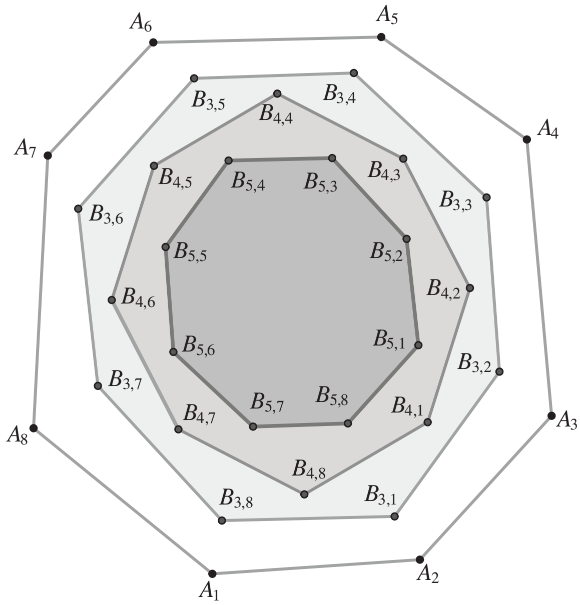

FIGURE 2

References

1. D. Wells, *The Penguin dictionary of curious and interesting geometry*, (1991) p. 53.
2. N. Lord, Maths bite: averaging polygons, *Math. Gaz*. **92** (March 2008) p. 134.

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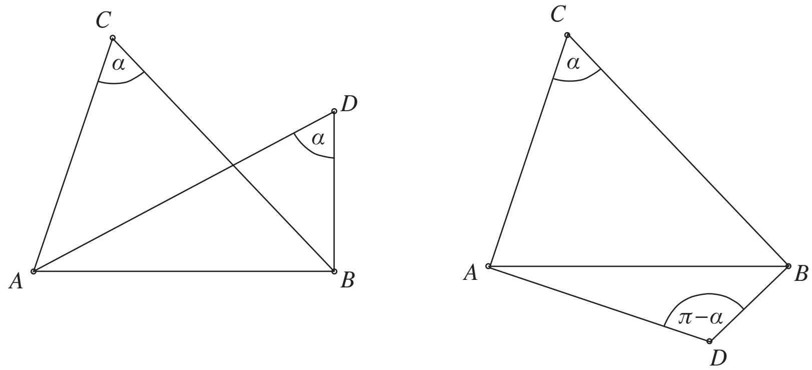
107.44 On converses of circle theorems

It has been said that one learns more from several proofs of the same result by different methods than from proofs of several results by the same method. With this in mind, we are going to look at the well-known circle theorems about angles in the same segment, and opposite angles of a cyclic quadrilateral, Euclid III.21 and III.22, or rather at their converses, which Euclid does not bother to prove. See [1].

So, suppose we are given distinct points A, B, C and D, with $\angle ACB = \alpha$ and either (i) $\angle ADB = \alpha$ (in sign as well as magnitude) or else (ii) $\angle BDA = \pi - \alpha$ (ditto), that is, $\angle ADB = \alpha - \pi$. See Figure 1. What this amounts to, in either case, is that the line through A and D, rotated about

D through α, becomes the line through B and D. In case (i), C and D are on the same side of AB, and in case (ii) they are on opposite sides. We want to prove that, in either case, A, B, C and D are concyclic.



Case (i) Case (ii)

FIGURE 1

The first and most elementary method is to use the forward theorems. So, put in the circle through A, B and C, and suppose this meets the line AD again at D'. We need to show that $D' = D$. But in case (i), by Euclid III.21, and in case (ii), by Euclid III.22, the line AD' (that is, the line AD), rotated about D' through α, becomes the line BD'. But this means that $DB \,//\, D'B$, so that DB and $D'B$ are the same line. Thus D' lies on BD, and it also lies on AD, and AD, BD are distinct lines, so that $D' = D$, as required.

Our second method is by complex numbers. Let a, b, c and d be the complex numbers representing A, B, C and D respectively. Their *cross-ratio* $(a, b; c, d)$ is defined by

$$(a, b; c, d) = \frac{(a - c)(b - d)}{(a - d)(b - c)}.$$

But here, $\dfrac{b - c}{a - c} = re^{i\alpha}$; and in case (i), $\dfrac{b - d}{a - d} = se^{i\alpha}$, or in case (ii), $\dfrac{b - d}{a - d} = te^{i(\alpha - \pi)} = -te^{i\alpha}$, where $r, s, t \in \mathbb{R}$, so in either case, $(a, b; c, d) \in \mathbb{R}$.

Now if $(a, b; c, d) \in \mathbb{R}$, it follows that A, B, C and D are collinear or concyclic, and here the four points are not collinear, so they must be concyclic. For the general case, see [2, pp.226-227], [3, pp.78-80], or [4], but for our present purposes the following will suffice. Choose axes and scaling so that $a = 0$ and $b = 1$. Now

$$(0, 1; c, d) = \frac{(-c)(1 - d)}{(-d)(1 - c)} = \frac{d^{-1} - 1}{c^{-1} - 1} \in \mathbb{R},$$

so 1, c^{-1} and d^{-1} lie on a line ℓ. Then C does not lie on the line AB, which is the real axis. Thus $c \notin \mathbb{R}$, so that also $c^{-1} \notin \mathbb{R}$, whence ℓ does not pass

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through 0. Now the map $z \rightarrow z^{-1}$ is inversion in the circle $|z| = 1$ followed by reflection in the real axis, so it sends ℓ to a circle through 0. Thus 0, 1, c and d, that is, a, b, c and d, lie on this circle; done. (For inversion, see [5, p. 109].)

Finally, we give a proof by projective geometry. Let f be the spiral similarity with $f(C) = C$ and $f(A) = B$. Let m be a variable line through A, so that $f(m)$ is a line through B, and when $m = AC$ (respectively, AD), then $f(m) = BC$ (respectively, BD). Further, the lines m and $f(m)$ are in homographic correspondence, so if they meet at P then, by Steiner's theorem, the locus of P, as m varies, is a conic Γ, through A, B, C and D. But f, as a projective transformation, fixes the circular points at infinity, I and J. So if $m = AI$, then $f(m) = BI$, and thus I lies on Γ. Similarly for J, and so Γ, being a conic through I and J, is a circle.

(For the definition of homographic correspondence, or projectivity, see [6, p.28]. For Steiner's theorem on the projective generation of a conic, see [6, p.136]; see also [7, p.321], where it is taken as the *definition* of a conic. See also [8]. For the circular points at infinity, see [2, p.270 and p.297], or [9].)

Acknowledgement

I wish to thank the referee for several suggestions that have improved this Note.

References

1. http://aleph0.clarku.edu/~djoyce/elements/bookIII/bookIII.html
2. John R. Silvester, *Geometry ancient and modern*, Oxford (2001).
3. Lars V. Ahlfors, *Complex analysis* (third edn.), McGraw-Hill (1979), available online at
   https://people.math.gatech.edu/~mccuan/courses/6321/lars-ahlfors-complex-analysis-third-edition-mcgraw-hill-science_engineering_math-1979.pdf
4. https://en.wikipedia.org/wiki/Cross-ratio
5. H. S. M. Coxeter and S. L. Greitzer, *Geometry revisited*, Singer (1967).
6. J. G. Semple and G. T. Kneebone, *Algebraic projective geometry*, Oxford (1952).
7. D. Pedoe, *A course of geometry for schools and colleges*, Cambridge (1970).
8. https://en.wikipedia.org/wiki/Steiner_conic
9. https://en.wikipedia.org/wiki/Circular_points_at_infinity

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