

Definition 2.2. *A strict total order is a binary relation $\prec$ on a set X satisfying the following conditions for all $x,y,z\in X$:*

- ***Irreflexivity****: $x\not\prec x$ for all $x\in X$.*
- ***Comparability****: if $x\neq y$ then $x\prec y$ or $y\prec x$.*
- ***Transitivity****: if $x\prec y$ and $y\prec z$ then $x\prec z$.*

The associated total order is denoted by $x\preceq y$ which means $x\prec y$ or $x=y$. We refer to $(X,\preceq)$ as a totally ordered set, and $(X,\prec)$ as a strict-totally ordered set.

Remark 2.3. For a strict-totally ordered set $(X,\prec)$, an associated cyclic order on X is defined by: $[x,y,z]$ holds provided $x\prec y\prec z$ or $y\prec z\prec x$ or $z\prec x\prec y$.

Remark 2.4. Consider $[0,2\pi)$ with the usual total order. Identifying $[0,2\pi)$ with S^1 using $\theta\mapsto e^{\theta i}$, and applying Remark 2.3 provides a cyclic order on S^1.

3. Cyclic orders on trees

A *tree* is a non-empty, connected, acyclic, simplicial graph. An edge with vertices u,v is associated to two *directed edges*: (u,v) and (v,u).

In this section, we cyclically order the directed edges of a tree. We emphasize that each edge corresponds to two directed edges. The cyclic ordering arises from the following statement, which is illustrated in Figure 1.

Lemma 3.1. *Let T be a finite tree embedded in the plane. There is an induced cyclic ordering on the directed edges of T.*

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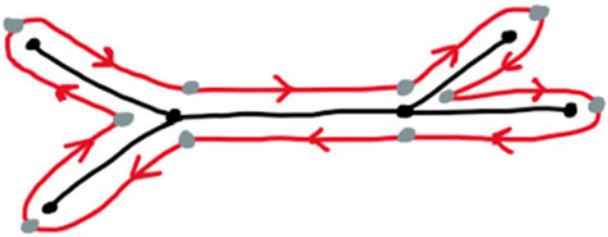

Figure 1. A clockwise boundary path cyclically orders the directed edges.

Proof. Regarding T as a disc diagram, the clockwise boundary path $\partial_{\mathsf{p}}(T)$ provides an embedding of the directed edges into S^1, hence inducing a cyclic order via Remark 2.4. Note that the boundary path traverses each edge twice: once in each direction. □

Definition 3.2. *A tree $T = (V, E)$ is a c-tree if link(v) has a cyclic order for each $v \in V$. Equivalently, there is a cyclic order on the edges adjacent to each vertex.*

We emphasize that *link*(v) has a point for each edge containing v. We are not considering directed edges here.

Definition 3.3. *An embedding $T \to \mathbb{R}^2$ of a locally finite c-tree is coordinated if for each $v \in V$ with adjacent edges $e_1 \prec e_2 \prec \cdots \prec e_n \prec e_1$, their images $\bar{e}_1 \prec \bar{e}_2 \prec \cdots \prec \bar{e}_n \prec \bar{e}_1$ are in the same clockwise order about $\bar{v} \in \mathbb{R}^2$.*

Lemma 3.4. *Each (locally) finite c-tree T has a coordinated embedding $T \to \mathbb{R}^2$.*

Proof. We produce a 'thickening' of T into 0-handles and 1-handles to obtain a disk as follows. Embed a valence n-vertex v with cyclically ordered edges $e_1, \ldots, e_n$ in a unit disk, by identifying v with 0 and identifying each edge with the segment joining 0 and $e^{\frac{2\pi}{n}i}$. Join disks for adjacent vertices along neighbourhoods (consistently orientated) to form a surface S homeomorphic to the unit disk, see Figure 2. □

Remark 3.5. The embedding of Lemma 3.4 is unique up to ambient isotopy. Hence, for any finite subtrees $T_a \subset T_b$, a coordinated embedding of T_a is essentially the same as an embedding of T_a induced by a coordinated embedding of T_b. Indeed, the way Lemma 3.4 embeds T_b induces the way it embeds T_a simply by 'forgetting' $T_b - T_a$.

For any two finite subtrees, their embeddings agree with a coordinated embedding of a larger finite tree containing them.

Theorem 3.6. *Let T be a c-tree. There is an induced cyclic order on the set of directed edges of T. It is uniquely determined by the cyclic orders on vertex links.*

Proof. For a c-tree, take a coordinated embedding of a finite subtree T'. Lemma 3.1 yields a cyclic order on the directed edges of T'. This cyclic order is consistent for $T' \subset T''$ whenever T'' is a larger finite subtree. Hence, it induces a cyclic order on all directed edges of T.

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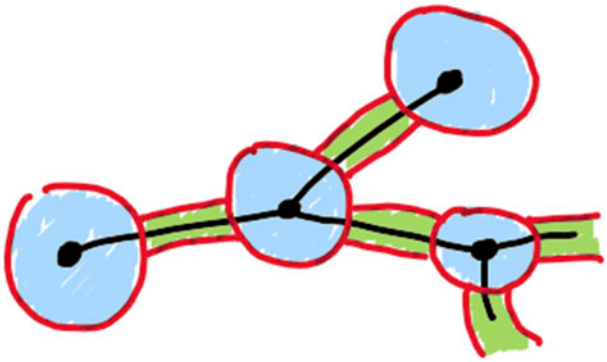

Figure 2. Handlebody decomposition of a tree in $\mathbb{R}^2$.

Uniqueness holds since the cyclic order on each link(v) is determined by the cyclic order of the outgoing directed edges at v. Note that in the cyclic ordering of the directed edges of star(v), for each edge, its two directed edges are consecutive. □

Lemma 3.7. (G-invariance). *Suppose G acts on a c-tree T so that cyclic orders on vertex links are G-invariant. Then the induced cyclic order on directed edges of T is G-invariant.*

Proof. This holds by Theorem 3.6 since the induced cyclic order on directed edges of T is determined by the cyclic orderings on vertex links. □

4. Cyclic orders and tree augmentation

We provide an alternate explanation of the cyclic ordering on directed edges of a c-tree given in § 3. This approach constructs a correspondence between directed edges and spurs.

Definition 4.1. *For vertices of a tree $x, y, z \in V$, the median $m(x, y, z)$ is the vertex equal to the intersection of geodesics $xy \cap yz \cap zx$.*

Lemma 4.2. *Let $T = (V, E)$ be a c-tree, there is a cyclic order on the set $L \subseteq V$ of spurs of T.*

Proof. When $x, y, z \in L$ are distinct, the median $m = m(x, y, z)$ has three distinct edges adjacent to m pointing to x, y and z. These edges e_x, e_y and e_z are cyclically ordered around m. Declare a cyclic order on L via:

$$[x, y, z] \text{ in } L \iff [e_x, e_y, e_z] \text{ in link}(m).$$

Non-degeneracy, cyclicity and asymmetry all follow immediately as the link of the median is cyclically ordered. For leaves $x, y, z, w \in L$, transitivity follows if $m(x, y, z) = m(x, z, w)$. Otherwise, let S be the smallest subtree containing $\{x, y, z, w\}$. S takes the form of an 'H' with two leaves at $m_1 = m(x, y, z)$ and two leaves at $m_2 = m(x, z, w)$.

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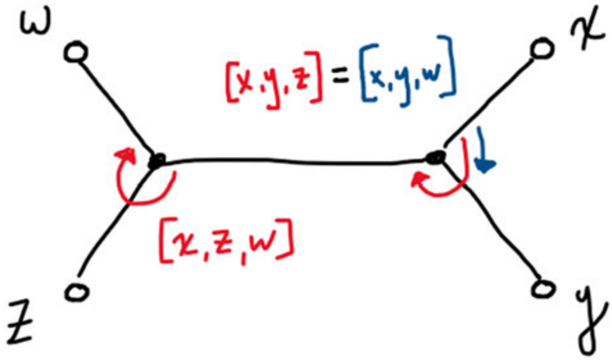

Figure 3. This explains transitivity for Lemma 4.2.

Figure 4. The direction of e determines the position of the spur.

Via Lemma 3.4, we can embed S into the plane so that links of vertices are cyclically ordered clockwise. If $[x, y, z]$ and $[x, z, w]$ hold, then $[x, y, w]$ also holds, see Figure 3. □

Definition 4.3. *(Augmented tree) Let T be a directed c-tree, the augmented tree $\overline{T}$ is obtained by adding an augmented edge e_{aug} at the barycentre of each directed edge e, see Figure 4. More precisely, for each edge $e \in E$, let b_e be its barycentre and cut e into two half edges, e_{out} and e_{in}. Orient the half edges so that e_{in} and e_{out} are incoming and outgoing at b_e. Under this construction links of vertices in the original tree T are unchanged, and the link of each barycentre vertex b_e is $\{e_{in}, e_{out}, e_{aug}\}$. Cyclically order link$(b_e)$ using the rule $[e_{in}, e_{out}, e_{aug}]$. Direct augmented edges away from barycentres, and note that the augmented tree $\overline{T}$ is now a directed c-tree.*

Theorem 4.4. *There is an induced cyclic order on the set of directed edges of a c-tree T.*

Proof. Construct the augmented tree $\overline{T}$ and note that each directed edge of T is associated to a spur of $\overline{T}$. Apply Lemma 4.2 to cyclically order these spurs. □

5. Ordered and cyclically ordered groups

Definition 5.1. (Left-ordered group). *A group G is left-ordered if there is a strict total order $(G, \prec)$ such that for all $x, y, g \in G$ we have:*

$$x \prec y \quad \Longrightarrow \quad gx \prec gy.$$

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G is left-ordered if and only if $G = P \sqcup \{1_G\} \sqcup N$ with $PP \subset P$ and $NN \subset N$ where $P = \{g \in G \ : \ 1_G \prec g\}$ and $N = \{g \in G \ : \ g \prec 1_G\}$. Then $g \prec h \iff g^{-1}h \in P$. The subset P is referred to as the positive cone.

Definition 5.2. (Left-cyclically ordered group). *A group G is left-cyclically ordered if there is a cyclic order on G that is left-invariant in the sense that:*

$$[a, b, c] \implies [ga, gb, gc].$$

Remark 5.3. Let G act freely on a cyclically ordered set X. Cyclically order G via:

$$[a, b, c] \text{ in } G \iff [ax, bx, cx] \text{ in } X.$$

The following well-known statements can be found in [8, § 1.1.3] and [11, Ex 2.116] respectively.

Lemma 5.4. *Let G act faithfully and order-preservingly on a strict-totally ordered set $(X, <)$. Then G has an induced left-order.*

Proof. Choose a well-ordering $\prec_w$ on X. For $g \neq h \in G$, let p be $\prec_w$-minimal with $gp \neq hp$. Declare $g \prec h$ if $gp < hp$.

This relation is irreflexive as $gp \not< gp$. Since G acts faithfully on X, for $g \neq h \in G$ there exists $x \in X$ with $gx \neq hx$, so comparability holds. G-invariance holds since $kgp < khp \iff gp < hp$. Let p_1 and p_2 be $\prec_w$-minimal with $xp_1 \neq yp_1$ and $yp_2 \neq zp_2$. If $p_1 = p_2$ we are done. If $p_1 \prec_w p_2$ then $yp_1 = zp_1$ and $xp_1 < zp_1$. If $p_2 \prec_w p_1$ then $xp_2 = yp_2$ and $xp_2 < zp_2$. Thus transitivity holds for $(G, \prec)$. □

Theorem 5.5. *Let G act faithfully and order-preservingly on a cyclically ordered set X. Then G has an induced left-cyclic order.*

Proof. Let $p \in X$ and $\dot{X} = X - \{p\}$. Observe that $\dot{X}$ is totally ordered and $H = stab(p)$ acts faithfully on $\dot{X}$. Via Lemma 5.4, H is left-ordered. There is a strict total order $(gH, \prec)$ for each left coset, by declaring $g\alpha \prec g\beta \iff \alpha \prec \beta$. This is independent of the choice g of representative, since $(H, \prec)$ is left H-invariant.

Our ordering on each coset provides a partial ordering on $G = \cup gH$. This partial ordering is G-invariant by definition. This partial ordering on G extends to a G-invariant cyclic ordering by cyclically ordering the left cosets using their bijection with Gp. Specifically $[a, b, c]$ holds if either:

(1) $a \prec b \prec c$ and $ap = bp = cp$,
(2) $a \prec b$ with $ap = bp \neq cp$, or $b \prec c$ with $bp = cp \neq ap$, or $c \prec a$ with $cp = ap \neq bp$,
(3) $[ap, bp, cp]$ in X. □

6. Ordering collections of cosets

Definition 6.1. (Convex subgroup). *A subgroup H of a left-ordered group $(G, \prec)$ is convex if for all $h_1, h_2 \in H$ and $g \in G$, if $h_1 \prec g \prec h_2$ then $g \in H$.*

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Definition 6.2. (c-convex subgroup). *A proper subgroup H of a left-cyclically ordered group G is c-convex if there is a G-invariant cyclic order on its cosets G/H.*

Two ways of defining c-convexity for subgroups of left-cyclically ordered groups appear in [5] and [7]. We refer to these as c'-convexity and c''-convexity and show they are both equivalent to c-convexity.

Definition 6.3. (c'-convex subgroup). *Let G be a left-cyclically ordered group and $H \subset G$ a proper subgroup. We say H is c'-convex if for every $g \notin H$ and $f \in G$ and $h_1, h_2 \in H$, if $[h_1, f, h_2]$ and $[h_1, h_2, g]$ then $f \in H$.*

The definition of c''-convexity requires the following preliminary notion.

Definition 6.4. *Let G be a left-cyclically ordered group. A proper subgroup $H \subset G$ is left-ordered by restriction if for each $h_1, h_2 \in H$, if $[h_1^{-1}, 1, h_1]$ and $[h_2^{-1}, 1, h_2]$ then $[h_1^{-1}h_2^{-1}, 1, h_2h_1]$. When $H \subset G$ is left-ordered by restriction, there is an induced left-order on H given by the following positive cone:*

$$P \;=\; \{\; h \in H \;:\; [h^{-1}, 1, h] \text{ holds in } G \;\}.$$

Definition 6.5. (c''-convex subgroup). *Let G be a left-cyclically ordered group. A proper subgroup $H \subset G$ is c''-convex if:*

(1) *Whenever $h_1, h_2 \in H$ and $f \in G$, if $[h_1, 1, h_2]$ and $[h_1, f, h_2]$ then $f \in H$.*
(2) *H is left-ordered by restriction.*

Theorem 6.6. *For a proper subgroup H of a left-cyclically ordered group G, the following are equivalent:*

(1) *c-convexity.*
(2) *c'-convexity.*
(3) *Property (1) of c''-convexity.*
(4) *c''-convexity.*

Proof of (1) $\implies$ (2). See [5, Lemma 5.1]. □

Proof of (2) $\implies$ (3). We argue by contradiction. If Property (1) fails, there exists $h_1, h_2 \in H$ and $g \notin H$ with $[h_2, 1, h_1]$ and $[h_2, g, h_1]$. Suppose $[1, g, h_1]$ and left-multiply by h_1^{-1} to get $[h_1^{-1}, h_1^{-1}g, 1]$. Since $[h_1, h_1^{-1}g, 1]$ and $[h_1, 1, g]$, c'-convexity implies that $h_1^{-1}g \in H$, a contradiction. The case $[h_2, g, 1]$ is analogous. □

Proof of (3) $\implies$ (1). See [7, Proposition 2.4]. We note that Property (2) of c''-convexity is not used in that proof. □

Proof of (2) $\implies$ (4). The proof that c'-convexity implies Property (2) of c''-convexity is shown in [5, Lemma 5.2]. □

Proof of (4) $\implies$ (3). This is immediate. □

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For subsets U, V of an ordered set $(X, \prec)$, declare $U \ll V$ if there exists $v \in V$ with $u \prec v$ for all $u \in U$. Note that within a left-ordered group $(G, \prec)$ we have $U \ll V \iff gU \ll gV$ for all $g \in G$.

The following property is well known.

Lemma 6.7. *Let $(G, \prec)$ be an ordered group and H a convex subgroup. The relation $\ll$ restricts to a G-invariant strict total order on the collection G/H of left cosets.*

Proof. Comparability of $(G/H, \ll)$ holds as cosets are disjoint and H is convex. Transitivity follows since $(G, \prec)$ is left-ordered. If $U \ll U$ for some $U \in G/H$, then there exists $v \in U$ with $u \prec v$ for all $u \in U$, so $v \prec v$ which is impossible. □

Lemma 6.8. *Let G be a left-cyclically ordered group. Let $H \subsetneq K \subsetneq G$ be convex subgroups. Then $H \ll K$ (in the induced order on K).*

Proof. Let $(K, \prec)$ be the induced left-order of Definition 6.4. Consider a coset $kH \neq H$. If $H \not\ll K$, there exists $h \in H$ with $k \prec h$. By Lemma 6.7, $k' \prec h'$ for all $k' \in kH$ and $h' \in H$. In particular, $k \prec 1$. Left multiplying gives $1 \prec k^{-1}$. Since $H \not\ll K$, we have $k^{-1} \prec h''$ for some $h'' \in H$. Finally, $1 \prec k^{-1} \prec h''$ implies $k^{-1} \in H$ by convexity, a contradiction as $k \notin H$. □

Lemma 6.9. *Suppose H and K are convex subgroups of the left-cyclically ordered group G. Either $H \subset K$ or $K \subset H$.*

Proof. If $H \not\subset K$ and $K \not\subset H$ then $H \cap K \subsetneq H$ and $H \cap K \subsetneq K$. Thus, $H \cap K \ll H$ and $H \cap K \ll K$ by Lemma 6.8. Thus there exists $h \in H$ and $k \in K$ with $\alpha \prec h$ and $\alpha \prec k$ for all $\alpha \in H \cap K$. Note that $h \neq k$, as otherwise $k \in H \cap K$ hence $k \prec k$. Without loss of generality, assume $[1, h, k]$. By convexity, $h \in K$. Thus, $h \in H \cap K$ so $h \prec h$, a contradiction. □

Corollary 6.10. *Suppose K and H are convex subgroups of a left-cyclically ordered group G. Let $x, y \in G$. If $xK \cap yH \neq \emptyset$ then either $xH \subset yK$ or $yK \subset xH$.*

Proof. This follows from Lemma 6.9. □

Definition 6.11. *It will be convenient to consider indexed collections of subsets $\{H_i\}_{i \in I}$ allowing 'repeats' in the sense that $H_i = H_j$ though $i \neq j$.*

Although we will not use it, it seems worth articulating the following special case of our preliminary goal, Theorem 6.13.

Lemma 6.12. *Let $(G, \prec)$ be a left-ordered group and $\{H_i\}_{i \in I}$ an indexed collection of convex subgroups. There is a G-invariant total order on the indexed collection of left cosets $\{gH_i \ : \ g \in G, i \in I\}$.*

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Proof. Choose a strict total order $\prec_I$ on I. Let $\ll_I$ denote the relation defined by:

$$g_1H_i \ll_I g_2H_j \iff \begin{cases} g_1H_i \neq g_2H_j \text{ and } g_1H_i \ll g_2H_j \\ g_1H_i = g_2H_j \text{ and } i \prec_I j. \end{cases}$$

Transitivity and comparability of $\ll_I$ hold since $(G, \prec)$ and $(I, \prec_I)$ are strict total orders. It is impossible for $g_1H_i \ll_I g_1H_i$, as this would imply $i \prec_I i$. Thus $\ll_I$ is irreflexive, and therefore a strict total order.

Let $g_1H_i \ll_I g_2H_j$. If $g_1H_i \neq g_2H_j$, then G-invariance of $(G, \prec)$ ensures $\alpha g_1H_i \ll_I \alpha g_2H_j$ for all $\alpha \in G$. If $g_1H_i = g_2H_j$, the order depends only on $(I, \prec_I)$, and $\alpha g_1H_i \ll_I \alpha g_2H_j$ for all $\alpha \in G$. Thus, $\ll_I$ is G-invariant. □

Theorem 6.13. *Let G be a left-cyclically ordered group and let $\{H_i\}_{i\in I}$ be an indexed collection of c-convex subgroups. There is a G-invariant cyclic order on the indexed collection of left cosets $\{gH_i : g \in G, i \in I\}$.*

Proof. Choose a strict total order $\prec_I$ on I. For any finite subcollection of c-convex subgroups $\{H_j\}_{j\in J} \subseteq \{H_i\}_{i\in I}$, by Lemma 6.9 there is a chain of inclusions. (We abuse notation and regard $J = \{0, 1, \ldots, n\}$.)

$$G = H_0 \supset H_1 \supseteq \cdots \supseteq H_n.$$

This chain of inclusions determines a graph of groups, whose underlying graph is a length-n subdivided interval. Direct all edges away from the root vertex v_0, whose vertex group is G. The edge e_i terminates at the vertex v_i, and $G_{e_i} = G_{v_i} = H_i$. As this graph of groups is telescopic its fundamental group is G.

Let $T = (V, E)$ be the Bass–Serre tree corresponding to this graph of groups. The vertex set $V = \sqcup_{i=0}^{n}\{gH_i : g \in G\}$ consists of the indexed collection of left cosets of vertex groups, see Figure 5.

There is a directed edge from g_1H_k to g_2H_{k+1} when $g_1H_k \supset g_2H_{k+1}$. Under this construction, each left coset gH_j for $j > 0$ is represented by a directed edge.

We turn T into a directed c-tree. For the root vertex G, note that $\text{link}(G)$ corresponds to G/H_1 which has a G-invariant cyclic order by c-convexity. For any other vertex gH_k, there is one incoming parent edge of $\text{link}(gH_k)$ and has outgoing edges representing containment of left-subcosets of H_{k+1}. By Lemma 6.7, $(H_k, \ll)$ induces a strict total order on H_k/H_{k+1}. This extends to a strict total order on $\{H_k\} \sqcup H_k/H_{k+1}$ by declaring H_k minimal. Translating by g provides a total order on gH_k and its left H_{k+1} cosets. This provides a cyclic order on $\text{link}(gH_k)$ by Remark 2.3.

Theorem 3.6 provides a cyclic order on directed edges of the c-tree T. Hence, this gives a cyclic order on left cosets of $\{H_j\}_{j\in J}$. This holds for any finite collection of convex subgroups. The cyclic order is consistent for graphs of groups $\mathcal{G}' \subset \mathcal{G}''$ as defined above. Hence, this induces a cyclic order on all left cosets in $\{gH_i : g \in G, i \in I\}$. As G is the fundamental group of this graph of groups, the cyclic order on the link of each vertex is G-invariant. Hence, by Lemma 3.7 the cyclic order on left cosets is G-invariant. □

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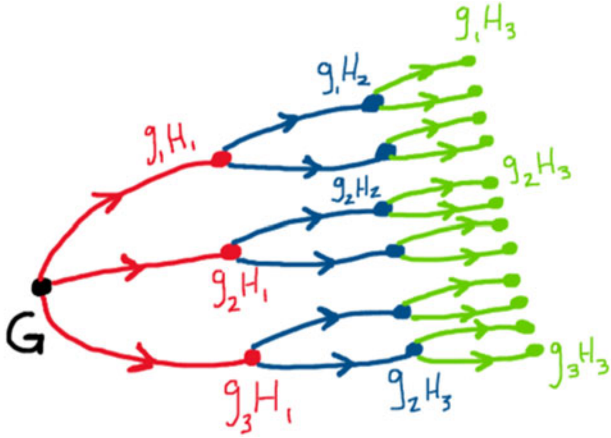

Figure 5. Part of a finite coset tree.

Remark 6.14. The referee suggests a more self-contained proof, using the fact that G has a maximal c-convex subgroup H. The left cosets within H have a left-invariant left-order by Lemma 6.12. And this can be extended to a cyclic order on the cosets in G by combining this with cyclic order on G/H.

7. Groups acting on trees

7.1. Action on tree

Definition 7.1. *An inclusion $H \to K$ of a left-ordered group into a left-cyclically ordered group is order-preserving if*

$$a \prec b \prec c \text{ in } H \quad \Longrightarrow \quad [a, b, c] \text{ in } K.$$

Theorem 7.2. *Let G act without inversions on a tree $T = (V, E)$. Suppose:*

(1) *The stabilizer G_v is left-cyclically ordered for each vertex $v \in V$.*
(2) *The stabilizer G_e is left-ordered for each edge $e \in E$.*
(3) *The inclusion $G_e \subset G_v$ is c-convex whenever v is a vertex of e.*

Then there is a c-tree $\widetilde{T} = (\widetilde{V}, \widetilde{E})$ such that:

(1) *There exists a spur $\tilde{e} \in \widetilde{E}$ such that $G\tilde{e}$ is a free orbit.*
(2) *There is a G-invariant cyclic order on the orbit $G\tilde{e}$ that induces a cyclic order on G.*
(3) *For each $e \in E$, the order on G_e is induced by the action of G_e on $\widetilde{T}$.*
(4) *For each $v \in V$, the cyclic order on G_v is induced by the action of G_v on $\widetilde{T}$.*

Proof. Build $\widetilde{T}$ from T as follows. For each $v \in V$ add a spur to v for each element of the stabilizer G_v. These spurs are in correspondence with cosets of the trivial subgroup

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of G_v which is a c-convex subgroup. $G_e \subset G_v$ is c-convex by hypothesis. For each $v \in V$, cyclically order link(v) via Theorem 6.13. Thus $\widetilde{T}$ is a c-tree.

Let $\tilde{e}$ be an added spur. Cyclically order the spurs and hence $G\tilde{e}$ by Theorem 4.4. By Lemma 3.7, the cyclic order on $G\tilde{e}$ is G-invariant. Finally, since G acts freely on $G\tilde{e}$, Remark 5.3 provides a left-cyclic order on G. □

7.2. Graph of groups statement

Corollary 7.3. *Let G split as a graph Γ of groups. Suppose each vertex group G_v is left-cyclically ordered, and each edge group G_e is left-ordered. Suppose each inclusion $G_e \hookrightarrow G_v$ of an edge group is c-convex. Then G has a left-cyclic order that restricts to the cyclic order of each vertex group G_v.*

Proof. Let $T = (V, E)$ be the Bass–Serre tree over Γ, which we assume to be directed. V consists of all left cosets of vertex groups of Γ in G, and E consists of left cosets of edge groups of Γ in G. That is, allowing for repeats (of edge or vertex groups):

$$V = \{gG_v \ : \ g \in G,\ v \in \text{Vertices}(\Gamma)\}$$
$$E = \{gG_e \ : \ g \in G,\ e \in \text{Edges}(\Gamma)\}.$$

Varying $g \in G$, there is an edge gG_e directed from gG_u to gG_v in T precisely when e is directed from u to v in Γ.

The stabilizer of a vertex gG_v equals gG_vg^{-1}, and similarly the stabilizer of an edge gG_e equals gG_eg^{-1}. Conjugation preserves the cyclic orders on G_v for each vertex, and similarly preserves the orderings on G_e for each edge, thus vertex and edge stabilizers are cyclically ordered. Let $\widetilde{T}$ be the c-tree obtained from T by Theorem 7.2 and note that the cyclic order on each vertex group is induced by its $\widetilde{T}$ action. $\widetilde{T}$ has a spur $\tilde{e}$ with a free G-orbit which provides a cyclic order on G by Remark 5.3. □

We note that [6] contains an analogous result to Corollary 7.3 for a graph of groups with left-orderable vertex groups and convex edge groups.

Remark 7.4. Every group acting faithfully without inversions on a c-tree arises as in Corollary 7.3. The edge stabilizers are c-convex subgroups of the vertex stabilizers. Indeed, for each edge e at a vertex v, the left cosets of $stab(e)$ in G_v correspond G_v-equivariantly to the edges in the G_v-orbit of e. The G_v-invariant cyclic order on the edges yields a G_v-invariant cyclic ordering on the cosets. Finally, every action on a tree arises as the Bass–Serre tree of a graph of groups.

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