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MEASURES BELOW OUTER MEASURES

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ABSTRACT. We give a simple proof that, for any $\epsilon > 0$, there is an outer measure μ^* on a finite set X such that, for any measure $\mu \leq \mu^*$, $\frac{\mu(X)}{\mu^*(X)} < \epsilon$. Thus there is a non-zero outer (finitely subadditive) measure ν^* on the clopen subsets of the Cantor set such that, if $\nu \leq \nu^*$ is a finitely additive measure on the clopen subsets of the Cantor set, then $\nu \equiv 0$.

An outer measure μ^* (a subadditive measure) on a finite set X may fail to be a (additive) measure. Nevertheless, it may be possible to find a measure $\mu \leq \mu^*$ which retains some of the properties of μ^*. Of course, the measure μ_0 obtained by dividing the counting measure on those elements of X which do not have zero measure by a suitable constant satisfies $\mu_0 \leq \mu^*$ but is completely unrelated to μ^*. The reason for this lack of connection is that the total measure of μ_0 is small compared to the total measure of μ^*. Thus we can take $\frac{\mu(X)}{\mu^*(X)}$ as a indicator of how much of μ^* has been retained by passage to μ. In this short note, we answer a question of Burke and Just by showing that, in general, we cannot expect this indicator to be positive and thus that there are outer measures on a finite set which do not have nice measures below them. This fact about measures on finite sets can be used to construct a non-zero outer (finitely subadditive) measure on the clopen subsets of the Cantor set such that the only smaller finitely additive measure on the clopen subsets of the Cantor set is the zero measure.

THEOREM 1. *For any $\epsilon > 0$, there is an outer measure μ^* on a finite set X such that, for any measure $\mu \leq \mu^*$, $\frac{\mu(X)}{\mu^*(X)} < \epsilon$.*

PROOF. Let $\epsilon > 0$. Choose $k > (1/\epsilon)+1$. Choose n such that $(1-1/k)^n < 1/k$. Define $A \subset k^n$ to be "small" if and only if $\exists\{a_i : i < n\} \subset k$ such that $a \in A \Rightarrow (\exists i < n) a(i) = a_i$. Define $\mu^*(A)$ where $A \subset k^n$ to be minimal number of small subsets whose union is A. Note that $\mu^*(k^n) = k$. Suppose $\mu \leq \mu^*$ is a measure. Inductively, find $\{a_i : i < n\}$ such that

$$\frac{\mu\big(\{a \in k^n : (\forall j < i) a(j) \neq a_j, a(i) = a_i\}\big)}{\mu\big(\{a \in k^n : (\forall j < i) a(j) \neq a_j\}\big)} > \frac{1}{k}.$$

Let $A_0 = \{a \in k^n : \exists i : a(i) = a_i\}$. We calculate

$$\frac{\mu(A_0)}{\mu(k^n)} > 1 - \left(1 - \frac{1}{k}\right)^n$$

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and $\mu(A_0) \leq \mu^*(A_0) = 1$. Thus

$$\frac{\mu(k^n)}{\mu^*(k^n)} < \frac{\left(\frac{1}{1-(1-\frac{1}{k})^n}\right)}{k} < \frac{\frac{1}{1-\frac{1}{k}}}{k} = \frac{1}{k-1} < \epsilon.$$

COROLLARY 1. *There is a non-zero outer (finitely subadditive) measure ν^* on the clopen subsets of the Cantor set such that, if $\nu \leq \nu^*$ is a finitely additive measure on the clopen subsets of the Cantor set, then $\nu \equiv 0$.*

PROOF. Let μ_n^* be an outer measure on X_n which satisfies the theorem for $\epsilon = 2^{-n}$. Define an outer measure ν^* on $\prod\{X_n : n \geq 0\}$ by declaring $\nu^* \leq \mu_n^* \circ \pi_n^{-1}$ for each $n \geq 0$.

Winfried Just and Maxim Burke had proved earlier that, for any $\epsilon > 0$, there is an outer measure μ^* on a finite set X such that, for any measure $\mu \leq \mu^*$, $\frac{\mu(X)}{\mu^*(X)} < 1/2 + \epsilon$. Just and Burke had also proved that for any outer measure μ^* on a finite set X which takes the same value on any two sets of equal cardinality, there is a measure $\mu \leq \mu^*$ such that $\frac{\mu(X)}{\mu^*(X)} > 1/2$.

David Fremlin observed independently that Corollary 1 is a consequence of the theorem.

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