EQUILIBRIUM POINTS FOR OPEN ACYCLIC RELATIONS

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1. Existence of balanced points. A formulation of a fixed point theorem, which can be applied conveniently to non-cooperative games and cooperative games, is suggested in this note.

Let $N_1, \ldots, N_m$ be m non-empty, finite disjoint sets. For $k = 1, \ldots, m$ we denote by S_k the simplex the coordinates of whose points are indexed by the members of N_k; thus S_k is the collection of all real functions x^k defined on N_k which satisfy:

(1.1) $$x^k(i) \geqslant 0, \text{ for all } i \in N_k,$$

(1.2) $$\sum_{i \in N_k} x^k(i) = 1.$$

Let $S = S_1 \times \ldots \times S_m$. We assume that for each $x \in S$ m binary relations $R^1(x), \ldots, R^m(x)$ are defined on $N_1, \ldots, N_m$ respectively. We further assume that

(1.3) $R^k(x)$ is *acyclic* (i.e., its (oriented) graph contains no circuits), for all $k = 1, \ldots, m$ and for all $x \in S$.

(1.4) R^k is *open* (*continuous*) for $k = 1, \ldots, m$; i.e., for each pair $i, j \in N_k$ the set $\{y | iR^k(y)j\}$ is open in S (when S is regarded as a subset of a proper euclidean space).

(1.5) if $x \in S$, $x = (x^1, \ldots, x^m)$, $x^j \in S_j$, $j = 1, \ldots, m$, and $x^k(i) = 0$ then there exists no $h \in N_k$ such that $hR^k(x)\, i$.

(1.5) is called the *immunity assumption*.

The following are simple results from the assumptions.

LEMMA 1. *If $i \in N_k$, $1 \leqslant k \leqslant m$, then the set*

(1.6) $$M_i^k = \{x | \text{ there is no } j \in N_k \text{ such that } iR^k(x)j\}$$

is non-empty and closed.

Proof. (1.5), (1.3), and (1.4).

LEMMA 2. *If $x \in S$, then for each k, $1 \leqslant k \leqslant m$, there exists an $i \in N_k$ such that*

(1.7) $$x \in M_i^k,$$

(1.8) $$x^k(i) > 0.$$

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Proof. (1.3) and (1.5).

A point $x \in S$ is *balanced* if $R^k(x) = \emptyset$ for $k = 1, \ldots, m$ (here $\emptyset$ denotes the empty set). It is clear that $x \in S$ is balanced if and only if:

$$x \in \bigcap_{k=1}^{m} \bigcap_{i \in N_k} M_i^k. \tag{1.9}$$

THEOREM. *There exists a balanced point in S.*

Proof. For $x \in S$ and $i \in N_k$, $k = 1, \ldots, m$, define

$$c_i^k(x) = d(x, M_i^k) \tag{1.10}$$

where $d(x, M_i^k)$ is the (euclidean) distance between x and M_i^k.

The functions $c_i^k(x)$ are non-negative continuous functions of x. It follows from (1.9), Lemma 1, and (1.10) that

(1.11) $x \in S$ is balanced if and only if $c_i^k(x) = 0$, for all $i \in N_k$ and for $k = 1, \ldots, m$.

We now define a mapping $f: S \rightarrow S$ by setting, for $x \in S$ and $i \in N_k$, $k = 1, \ldots, m$,

$$(f(x))^k(i) = \{x^k(i) + c_i^k(x)\} \Big/ \left(1 + \sum_{j \in N_k} c_j^k(x)\right). \tag{1.12}$$

We claim that,

(1.13) $y \in S$ is a fixed point of f if and only if $c_i^k(y) = 0$ for all $i \in N_k$ and for $k = 1, \ldots, m$.

The sufficiency part of (1.13) is immediate. To prove necessity let $y \in S$ satisfy $y = f(y)$. For each $1 \leqslant k \leqslant m$ there exists an $i \in N_k$ such that $y \in M_i^k$ and $y^k(i) > 0$ (see Lemma 2). Hence $c_i^k(y) = 0$ and

$$y^k(i) = y^k(i) \Big/ \left(1 + \sum_{j \in N_k} c_j^k(y)\right). \tag{1.14}$$

Since $y^k(i) > 0$ and $c_j^k(y) \geqslant 0$ for $j \in N_k$, we conclude that $c_j^k(y) = 0$ for all $j \in N_k$.

By Brouwer's fixed point theorem f has a fixed point. The proof now follows from (1.11) and (1.13).

We are now able to generalize a result of Knaster, Kuratowski, and Mazurkiewicz.

COROLLARY. *Let C_i^k, $i \in N_k$, $k = 1, \ldots, m$, be closed subsets of S, such that for each $Q \subset N_k$, $k = 1, \ldots, m$,*

$$\bigcup_{j \in Q} C_j^k \supset \{x | x \in S \text{ and } x^k(i) = 0 \text{ for all } i \in N_k - Q\}. \tag{1.15}$$

Then

$$\bigcap_{k=1}^{m} \bigcap_{i \in N} C_i^k \neq \emptyset.$$

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Proof. For $x \in S$ and $i, j \in N_k$ define:

(1.16) $$iR^k(x)j \Leftrightarrow d(x, C_i^k) > d(x, C_j^k) \quad \text{and} \quad x^k(j) > 0,$$

where $d(x, C_i^k)$ $(d(x, C_j^k))$ is the distance between x and C_i^k (C_j^k). The balanced points of the relations defined by (1.16) belong to the intersection of all the C_i^k.

2. Applications

2.1. Nash's equilibrium points (3). Let $\{S_1, \ldots, S_n; H_1, \ldots, H_n\}$ be a finite n-person game in normalized form; here $S_1, \ldots, S_n$ are the sets of mixed strategies, and $H_1, \ldots, H_n$ are the payoff functions of the players $1, \ldots, n$ respectively. If $x = (x^1, \ldots, x^n) \in S = S_1 \times \ldots \times S_n$ is an n-tuple of mixed strategies and $y^k \in S_k$, then we define:

$$x|y^k = (x^1, \ldots, x^{k-1}, y^k, x^{k+1}, \ldots, x^n).$$

$x \in S$ is an equilibrium point if

(2.1) $$H_k(x) \geqslant H_k(x|y^k) \qquad \text{for all } y^k \in S_k, k = 1, \ldots, n.$$

Let $N_1, \ldots, N_n$ be the sets of pure strategies of the players $1, \ldots, n$ respectively. For $x \in S$ and $i, j \in N_k$, $1 \leqslant k \leqslant n$, we define

(2.2) $$iR^k(x)j \Leftrightarrow H_k(x|i) > H_k(x|j) \text{ and } x^k(j) > 0.$$

Interpretation. Player k "prefers" his pure strategy i to j, when x is played, if (a) i is better than j against the strategies $x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^n$, and (b) he uses j with positive probability in x^k.

It is a straightforward matter to show that Nash's equilibrium points are exactly the balanced points of S, and that the results of the previous section can be applied to yield the existence of balanced points in S.

2.2. The kernel of a cooperative game (1). Let $G = (N, v)$ be a cooperative game; here $N = \{1, \ldots, n\}$ is the set of players of G, and v is the characteristic function. We assume that v satisfies:

(2.3) $$v(S) \geqslant 0, \qquad \text{for all } S \subseteq N;$$

(2.4) $$v(\{i\}) = 0, \qquad \text{for } i = 1, \ldots, n.$$

An outcome of G is a pair $(x; \beta)$, where $\beta = \{B_1, \ldots, B_m\}$ is a partition of the set of players, and $x = (x_1, \ldots, x_n)$ is a payoff distribution to the players, which satisfies:

(2.5) $$x_i \geqslant 0, \qquad \text{for } i = 1, \ldots, n;$$

(2.6) $$\textstyle\sum_{i \in B_j} x_i = v(B_j), \qquad \text{for } j = 1, \ldots, m.$$

Let β be a partition of N. We set,

(2.7) $$X(\beta) = \{x | (x, \beta) \text{ is an outcome for } G\}.$$

Let $x \in X(\beta)$ and $i, j \in B_k \in \beta$, $i \neq j$. We use the notation

$$s_{ij}(x) = \max\{v(S) - \textstyle\sum_{h \in S} x_h | S \subset N,\ i \in S, \quad \text{and} \quad j \notin S\}. \tag{2.8}$$

The relations associated with x are defined by

$$iR^k(x)j \Leftrightarrow s_{ij}(x) > s_{ji}(x) \text{ and } x_j > 0. \tag{2.9}$$

By definition x is balanced (according to our definition) if and only if it belongs to the kernel of G (for the partition β of the players). It is proved in **(1)** that the relations defined in (2.9) are transitive; since (1.4) and (1.5) are obvious in this case, the non-emptiness of the kernel follows from the theorem in the first section (with obvious modifications).

We remark that our results can also be applied to yield a direct existence proof for the bargaining set $M_1^{(i)}$ **(2; 4)**.

References

1. M. Davis and M. Maschler, *The kernel of a cooperative game*, Econometric Research Program, Research Memorandum No. 58 (1963), Princeton University, Princeton, N.J.; to appear in Naval Logistics Quarterly.

2. ——— *Existence of stable payoff configurations for cooperative games*, Bull. Amer. Math. Soc., *69* (1963), 106–108.

3. J. F. Nash, *Non cooperative games*, Ann. Math., *54* (1951), 286–295.

4. B. Peleg, *Existence theorem for the bargaining set* $M_1^{(i)}$, Bull. Amer. Math. Soc., *69* (1963), 109–110.

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