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ARTICLE

Couplings and Poisson approximation for stabilising functionals of determinantal point processes

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Abstract

We prove a Poisson process approximation result for stabilising functionals of a determinantal point process. Our results use concrete couplings of determinantal processes with different Palm measures and exploit their association properties. Second, we focus on the Ginibre process and show in the asymptotic scenario of an increasing observation window that the process of points with a large nearest neighbour distance converges after a suitable scaling to a Poisson point process. As a corollary, we obtain the scaling of the maximum nearest neighbour distance in the Ginibre process, which turns out to be different from its analogue for independent points.

Keywords: Coupling method; determinantal process; Ginibre process; Kantorovich–Rubinstein distance; Palm calculus; Poisson approximation; scaling limit

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1. Introduction

Determinantal point processes (DPPs) were introduced in quantum mechanics to study configurations of fermions [20]. Due to their repulsive nature, they play a fundamental role in applied sciences, for example, as a model for base stations in a wireless network [21]. In mathematics, DPPs arise naturally in different fields, such as eigenvalues of random matrices [11] and random spanning trees [5]. DPPs have notable probabilistic properties. Amongst others, the (reduced) Palm process of a DPP is again a determinantal process, and important quantities such as the Laplace transform and Janossy densities admit closed-form expressions (see, e.g. [9]). A DPP on $\mathbb{R}^d$ is determined by its correlation kernel K, which is a Hermitian function from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{C}$. An important DPP is the Ginibre process on $\mathbb{R}^2$ with a Gaussian kernel given in Section 2.

We study the following model. Let η be a stationary DPP on $\mathbb{R}^d$, and let g be a measurable function from $\mathbb{R}^d \times \mathbf{N}$ to $\{0, 1\}$, where we write $\mathbf{N}$ for the set of σ-finite point configurations on $\mathbb{R}^d$. For some measurable $W \subset \mathbb{R}^d$, let

$$\Xi[\eta] := \sum_{x \in \eta \cap W} g(x, \eta)\delta_x,$$

where δ_x denotes the Dirac measure in x. Here, the function g has the effect of a thinning of η, in the sense that Ξ is the point process of all points $x \in \eta$ in the set W, which satisfy $g(x, \eta) = 1$. The random measure Ξ is a flexible model, which appears in the study of random spatial graphs, stochastic topology, and geometric extreme value theory.

In this article, we study the distance (in an appropriate metric of point processes) of Ξ and a Poisson point process. To the best of our knowledge, this is the first paper that systematically studies Poisson approximation for determinantal processes. This continues the studies for stabilising functionals of Poisson point processes [4,10,24], Poisson hyperplane processes [23], and Gibbs point processes [16]. However, the repulsive behaviour of a DPP requires additional tools that are not needed if Poisson input is considered in Ξ. As main contributions, this article shows:

(i) If the correlation kernel K is fast decaying and if the thinning function g is stabilising and satisfies some natural assumptions, the bound of the distance of Ξ and a Poisson process is comparable to the bounds obtained for thinned Poisson point processes (see [4] and [23]).

(ii) If η is the Ginibre process and if Ξ is the point process of elements in $\eta \cap W$ with a large distance to its nearest neighbour, we prove, in an asymptotic scenario where the volume of W tends to infinity, that an appropriate scaling of Ξ converges to a Poisson process.

Our paper is organised as follows. In Section 2, we introduce DPPs and state Theorems 2.1 and 2.2 as our two main results. In Section 3, we introduce important notions such as Palm theory, negative association, and correlation decay. The proof of Theorem 2.1 is given in Section 4. In Section 5, we provide the proof of Theorem 2.2.

2. Model and main results

We work on the Euclidean space $\mathbb{R}^d$ ($d \geqslant 1$) equipped with its Borel σ-field $\mathcal{B}^d$, Lebesgue measure λ, and Euclidean norm $\|\cdot\|$. We denote by $\mathbf{N}$ the space of all σ-finite counting measures on $\mathbb{R}^d$ and by $\widehat{\mathbf{N}}$ the space of all finite counting measures on $\mathbb{R}^d$ and equip $\mathbf{N}$ and $\widehat{\mathbf{N}}$ with their corresponding σ-fields $\mathcal{N}$ and $\widehat{\mathcal{N}}$, which are induced by the maps $\omega \mapsto \omega(B)$ for all $B \in \mathcal{B}^d$. A *point process* is a random element η of $\mathbf{N}$, defined over some fixed probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The *intensity measure* of η is the measure $\mathbb{E}[\eta]$ defined by $\mathbb{E}[\eta](B) := \mathbb{E}[\eta(B)]$, $B \in \mathcal{B}^d$. For $z \in \mathbb{R}^d$ and $r > 0$, let $B_r(z)$ be the closed Euclidean ball with radius r around z. We denote $|B| := \lambda(B)$ and write $A \oplus B$ for the Minkowski sum of $A, B \subset \mathbb{R}^d$.

Let $K : (\mathbb{R}^d)^2 \to \mathbb{C}$ be a complex function. We say that η is a *determinantal point process with correlation kernel* K, if for every $n \in \mathbb{N}$ and pairwise disjoint $A_1, \ldots, A_n \in \mathcal{B}^d$, we have that

$$\mathbb{E}[\eta(A_1) \cdots \eta(A_n)] = \int_{A_1 \times \cdots \times A_n} \det(K(x_i, x_j))_{i,j=1}^n d(x_1, \ldots x_n),$$

where $d \ldots$ denotes integration with respect to λ, $(K(x_i, x_j))_{i,j=1}^m$ is the $m \times m$-matrix with entry $K(x_i, x_j)$ at position (i, j), and $\det M$ is the determinant of the complex-valued $m \times m$-matrix M. This says that η has correlation functions of all orders, that the mth order correlation function $\rho^{(m)}$ is given by

$$\rho^{(m)}(x_1, \ldots, x_m) = \det(K(x_i, x_j))_{i,j=1}^m, \quad x_1, \ldots, x_m \in \mathbb{R}^d, \quad m \in \mathbb{N},$$

and that it is locally integrable. In this article, we assume that K satisfies the following assumptions (i)–(iv):

(i) K is Hermitian, that is, $K(x, y) = \overline{K(y, x)}$, $x, y \in \mathbb{R}^d$.

(ii) K is locally square integrable, that is, for every compact $B \in \mathcal{B}^d$, the integral

$$\int_B \int_B |K(x, y)|^2 dy dx$$

is finite.

(iii) K is locally of trace class, that is, for every compact $B \in \mathcal{B}^d$, the integral $\int_B K(x, x) dx$ is finite.

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Under the assumptions (i)–(iii), it follows from Mercer's theorem that for every compact $B \subset \mathbb{R}^d$, the restriction η_B of η to B is a determinantal point process whose kernel K_B is for almost all $(x, y) \in B \times B$ given by

$$K_B(x, y) = \sum_{k=1}^{\infty} \lambda_k^B \phi_k^B(x)\overline{\phi_k^B(y)},$$

where $\lambda_k^B \in \mathbb{R}$, $k \in \mathbb{N}$, and the functions ϕ_k^B, $k \in \mathbb{N}$, form on orthonormal base of $L^2(B)$. Finally, we assume that

(iv) $0 \leqslant \lambda_k^B \leqslant 1$ for all $k \in \mathbb{N}$ and all compact $B \in \mathcal{B}^d$.

Under the assumptions (i)–(iv), there exists a unique (in distribution) determinantal point process with correlation kernel K (see [[28], Theorem 3]).

For $x \in \mathbb{R}^d$, we call η^x a Palm version of the point process η at x, if for all measurable $f: \mathbb{R}^d \times \mathbf{N} \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\int f(x, \eta)\eta(dx)\Big] = \int \mathbb{E}[f(x, \eta^x)]\mathbb{E}[\eta](dx). \tag{1}$$

Later, we will generalise this definition and define Palm processes of η with respect to Ξ.

Let η be a stationary determinantal process satisfying (i)–(iv) with intensity $\rho > 0$. Let $g : \mathbb{R}^d \times \mathbf{N} \to \{0, 1\}$ be a measurable function (called *score function*), and let $W \in \mathcal{B}^d$. Recall from the Introduction that

$$\Xi[\omega] = \sum_{x \in \omega \cap W} g(x, \omega)\delta_x, \tag{2}$$

and set $\Xi := \Xi[\eta]$. Note that by (1), the intensity measure $\mathbf{L}$ of Ξ is given by

$$\mathbf{L}(A) = \rho \int_{W \cap A} \mathbb{E}[g(x, \eta^x)]\, dx, \quad A \in \mathcal{B}^d.$$

In this article, we study the Kantorovich–Rubinstein (KR) distance of Ξ and a finite Poisson process. We recall the definition of the KR distance from [10]. For finite point processes ζ and η on $\mathbb{R}^d$, let

$$\mathbf{d_{KR}}(\zeta, \eta) := \sup_{h \in \mathrm{Lip}} |\mathbb{E}h(\zeta) - \mathbb{E}h(\eta)|,$$

where Lip is the class of all measurable 1-Lipschitz functions $h : \widehat{\mathbf{N}} \to \mathbb{R}$ with respect to the total variation between measures ω_1, ω_2 on $\mathbb{R}^d$ given by

$$d_{\mathrm{TV}}(\omega_1, \omega_2) := \sup |\omega_1(A) - \omega_2(A)|,$$

where the supremum is taken over all $A \in \mathcal{B}^d$ with $\omega_1(A), \omega_2(A) < \infty$. Under appropriate conditions on η and g, we prove that Ξ can be approximated by a Poisson process.

We suppose that there exists $\alpha \in (0, \infty)$ such that for all $A \in \mathcal{B}^d$ and all $\omega \in \mathbf{N}$,

$$\sum_{x \in \omega \cap A} g(x, \omega) < \alpha |A|, \tag{3}$$

and assume that g is monotonic in the sense that for all $x \in W$, we have

$$g(x, \omega_1) \leqslant g(x, \omega_2) \quad \text{or} \quad g(x, \omega_1) \geqslant g(x, \omega_2), \qquad \omega_1 \subset \omega_2. \tag{4}$$

We further assume that g is stabilising with respect to a Borel set $S \subset \mathbb{R}^d$, by which we mean that

$$g(x, \omega) = g(x, \omega \cap S_x)$$

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holds for any $\omega \in \mathbf{N}$ and any $x \in \mathbb{R}^d$, where $S_x := x + S$.

Moreover, we assume that the kernel $K : (\mathbb{R}^d)^2 \to \mathbb{C}$ satisfies

$$|K(x, y)| \leqslant \phi(\|x - y\|), \quad x, y \in \mathbb{R}^d, \tag{5}$$

for some decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to\infty} \phi(r) = 0$.

Theorem 2.1. *Let Ξ be the point process defined at (2) with compact $W \subset \mathbb{R}^d$. Let $S, T \subset \mathbb{R}^d$ be closed with $o \in S$ and $(S \oplus T^c) \cap S = \varnothing$. Suppose that g is stabilising with respect to S and satisfies (3) and (4) with $\alpha > 0$ and that (5) holds. Let ζ be a finite Poisson process on $\mathbb{R}^d$ with intensity measure $\mathbf{M}$. Then,*

$$\mathbf{d_{KR}}(\Xi, \zeta) \leqslant d_{TV}(\mathbf{L}, \mathbf{M}) + 2(E_1 + E_2 + F)$$

with

$$E_1 := \rho^2 \int_W \int_{W\cap T_x} \mathbb{E}[g(x, \eta^x)]\mathbb{E}[g(y, \eta^y)]\, dydx,$$
$$E_2 := \int_W \int_{W\cap T_x} \mathbb{E}[g(x, \eta^{x,y})g(y, \eta^{x,y})]\rho^{(2)}(x, y)dydx,$$
$$F := \|K\|(1 + \rho + 3^{5/2}\|K\|)(\alpha|S| + 1) \max(|S|, 1)|W \oplus S|^2\phi(d(S, T^c)),$$

where $T_x := x + T$, $\|K\| = \sup_{x,y\in\mathbb{R}^d} |K(x, y)|$, and $d(S, T^c)$ is the Hausdorff distance of S and T^c.

In the second part of this paper, we give an application of Theorem 2.1 for a concrete choice of η and of g. Let η be the (infinite) Ginibre process ξ, which is a stationary determinantal point process on $\mathbb{C}$ with correlation kernel given by

$$K(z, w) = \pi^{-1}e^{-(|z|^2+|w|^2)/2}e^{z\overline{w}}, \quad z, w \in \mathbb{C}.$$

Hence, ξ has intensity $\rho = \pi^{-1}$, and it holds that $|K(z, w)| \leqslant \phi(\|z - w\|)$ with $\phi(r) := \pi^{-1} \exp(-r^2/2)$ for $r > 0$ and $z, w \in \mathbb{C}$.

In Theorem 2.2 below, we choose g depending on $n \in \mathbb{N}$. Let g_n be the indicator function, which is one if and only if the process $\xi \setminus \{x\}$ is empty in a ball with a certain radius v_n (chosen such that $v_n \to \infty$ as $n \to \infty$) around x. This choice leads to the study of large nearest neighbour balls. It is also an important prototype for more sophisticated models in stochastic geometry and has been studied extensively for different point processes in various spaces (see [25]).

We consider ξ as a random set in $\mathbb{R}^2$. Let $B_n := B_n(o)$ the closed ball with radius $n > 0$ in $\mathbb{R}^2$ centred at the origin o. We consider the process

$$\Xi_n := \sum_{x\in\xi\cap B_n} \mathbb{1}\{\xi(B_{v_n}(x) \setminus \{x\}) = 0\}\, \delta_x. \tag{6}$$

as well as the scaled process

$$\Psi_n := \sum_{y\in\Xi_n} \delta_{y/n} = \sum_{x\in\xi\cap B_n} \mathbb{1}\{\xi(B_{v_n}(x) \setminus \{x\}) = 0\}\, \delta_{x/n}. \tag{7}$$

In the following theorem, we compare Ψ_n with a Poisson process on the unit ball B_1 in $\mathbb{R}^2$.

Theorem 2.2. *Let ν be a stationary Poisson process on $\mathbb{R}^2$ with intensity $\tau > 0$. There exists a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n^4 \sim 8 \log n$ as $n \to \infty$ and a constant $C > 0$ such that for all $n \in \mathbb{N}$ and any $\varepsilon > 0$,*

$$\mathbf{d_{KR}}(\Psi_n, \nu \cap B_1) \leqslant Cn^{\varepsilon-1/16}.$$

As an application of the above theorem, we consider largest distances to the nearest neighbour.

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Corollary 2.3. *We have as $n \to \infty$,*

$$\frac{1}{8 \log n} \max_{x \in \xi \cap B_n} \min_{y \in \xi \setminus \{x\}} \|x - y\|^4 \xrightarrow{\mathbb{P}} 1.$$

The proof of Corollary 2.3 is quite standard (see, e.g. [[24], Corollary 4.2] or [[7], Corollary 1]) and therefore omitted.

Remark 2.4. *(i) One should compare Theorem 2.1 with [[4], Theorem 4.1] (or the refined version [[23], Theorem 4.1]) that discusses Poisson process approximation for score sums built on a Poisson process. The terms E_1, E_2, and E_3 in Theorem 2.1 are the analogues to the terms E_1, E_2, and E_3 in [[4], Theorem 4.1]. Due to the spatial independence property of the Poisson process, there is no analogue of the term F (which reflects the correlation decay of the determinantal process η) in [4] and [23]. Note also that for a wide class of DPPs (including the Ginibre process), the β-mixing coefficient does not decay exponentially fast (see [[26], Proposition 4.2]). Therefore, the exponential decay dependence property from [8] is violated, and general results for Poisson approximation of strongly mixing processes do not apply.*

(ii) Note that the scaling of the maximum nearest neighbour ball in Corollary 2.3 is different from its analogue for independent points, where the second power is proportional to $\log n$ as $n \to \infty$ (see [6]). It seems interesting to investigate whether Theorem 2.2 can be extended to k-nearest neighbour distances ($k \geqslant 2$). However, this extension would require delicate estimates on empty-space probabilities of the Ginibre process, which are beyond the scope of this article.

3. Preliminaries

3.1 Palm calculus and negative associations

Following [[14], Chapter 6], we next introduce Palm measures and thereby generalise the definition given in (1). Let η be a stationary determinantal process as introduced in Section 2, and let Ξ be its thinned process with intensity measure $\mathbf{L}$. Then there are point processes $\eta^{x,\Xi}$, $x \in \mathbb{R}^d$, such that for all measurable mappings f from $\mathbb{R}^d \times \mathbf{N}$ to $[0, \infty)$,

$$\mathbb{E}\Big[\int f(x, \eta)\, \Xi(\mathrm{d}x)\Big] = \int \mathbb{E}[f(x, \eta^{x,\Xi})]\mathbf{L}(dx). \tag{8}$$

The processes $\eta^{x,\Xi}, x \in \mathbb{R}^d$, are called *Palm processes* of η with respect to Ξ at x, and the distribution $P^{x,\Xi}$ is called the *Palm measure* of η with respect to Ξ. Since Ξ is simple, $\eta^{x,\Xi}$ can be interpreted as the process η seen from x and conditioned on Ξ having a point in x. Since $\Xi \subset \eta$, it follows from [[14], Lemma 6.2 (ii)] that $\delta_x \in \eta^{x,\Xi}$ a.s. This allows us to define the reduced Palm process $\eta^{x!,\Xi} := \eta^{x,\Xi} - \delta_x$ with distribution $P^{x!,\Xi}$. If $\eta = \Xi$ (i.e. $g \equiv 1$ and $W = \mathbb{R}^d$), we write η^x for a Palm process of η (with respect to itself) at x (c.f. [1]) and $\eta^{x!}$ for a reduced Palm process.

Recall that K is the correlation kernel of η and write P for its distribution. For $x \in \mathbb{R}^d$, let $\eta^{x!}$ be reduced Palm processes of η at x and denote their distribution by $P^{x!}$. Then $\eta^{x!}$, $x \in \mathbb{R}^d$, are determinantal processes with correlation kernel K^x, $x \in \mathbb{R}^d$, given by

$$K^x(z, w) = K(z, w) - \frac{K(z, x)K(x, w)}{\rho}, \quad z, w \in \mathbb{R}^d, \tag{9}$$

(see [[27], Theorem 1.7]). By [[12], Theorem 3] (see also [22]), the process $\eta^{x!}$ is stochastically dominated by η (denoted by $P^{x!} \leqslant P$) which means that

$$\mathbb{E}[F(\eta^{x!})] \leqslant \mathbb{E}[F(\eta)] \tag{10}$$

for each measurable $F : \mathbf{N} \to \mathbb{R}$, which is bounded and increasing, by which we mean that $F(\omega_1) \leqslant F(\omega_2)$ if $\omega_1 \subset \omega_2$.

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For $x \in \mathbb{R}^d$ let, η^x be a Palm process of η at x and $\eta^{x,\Xi}$ a Palm process of η with respect to Ξ at x. Then we have

$$\mathbb{E}\Big[\int f(x,\eta)\ \Xi(dx)\Big] = \mathbb{E}\Big[\int_W f(x,\eta)g(x,\eta)\eta(\mathrm{d}x)\Big] = \int_W \mathbb{E}[f(x,\eta^x)g(x,\eta^x)]\rho(x)\lambda(\mathrm{d}x)$$
$$= \int_W \mathbb{E}[f(x,\eta^{x,\Xi})]\mathbb{E}[g(x,\eta^x)]\rho(x)\lambda(\mathrm{d}x). \tag{11}$$

An important property of DPPs is that they have negative associations (see [[19], Theorem 3.7] or [[17], Theorem 3.2]), by which we mean that

$$\mathbb{E}[F(\eta)G(\eta)] \leqslant \mathbb{E}[F(\eta)]\mathbb{E}[G(\eta)], \tag{12}$$

for any real, bounded, and increasing functions $F, G : \mathbf{N} \to \mathbb{R}$ that are measurable with respect to complementary subsets (see [17]).

Let $F : \mathbf{N} \to \mathbb{R}$ be measurable, bounded, and increasing and assume that g is measurable with respect to S and increasing in the second argument. Then we find for almost all $x \in W$ from (11) and (12) (applied to the determinantal point process $\eta^{x!}$) that

$$\mathbb{E}[F(\eta^{x,\Xi} \cap S_x^c)]\mathbb{E}[g(x,\eta^x)] = \mathbb{E}[F(\eta^x \cap S_x^c)g(x,\eta^x)] \leqslant \mathbb{E}[F(\eta^x \cap S_x^c)]\mathbb{E}[g(x,\eta^x)], \tag{13}$$

implying that $P^{x!,\Xi}|_{S_x^c} \leqslant P^{x!}|_{S_x^c}$ for λ-a.a. $x \in W$. On the other hand, if g is measurable with respect to S and decreasing in the second argument, then we find by taking $-g$ in (12) that

$$\mathbb{E}[F(\eta^{x,\Xi} \cap S_x^c)]\mathbb{E}[g(x,\eta^x)] = \mathbb{E}[F(\eta^x \cap S_x^c)g(x,\eta^x)] \geqslant \mathbb{E}[F(\eta^x \cap S_x^c)]\mathbb{E}[g(x,\eta^x)], \tag{14}$$

implying that $P^{x!}|_{S_x^c} \leqslant P^{x!,\Xi}|_{S_x^c}$ for λ-a.a. $x \in W$.

3.2 Fast decay of correlation

Let η be a stationary determinantal process on $\mathbb{R}^d$ with covariance kernel K that satisfies the conditions (i)–(iv) and $|K(x,y)| \leqslant \phi(\|x-y\|)$ for some exponentially decreasing function ϕ (see [5]). Then we have from [[3], Lemma 1.3] that the correlation functions $\rho^{(m)}$, $m \in \mathbb{N}$, of η satisfy

$$|\rho^{(p+q)}(x_1,\ldots,x_{p+q}) - \rho^{(p)}(x_1,\ldots,x_p)\rho^{(q)}(x_{p+1},\ldots,x_{p+q})| \leqslant m^{1+\frac{m}{2}}\phi(s)\|K\|^{m-1}, \tag{15}$$

where $m := p+q$, $s := d(\{x_1,\ldots,x_p\},\{x_{p+1},\ldots,x_{p+q}\}) := \inf_{i\in\{1,\ldots,p\},j\in\{p+1,\ldots,p+q\}} |x_i - x_j|$, and $\|K\| := \sup_{x,y\in\mathbb{R}^d} |K(x,y)|$.

3.3 Poisson process approximation

The following Poisson approximation result is inspired by [[4], Theorem 3.1].

Proposition 3.1. *Let the assumptions of Theorem 2.1 prevail. For $x, y \in W$, let η^x be a Palm version of η at x, let $\eta^{x,y}$ be a Palm version of η^x at y, and let $\eta^{x,\Xi} \sim P^{x,\Xi}|_{S_x^c}$ be a Palm version of η with respect to Ξ at x, restricted to S_x^c. Let ζ be a finite Poisson process with intensity measure $\mathbf{M}$. Then we have*

$$\mathbf{d_{KR}}(\Xi,\zeta) \leqslant d_{TV}(\mathbf{L},\mathbf{M}) + 2(R_1+R_2+R_3+R_4)$$

with

$$R_1 := \int_W \int_{T_x} \mathbb{E}[g(x,\eta^x)]\mathbb{E}[g(y,\eta^y)]\rho^2 dydx,$$
$$R_2 := \int_W \int_{T_x} \mathbb{E}[g(x,\eta^{x,y})g(y,\eta^{x,y})]\rho^{(2)}(x,y)dydx,$$
$$R_3 := \rho|W|\text{ess sup}_{x\in W}\mathbb{E}[g(x,\eta^x)]\mathbb{E}[(\eta\Delta\eta^{x,\Xi})(W\setminus T_x)],$$
$$R_4 := \rho|W|\text{ess sup}_{x\in W}\mathbb{E}[g(x,\eta^x)]\mathbb{E}\Big[\sum_{y\in\eta\cap\eta^{x,\Xi}\cap W\setminus T_x}|g(y,\eta)-g(y,\eta^{x,\Xi})|\Big],$$

where Δ *stands for the symmetric difference.*

Proof. Without loss of generality, we can assume that $\mathbf{L}=\mathbf{M}$ (otherwise apply [[4], (2.6)]). We adapt the proof of [[4], Theorem 3.1] to our setting. Let $\mathcal{L}$ be the generator of the Glauber dynamics from [[4], (2.7)] with associated Markov semigroup P_s, and let $D_xP_sh(\omega) := P_sh(\omega+\delta_x)-P_sh(\omega)$ for $\omega\in\widehat{\mathbf{N}}$ and $h\in\text{Lip}$. By definition of KR distance and [[4], (3.1)], we have

$$\begin{aligned}\mathbf{d_{KR}}(\Xi,\zeta) &= \sup_{h\in\text{Lip}}\Big|\int_0^\infty \mathbb{E}[\mathcal{L}P_sh(\Xi)]ds\Big|\\ &= \sup_{h\in\text{Lip}}\Big|\int_0^\infty\int_W\Big(\mathbb{E}[D_xP_sh(\Xi)]-\mathbb{E}[D_xP_sh(\Xi^{x!})]\Big)\mathbf{L}(dx)ds\Big|.\end{aligned}$$

We can bound the absolute value of the last integrand by

$$\begin{aligned}&\mathbb{E}[|D_xP_sh(\Xi)-D_xP_sh(\Xi^{x!})|]=\mathbb{E}[|P_sh(\Xi+\delta_x)-P_sh(\Xi)-P_sh(\Xi^x)+P_sh(\Xi^{x!}|]\\ &\quad\leqslant\mathbb{E}[|P_sh(\Xi+\delta_x)-P_sh(\Xi\cap T_x^c+\delta_x)|+|P_sh(\Xi)-P_sh(\Xi\cap T_x^c)|]\\ &\qquad+\mathbb{E}[|P_sh(\Xi^{x!}+\delta_x)-P_sh(\Xi^{x!}\cap T_x^c+\delta_x)|+|P_sh(\Xi^{x!})-P_sh(\Xi^{x!}\cap T_x^c)|]\\ &\qquad+\mathbb{E}[|P_sh(\Xi\cap T_x^c+\delta_x)-P_sh(\Xi^{x!}\cap T_x^c+\delta_x)|+|P_sh(\Xi\cap T_x^c)-P_sh(\Xi^{x!}\cap T_x^c)|].\end{aligned}\tag{16}$$

By [[4], (2.9)] we have that $|P_sh(\omega_1)-P_sh(\omega_1)|\leqslant e^{-s}(\omega_1\Delta\omega_2)(W)$ for $\omega_1,\omega_2\in\widehat{\mathbf{N}}$. Hence,

$$\mathbb{E}[|P_sh(\Xi+\delta_x)-P_sh(\Xi\cap T_x^c+\delta_x)|+|P_sh(\Xi)-P_sh(\Xi\cap T_x^c)|]\leqslant 2e^{-s}\mathbb{E}[\Xi(T_x)],$$
$$\mathbb{E}[|P_sh(\Xi^{x!}+\delta_x)-P_sh(\Xi^{x!}\cap T_x^c+\delta_x)|+|P_sh(\Xi^{x!})-P_sh(\Xi^{x!}\cap T_x^c)|]\leqslant 2e^{-s}\mathbb{E}[\Xi^{x!}(T_x)].$$

We now specify a coupling of Ξ and $\Xi^{x!}$ that we use to bound the third term in the right-hand side of (16). Let $\eta\sim P$ and let $\eta^{x,\Xi}\sim P^{x,\Xi}|_{S_x^c}$ be a Palm version of η with respect to Ξ at x, restricted to S_x^c. Since $(S\oplus T^c)\cap S=\varnothing$, we have $S_y\cap S_x=\varnothing$ for all $y\in T_x^c$. Hence, using that g stabilises with respect to S, $\Xi[\eta^{x,\Xi}]\cap T_x^c$ and $\Xi[\hat{\eta}^{x,\Xi}]\cap T_x^c$ agree in distribution, where $\hat{\eta}^{x,\Xi}\sim P^{x,\Xi}$. Therefore, it follows from the definition of $P^{x,\Xi}$ that $\Xi[\eta^{x,\Xi}]\cap T_x^c$ and $\Xi^{x!}\cap T_x^c$ agree in distribution. This gives

$$\mathbb{E}[|P_sh(\Xi\cap T_x^c+\delta_x)-P_sh(\Xi^{x!}\cap T_x^c+\delta_x)|]\leqslant e^{-s}\mathbb{E}[(\Xi[\eta]\Delta\Xi[\eta^{x,\Xi}])(T_x^c)],$$
$$\mathbb{E}[|P_sh(\Xi\cap T_x^c)-P_sh(\Xi^{x!}\cap T_x^c)|]\leqslant e^{-s}\mathbb{E}[(\Xi[\eta]\Delta\Xi[\eta^{x,\Xi}])(T_x^c)].$$

From the particular form of the score functional Ξ, we obtain for the last term on the right-hand side above,

$$(\Xi[\eta]\Delta\Xi[\eta^{x,\Xi}])(T_x^c)\leqslant(\eta\Delta\eta^{x,\Xi})(W\setminus T_x)+\sum_{y\in\eta\cap\eta^{x,\Xi}\cap W\setminus T_x}|g(y,\eta)-g(y,\eta^{x!,\Xi})|\quad\text{a.s.}$$

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Hence, we have shown that

$$\mathbf{d_{KR}}(\Xi,\zeta)\leqslant 2\Big\{\int_W \mathbb{E}[\Xi(T_x)]+\mathbb{E}[\Xi^{x!}(T_x)]\,\mathbf{L}(dx)$$
$$+\rho|W|\text{ess sup}_{x\in W}\mathbb{E}[g(x,\eta^x)]\mathbb{E}[(\eta\Delta\eta^{x,\Xi})(W\setminus T_x)]$$
$$+\rho|W|\text{ess sup}_{x\in W}\mathbb{E}[g(x,\eta^x)]\mathbb{E}\Big[\sum_{y\in\eta\cap\eta^{x,\Xi}\cap W\setminus T_x}|g(y,\eta)-g(y,\eta^{x,\Xi})|\Big]\Big\}.$$

From here, the asserted bound follows from the observations that $\mathbf{L}(dx)=\mathbb{1}\{x\in W\}\mathbb{E}[g(x,\eta^x)]\rho dx$ and that

$$\begin{aligned}\int_W \mathbb{E}[\Xi^{x!}(T_x)]\,\mathbf{L}(dx)&=\int_W \mathbb{E}[(\Xi[\eta^{x,\Xi}]-\delta_x)(T_x)]\,\mathbf{L}(dx)\\&=\mathbb{E}\Big[\int_W (\Xi[\eta]-\delta_x)(T_x)\;\Xi(dx)\Big]\\&=\int_W\int_{T_x}\mathbb{E}[g(x,\eta^{x,y})g(y,\eta^{x,y})]\rho^{(2)}(x,y)dydx.\end{aligned}$$

□

4. Proof of Theorem 2.1

Proof of Theorem 2.1. For $x\in W$, let $S_x:=x+S$ and $T_x:=x+T$. Let $\mathbf{L}$ be the intensity measure of Ξ. We apply Proposition 3.1. Since the terms R_1 and R_2 directly translate into E_1 and E_2, it remains to bound R_3 and R_4.

Bounding R_3.

Recall that P is the distribution of a determinantal process with correlation kernel K, that P^x is its Palm measure and that $P^{x,\Xi}$ is the Palm measure with respect to Ξ. The idea is to show the existence of a construct $(\eta,\eta^{x,\Xi})$ of P and $P^{x,\Xi}$ for each $x\in W$ with a 'small' symmetric difference $(\eta\Delta\eta^{x,\Xi})(W\setminus T_x)$. We discuss the coupling for increasing and decreasing scores separately.

(i) *Increasing scores.* If $g(x,\omega_1)\leqslant g(x,\omega_2)$ for $\omega_1\subset\omega_2$, we have by (10) and (13) that $P^{x!}\leqslant P$ and $P^{x!,\Xi}|_{S_x^c}\leqslant P^{x!}$ for $\mathbf{L}$-a.a. $x\in W$, implying that $P^{x!,\Xi}|_{S_x^c}\leqslant P$. By Strassen's theorem (see [18]), this implies that there are processes $\eta\sim P$ and $\eta^{x!,\Xi}\sim P^{x!,\Xi}|_{S_x^c}$ such that $\eta^{x!,\Xi}\subset\eta$. Thus, we have

$$\begin{aligned}\mathbb{E}[(\eta\Delta\eta^{x!,\Xi})(W\setminus T_x)]&=\mathbb{E}[\eta(W\setminus T_x)]-\mathbb{E}[\eta^{x!,\Xi}(W\setminus T_x)]\\&=\big\{\mathbb{E}[\eta(W\setminus T_x)]-\mathbb{E}[\eta^{x!}(W\setminus T_x)]\big\}+\mathbb{E}[\eta^{x!}(W\setminus T_x)]-\mathbb{E}[\eta^{x!,\Xi}(W\setminus T_x)].\end{aligned}\tag{17}$$

The term in $\{\cdots\}$ on the right-hand side above is by (9), (5) and since ϕ is decreasing, given by

$$\rho^{-1}\int_{W\setminus T_x}|K(x,y)|^2dy\leqslant\rho^{-1}\|K\|\int_{W\setminus T_x}\phi(\|x-y\|)dy\leqslant\rho^{-1}\|K\||W|\sup_{y\in T^c}\phi(\|y\|),\quad x\in W.\tag{18}$$

Next we consider the second term on the right-hand side in (17). By definition of the reduced Palm process $\eta^{x!,\Xi}$, we have for λ-almost all $x\in W$,

$$\begin{aligned}&\mathbb{E}[g(x,\eta^x)]\big\{\mathbb{E}[\eta^{x!}(W\setminus T_x)]-\mathbb{E}[\eta^{x!,\Xi}(W\setminus T_x)]\big\}\\&\quad=\mathbb{E}[\eta^x(W\setminus T_x)]\mathbb{E}[g(x,\eta^x)]-\mathbb{E}[\eta^x(W\setminus T_x)g(x,\eta^x)]\\&\quad=-\mathtt{Cov}(\eta^x(W\setminus T_x),g(x,\eta^x)).\end{aligned}\tag{19}$$

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Now we use that the reduced Palm process $\eta^{x!}$ is a determinantal process itself and therefore has negative associations (see (12)). For $k \in \mathbb{N}$, we consider the auxiliary functions

$$f^{(k)}(\omega) := \min\{k, \omega(S_x) - g(x, \omega)\}, \quad f(\omega) := \omega(S_x) - g(x, \omega), \quad \omega \in \mathbf{N}.$$

It is easy to see that $f^{(k)}$, $k \in \mathbb{N}$, and f are bounded and increasing. Since $\eta^{x!}$ has negative associations, we have that

$$\mathtt{Cov}(\min\{k, \eta^{x!}(W \setminus T_x)\}, f^{(k)}(\eta^{x!})) \leqslant 0.$$

Hence, by monotone convergence,

$$\begin{aligned}&\mathtt{Cov}(\eta^{x!}(W \setminus T_x), \eta^{x!}(S_x)) - \mathtt{Cov}(\eta^{x!}(W \setminus T_x), g(x, \eta^x))\\ &= \mathtt{Cov}(\eta^{x!}(W \setminus T_x), f(\eta^{x!})) = \lim_{k\to\infty} \mathtt{Cov}(\min\{k, \eta^{x!}(W\setminus T_x)\}, f^{(k)}(\eta^{x!})) \leqslant 0.\end{aligned}$$

This shows that (19) is bounded by

$$\begin{aligned}&-\mathtt{Cov}(\eta^{x!}(W \setminus T_x), \eta^{x!}(S_x))\\ &\quad= \mathbb{E}[\eta^{x!}(W \setminus T_x)]\mathbb{E}[\eta^{x!}(S_x)] - \mathbb{E}[\eta^{x!}(W \setminus T_x)\eta^{x!}(S_x)]\\ &\quad\leqslant \big|\mathbb{E}[\eta^{x!}(W \setminus T_x)] - \mathbb{E}[\eta(W \setminus T_x)]\big|\mathbb{E}[\eta^{x!}(S_x)] \qquad (20)\\ &\qquad+ \big|\mathbb{E}[\eta^{x!}(W \setminus T_x)\eta^{x!}(S_x)] - \mathbb{E}[\eta(W \setminus T_x)]\mathbb{E}[\eta^{x!}(S_x)]\big|. \qquad (21)\end{aligned}$$

Here, we use (18) and $\mathbb{E}[\eta^{x!}(S_x)] \leqslant \mathbb{E}[\eta(S_x)] = \rho|S|$ to obtain for (20)

$$\big|\mathbb{E}[\eta^{x!}(W \setminus T_x)] - \mathbb{E}[\eta(W \setminus T_x)]\big|\mathbb{E}[\eta^{x!}(S_x)] \leqslant \|K\||W||S| \sup_{y\in T^c} \phi(\|y\|).$$

Next we consider (21). We write $\rho_x^{(m)}$ for the m-th correlation function of $\eta^{x!}$ and find by [[27], Lemma 6.4], by the definition of $\eta^{x!}$ and by (15) that

$$\begin{aligned}&\rho\Big(\mathbb{E}[\eta^{x!}(W \setminus T_x)\eta^{x!}(S_x)] - \mathbb{E}[\eta(W \setminus T_x)]\mathbb{E}[\eta^{x!}(S_x)]\Big)\\ &\quad= \rho \int_{S_x}\int_{W\setminus T_x} (\rho_x^{(2)}(y, z) - \rho_x(y)\rho)\, dzdy\\ &\quad= \int_{S_x}\int_{W\setminus T_x} (\rho^{(3)}(x, y, z) - \rho^{(2)}(x, y)\rho)\, dzdy\\ &\quad\leqslant 3^{5/2}\|K\|^2 \int_{S_x}\int_{W\setminus T_x} \phi(d(\{x, y\}, \{z\}))\, dzdy\\ &\quad\leqslant 3^{5/2}\|K\|^2|W||S|\phi(d(S, T^c)).\end{aligned}$$

Thus, since $\mathbf{L}(W) \leqslant \rho|W|$, we can conclude that

$$\begin{aligned}R_3 &= \rho|W|\operatorname{ess\,sup}_{x\in W}\mathbb{E}[g(x, \eta^x)]\mathbb{E}[(\eta^x\Delta\eta^{x!,\Xi})(W \setminus T_x)]\\ &\leqslant (1 + \rho|S|)\|K\||W|^2 \sup_{y\in T^c} \phi(\|y\|) + 3^{5/2}\|K\|^2|W|^2|S|\phi(d(S, T^c))\\ &\leqslant \|K\|(1 + \rho + 3^{5/2}\|K\|) \max(|S|, 1)|W|^2\phi(d(S, T^c)). \qquad (22)\end{aligned}$$

(ii) *Decreasing scores.* If $g(x, \omega_1) \geqslant g(x, \omega_2)$ for $\omega_1 \subset \omega_2$, we have by (10) and (14) that $P^{x!} \leqslant P$ and $P^{x!}|_{S_x^c} \leqslant P^{x!,\Xi}|_{S_x^c}$ for λ-a.a. $x \in W$. Let $\eta^x \sim P^x$. By Strassen's theorem and [[13], Theorem 2.15], there exist point processes $\eta \sim P$ and $\eta^{x,\Xi} \sim P^{x,\Xi}|_{S_x^c}$ such that $\eta^{x!} \subset \eta$ and $\eta^{x!} \cap S_x^c \subset \eta^{x!,\Xi}$. This gives

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$$\begin{aligned}\mathbb{E}[(\eta\Delta\eta^{x,\Xi})(W\setminus T_x)] &\leqslant \mathbb{E}[(\eta\setminus\eta^{x!})(W\setminus T_x)]+\mathbb{E}[(\eta^{x!,\Xi}\setminus\eta^{x!})(W\setminus T_x)]\\ &=\big\{\mathbb{E}[\eta(W\setminus T_x)]-\mathbb{E}[\eta^{x!}(W\setminus T_x)]\big\}+\mathbb{E}[\eta^{x!,\Xi}(W\setminus T_x)]-\mathbb{E}[\eta^{x!}(W\setminus T_x)].\end{aligned} \tag{23}$$

Here we bound the term in $\{\cdots\}$ as in (18). Moreover, we obtain

$$\begin{aligned}&\mathbb{E}[g(x,\eta^x)]\big\{\mathbb{E}[\eta^{x!,\Xi}(W\setminus T_x)]-\mathbb{E}[\eta^{x!}(W\setminus T_x)]\big\}\\ &\quad=\mathbb{E}[\eta^x(W\setminus T_x)g(x,\eta^x)]-\mathbb{E}[\eta^x(W\setminus T_x)]\mathbb{E}[g(x,\eta^x)]\\ &\quad=\mathtt{Cov}(\eta^x(W\setminus T_x),g(x,\eta^x)).\end{aligned} \tag{24}$$

Now we use that the reduced Palm process $\eta^{x!}$ is a determinantal process itself and therefore has negative associations (see [12]). For $k\in\mathbb{N}$, we consider the auxiliary functions

$$f^{(k)}(\omega):=\min\{k,\omega(S_x)+g(x,\omega)\},\quad f(\omega):=\omega(S_x)+g(x,\omega),\quad \omega\in\mathbf{N}.$$

It is easy to see that $f^{(k)}$, $k\in\mathbb{N}$, and f are bounded and increasing. Since $\eta^{x!}$ has negative associations, we have that

$$\mathtt{Cov}(\min\{k,\eta^{x!}(W\setminus T_x)\},f^{(k)}(\eta^{x!}))\leqslant 0.$$

Hence, by monotone convergence,

$$\begin{aligned}&\mathtt{Cov}(\eta^{x!}(W\setminus T_x),\eta^{x!}(S_x))+\mathtt{Cov}(\eta^{x!}(W\setminus T_x),g(x,\eta^{x!}))\\ &\quad=\mathtt{Cov}(\eta^{x!}(W\setminus T_x),f(\eta^{x!}))\\ &\quad=\lim_{k\to\infty}\mathtt{Cov}(\min\{k,\eta^{x!}(W\setminus T_x)\},f^{(k)}(\eta^{x!}))\leqslant 0.\end{aligned}$$

This shows that (24) is bounded by $-\mathtt{Cov}(\eta^{x!}(W\setminus T_x),\eta^{x!}(S_x))$. Therefore, we can proceed as for increasing scores and obtain the same bound for R_3 as in (22).

Bounding R_4.

For each $x\in W$, we have

$$\begin{aligned}&\mathbb{E}\Big[\sum_{y\in\eta^{x,\Xi}\cap\eta\cap W\setminus T_x}|g(y,\eta^{x,\Xi})-g(y,\eta)|\Big]\\ &\quad=\mathbb{E}\Big[\sum_{y\in\eta^{x,\Xi}\cap\eta\cap W\setminus T_x}\mathbb{1}\{(\eta^{x,\Xi}\Delta\eta)\cap S_x\neq\emptyset\}|g(y,\eta^{x,\Xi})-g(y,\eta)|\Big]\\ &\quad\leqslant\mathbb{E}\Big[\sum_{z\in(\eta^{x,\Xi}\Delta\eta)\cap(W\oplus S_x)\setminus T_x}\ \sum_{y\in\eta^{x,\Xi}\cap\eta\cap S_z}|g(y,\eta^{x,\Xi})-g(y,\eta)|\Big]\\ &\quad\leqslant\mathbb{E}\Big[\sum_{z\in(\eta^{x,\Xi}\Delta\eta)\cap(W\oplus S_x)\setminus T_x}\ \max_{\omega\in\{\eta^{x,\Xi},\eta\}}\sum_{y\in\omega\cap S_z}g(y,\omega)\Big].\end{aligned}$$

Here we obtain from Condition (3) that the above is bounded by

$$\alpha|S|\mathbb{E}\Big[(\eta^{x,\Xi}\Delta\eta)((W\oplus S)\setminus T_x)\Big].$$

Hence, we obtain from the estimate in (22) (with W replaced by $W\oplus S$) that

$$\begin{aligned}R_4&=\rho|W|\text{ess sup}_{x\in W}\mathbb{E}[g(x,\eta^x)]\mathbb{E}\Big[\sum_{x\in\eta^{x,\Xi}\cap\eta\cap W\setminus T_x}|g(x,\eta^{x,\Xi})-g(x,\eta)|\Big]\\ &\leqslant\alpha|S|\|K\|(1+\rho+3^{5/2}\|K\|)\max(|S|,1)|W\oplus S|^2\phi(d(S,T^c)).\end{aligned} \tag{25}$$

$\square$

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5. Proof of Theorem 2.2

In the proof of Theorem 2.2, we repeatedly use that the set of absolute values of the points of the (infinite) Ginibre process ξ has the same distribution as a sequence $(X_i)_{i\in\mathbb{N}}$ of independent random variables with $X_i^2 \sim \mathrm{Gamma}(i, 1)$ (see [15] or [[1], Theorem 26]). This implies that the mapping $r \mapsto \mathbb{P}(\xi(B_r)=0)$ is continuous and that $\mathbb{P}(\xi(B_r)=0) \downarrow 0$ as $r \to \infty$. Hence, for all $\tau > 0$, there exists an unbounded increasing sequence $(v_n)_{n\in\mathbb{N}}$ such that

$$\mathbf{L}_n(A) := \mathbb{E}[\Xi_n(A)] = \frac{|A\cap B_n|}{\pi}\mathbb{P}(\xi^{o!}(B_{v_n})=0) = \frac{\tau|A\cap B_n|}{n^2}, \quad n\in\mathbb{N},\ A\in\mathcal{B}^2. \tag{26}$$

To determine the asymptotic behaviour of v_n as $n\to\infty$, we use that by [[1], Theorem 26],

$$\mathbb{P}(\xi^{o!}(B_{v_n})=0) = e^{v_n^2}\mathbb{P}(\xi(B_{v_n})=0). \tag{27}$$

Moreover, by [[2], Proposition 7.2.1],

$$\lim_{r\to\infty}\frac{1}{r^4}\log\mathbb{P}(\xi(B_r)=0) = -\frac{1}{4}.$$

Therefore, re-writing $\log\mathbb{P}(\xi(B_{v_n})=0)$ as

$$\log(n^2 e^{v_n^2}\mathbb{P}(\xi(B_{v_n})=0)) - 2\log n - v_n^2,$$

we find from (26) and (27) that $\frac{v_n^4}{\log n} \to 8$ as $n\to\infty$.

***Proof of Theorem* 2.2.** Given $n\in\mathbb{N}$, we choose ξ as the Ginibre process, let $g(x,\omega) := \mathbb{1}\{\omega(B_{v_n}(x)\backslash\{x\})=0\}$, $S := B_{v_n} := B_{v_n}(o)$ and $T := B_{\log n} := B_{\log n}(o)$. Note that g is stabilising with respect to S. The idea is to apply Theorem 2.1 to the process Ξ_n defined at (6), where we choose ζ as a stationary Poisson process with intensity $\frac{\tau}{n^2}$. Then, $\zeta\cap B_n$ has intensity measure $\mathbf{L}_n$ given at (26). First, we check the Conditions (3) and (4). Note that for all $\omega\in\mathbf{N}$ and $n\in\mathbb{N}$,

$$\bigcap_{x\in\Xi_n[\omega]} B_{v_n/2}(x) = \emptyset.$$

Therefore, (3) holds with $\alpha := \frac{1}{\pi(v_n/2)^2} = \frac{4}{\pi v_n^2} \leqslant \frac{4}{\pi v_1^2}$. Moreover, it clearly holds that $g(x,\omega_1) \geqslant g(x,\omega_2)$ for $\omega_1\subset\omega_2$, verifying (4).

Thus, by invariance of KR distance under scalings and Theorem 2.1, we have

$$\mathbf{d_{KR}}(\Psi_n, \nu\cap B_1) = \mathbf{d_{KR}}(\Xi_n, \zeta\cap B_n) \leqslant 2(E_{1,n}+E_{2,n}+F_n), \tag{28}$$

where the error terms $E_{1,n}$, $E_{2,n}$, and F_n depend on n.

Next we bound $E_{1,n}$, $E_{2,n}$, and F_n. Since $\|K\| = \pi^{-1}$ and $\phi(r) = \pi^{-1}\exp(-r^2/2)$ for $r>0$, we obtain

$$F_n = \frac{1}{\pi^2}\left(1+\frac{1+3^{5/2}}{\pi}\right)\left(\frac{4}{\pi v_1^2}|B_{v_n}|+1\right)\max(|B_{v_n}|,1)|B_{n+v_n}|^2\exp\left(-\frac{|\log n - v_n|^2}{2}\right).$$

Using here that $v_n^4 \sim 8\log n$, we obtain that $F_n \leqslant 1/n$ for n large enough. Next we bound $E_{1,n}$, where we recall that $\rho = \pi^{-1}$. From (26), we obtain

$$\begin{aligned} E_{1,n} &= \frac{1}{\pi^2}\int_{B_n}\int_{B_n\cap B_{\log n}(x)} \mathbb{P}(\xi^{x!}(B_{v_n}(x))=0)\mathbb{P}(\xi^{y!}(B_{v_n}(y))=0)dydx \\ &\leqslant \frac{\tau^2}{n^4}|B_n|\sup_{x\in B_n}|B_{\log n}(x)\cap B_n| \leqslant \frac{\tau^2\pi^2(\log n)^2}{n^2}. \end{aligned}$$

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Thus, it remains to bound

$$\begin{aligned} E_{2,n} &= \int_{B_n} \int_{B_n \cap B_{\log n}} \mathbb{E}[\mathbb{1}\{\xi^{x,y}(B_{v_n}(x) \setminus \{x\}) = \emptyset\}\mathbb{1}\{\xi^{x,y}(B_{v_n}(y) \setminus \{y\}) = \emptyset\}]\rho^{(2)}(x,y)dydx \\ &= \int_{B_n} \int_{B_n \cap B_{\log n}} \mathbb{E}[\mathbb{1}\{\xi^{x!,y!}(B_{v_n}(x) \cup B_{v_n}(y)) = \emptyset\}]\mathbb{1}\{\|x-y\| > v_n\}\rho^{(2)}(x,y)dydx, \end{aligned} \tag{29}$$

where $\xi^{x!,y!} := \xi^{x,y} \setminus \{x,y\}$. By [[27], Theorem 6.5], the reduced Palm process $\xi^{x!,y!}$ is a determinantal process itself. Hence, we can conclude from [[19], Theorem 3.7] that $\xi^{x!,y!}$ has negative associations, that is, that $\mathbb{E}[F(\xi^{x!,y!})G(\xi^{x!,y!})] \leqslant \mathbb{E}[F(\xi^{x!,y!})]\mathbb{E}[G(\xi^{x!,y!})]$ for every pair F, G of real bounded increasing (or decreasing) functions that are measurable with respect to complementary subsets of $\mathbb{R}^d$. We apply this with the decreasing functions $F(\mu) = \mathbb{1}\{\mu(B_{v_n}(x)) = 0\}$ and $G(\mu) = \mathbb{1}\{\mu(B_{v_n}(y) \setminus B_{v_n}(x)) = 0\}$. This gives

$$\mathbb{P}(\xi^{x!,y!}(B_{v_n}(x) \cup B_{v_n}(y)) = 0) \leqslant \mathbb{P}(\xi^{x!,y!}(B_{v_n}(x)) = 0)\, \mathbb{P}(\xi^{x!,y!}(B_{v_n}(y) \setminus B_{v_n}(x)) = 0). \tag{30}$$

To bound the first probability, we note that by [[12], Theorem 1], there is a reduced Palm process $\xi^{x!,y!}$ of $\xi^{x!}$ such that $\xi^{x!,y!} \subset \xi^{x!}$ and $|\xi^{x!} \setminus \xi^{x!,y!}| \leqslant 1$ a.s. This gives

$$\mathbb{P}(\xi^{x!,y!}(B_{v_n}(x)) = 0) \leqslant \mathbb{P}(\xi^{x!}(B_{v_n}(x)) \leqslant 1).$$

Now we apply the same argument to the determinantal process $\xi^{x!}$ and obtain the bound

$$\mathbb{P}(\xi^{x!}(B_{v_n}(x)) \leqslant 1) \leqslant \mathbb{P}(\xi(B_{v_n}(x)) \leqslant 2) = \mathbb{P}(\xi(B_{v_n}) \leqslant 2),$$

where the last equality holds due to the stationarity of ξ. As mentioned at the beginning of this section, the set of absolute values of the points of the Ginibre process ξ has the same distribution as a sequence $(X_i)_{i\in N}$ of independent random variables with $X_i^2 \sim \text{Gamma}(i, 1)$. Similarly to [[2], Section 7.2], this gives

$$\begin{aligned} \mathbb{P}(\xi(B_{v_n}) \leqslant 2) &= \mathbb{P}(\#\{j \in \mathbb{N} : X_j \leqslant v_n\} \leqslant 2) \\ &\leqslant \mathbb{P}(\#\{j \in \{1, \ldots, v_n^2\} : X_j \leqslant v_n\} \leqslant 2) \\ &= \mathbb{P}\Big(\bigcup_{i=1}^{v_n^2} \bigcup_{\substack{j=1 \\ j\neq i}}^{v_n^2} \{\forall k \in \{1, \ldots, v_n^2\} \setminus \{i,j\} : X_k > v_n\}\Big). \end{aligned}$$

In the above equation, with a slight abuse of notation, we have written v_n^2 instead of $\lfloor v_n^2 \rfloor$. The union bound yields that the above is bounded by

$$\sum_{i=1}^{v_n^2} \sum_{\substack{j=1 \\ j\neq i}}^{v_n^2} \mathbb{P}(\forall k \in \{1, \ldots, v_n^2\} \setminus \{i,j\} : X_k > v_n) = \sum_{i=1}^{v_n^2} \sum_{\substack{j=1 \\ j\neq i}}^{v_n^2} \prod_{\substack{k=1 \\ k\neq i,j}}^{v_n^2} \mathbb{P}(X_k^2 > v_n^2).$$

Let $t < 1$. The moment generating function $M_{X_k^2}(t) = \mathbb{E}[e^{tX_k^2}] = (1-t)^{-k}$ of X_k^2 exists, and we obtain from the Chernoff bound that

$$\mathbb{P}(X_k^2 > r^2) \leqslant e^{-tr^2}\mathbb{E}[e^{tX_k^2}] = e^{-tr^2}(1-t)^{-k}.$$

For $k < r^2$, this bound is maximised for $t = 1 - \frac{k}{r^2}$, which gives

$$\mathbb{P}(\xi(B_{v_n}) \leqslant 2) \leqslant \sum_{i=1}^{v_n^2} \sum_{\substack{j=1 \\ j\neq i}}^{v_n^2} \prod_{\substack{k=1 \\ k\neq i,j}}^{v_n^2} e^{-(1-\frac{k}{v_n^2})v_n^2 - k\log\left(\frac{k}{v_n^2}\right)} = \sum_{i=1}^{v_n^2} \sum_{\substack{j=1 \\ j\neq i}}^{v_n^2} \prod_{\substack{k=1 \\ k\neq i,j}}^{v_n^2} e^{-v_n^2 + k - k\log\left(\frac{k}{v_n^2}\right)}.$$

Using here that $u \mapsto u - u\log(u/r^2)$ is increasing for $u \leqslant r^2$, we find that

$$\begin{aligned}\mathbb{P}(\xi(B_{v_n}) \leqslant 2) &\leqslant \sum_{i=1}^{v_n^2} \sum_{\substack{j=1\\ j\neq i}}^{v_n^2} \prod_{k=3}^{v_n^2} e^{-v_n^2+k-k\log\left(\frac{k}{v_n^2}\right)} \\ &\leqslant v_n^4 \prod_{k=3}^{v_n^2} e^{-v_n^2+k-k\log\left(\frac{k}{v_n^2}\right)} \\ &= v_n^2 e^{-\frac{1}{2}(v_n^2-3)(v_n^2-2)-v_n^4\int_{3/v_n^2}^1 u\log(u)\mathrm{d}x + O(v_n^2\log v_n)} \\ &= e^{-\frac{1}{4}v_n^4(1+o(1))}\end{aligned}$$

as $n \to \infty$, where we have used that $\int_0^1 u\log(u)dx = -\frac{1}{4}$.

Next we bound the second probability in (30). By the same coupling argument as above, we find that

$$\mathbb{P}(\xi^{x!,y!}(B_{v_n}(y)\backslash B_{v_n}(x)) = 0) \leqslant \mathbb{P}(\xi(B_{v_n}(y)\backslash B_{v_n}(x)) \leqslant 2). \tag{31}$$

Next we note that $B_{v_n/2}\left(y + \frac{v_n(y-x)}{2|y-x|}\right) \subset B_{v_n}(y) \setminus B_{v_n}(x)$ if $\|x-y\| \geqslant v_n$. (This is sufficient, since the integrand of (29) vanishes if $\|x-y\| < v_n$.) Hence, (31) is for $\|x-y\| \geqslant v_n$ bounded by

$$\mathbb{P}\Big(\xi\big(B_{v_n/2}\big(y + \frac{v_n(y-x)}{2|y-x|}\big)\big) \leqslant 2\Big) = \mathbb{P}(\xi(B_{v_n/2}) \leqslant 2) \leqslant e^{-\frac{1}{4}(v_n/2)^4(1+o(1))}$$

by the same estimates as above (with $v_n/2$ instead of v_n). Since $\rho^{(2)}(x,y) \leqslant 1/\pi^2$ for all $x, y \in \mathbb{R}^2$, we arrive for all $\varepsilon > 0$ at the bound

$$\begin{aligned}E_2 &= \int_{B_n}\int_{B_n\cap B_{\log n}} \mathbb{E}[\mathbb{1}\{\xi^{x!,y!}(B_{v_n}(x)\cup B_{v_n}(y)) = \emptyset\}]\mathbb{1}\{\|x-y\| > v_n\}\rho^{(2)}(x,y)dydx \\ &\leqslant n^2(\log n)^2 e^{-\frac{1}{4}v_n^4(1+o(1))} e^{-\frac{1}{64}v_n^4(1+o(1))} \leqslant n^{\varepsilon-1/16},\end{aligned}$$

where we have used (26) and that $\frac{v_n^4}{\log n} \to 8$ as $n \to \infty$. Hence, the assertion follows from (28). $\square$

References

[1] Hough, J. Ben, Krishnapur, M., Peres, Y. and Virág, B. (2006) Determinantal Processes and Independence. *Probab. Surverys* **3** 206–229.

[2] Hough, J. B., Krishnapur, M., Peres, Y. and Virág, B. (2009) *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. Vol. **51** American Mathematical Soc.

[3] Blaszczyszyn, B., Yogeshwaran, D. and Yukich, J. (2019) Supplement to “Limit theory for geometric statistics of point processes having fast decay of correlations. *Ann. Probab.* **47** 835–895.

[4] Bobrowski, O., Schulte, M. and Yogeshwaran, D. (2022) Poisson process approximation under stabilization and Palm coupling. *Ann. Henri Lebesgue* **5** 1489–1534.

[5] Burton, R. and Pemantle, R. (1993) Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer impedances. *Ann. Probab.* **21** 1329–1371.

[6] Chenavier, N., Henze, N. and Otto, M. (2022) Limit laws for kth-nearest neighbor balls. *J. Appl. Probab.* **13** 880–894.

[7] Chenavier, N. and Hirsch, C. (2022) Extremal lifetimes of persistent cycles. *Extremes* **25** 299–330.

[8] Chenavier, N. and Otto, M. (2023) Compound Poisson process approximation under β-mixing and stabilization, Preprint, arXiv: 2310.15009.

[9] Decreusefond, L., Flint, I., Privault, N. and Torrisi, G. L. (2016) Determinantal point processes. In *Stochastic analysis for Poisson point processes*, (G. Peccati and M. Reitzner, eds), Springer Cham, pp. 311–341.

[10] Decreusefond, L., Schulte, M. and Thäle, C. (2016) Functional Poisson approximation in Kantorovich-Rubinstein distance with applications to U-statistics and stochastic geometry. *Ann. Probab.* **44** 2147–2197.

[11] Diaconis, P. (2003) Patterns in eigenvalues: the 70th Josiah Willard Gibbs lecture. *Bull. Amer. Math. Soc.* **40** 155–178.

https://doi.org/10.1017/S0963548325000082 Published online by Cambridge University Press

[12] Goldman, A. (2010) The Palm measure and the Voronoi tessellation for the Ginibre process. *Ann. Appl. Probab.* **20** 90–128.
[13] Kallenberg, O. (2002) *Foundations of Modern Probability*. Springer, 2nd edition.
[14] Kallenberg, O. (2017) *Random Measures, Theory and Applications*. Springer.
[15] Kostlan, E. (1992) On the spectra of Gaussian matrices. *Linear Algebra Appl.* **164** 385–388.
[16] Last, G. and Otto, M. (2023) Disagreement coupling of Gibbs processes with an application to Poisson approximation. *Ann. Appl. Probab.* **33** 4091–4126.
[17] Last, G. and Szekli, R. (2019) On negative association of some finite point processes on general state spaces. *J. Appl. Probab.* **56** 139–152.
[18] Lindvall, T. (1999) On Strassen's theorem on stochastic domination. *Electron. Comm. Probab.* **4** 51–59.
[19] Lyons, R. (2014) Determinantal probability: basic properties and conjectures, Preprint, arXiv: 1406.2707.
[20] Macchi, O. (1975) The coincidence approach to stochastic point processes. *Adv. Appl. Probab.* **7** 83–122.
[21] Miyoshi, N. and Shirai, T. (2014) Spatial stochastic models for analysis of heterogeneous cellular networks with repulsively deployed base stations. *Perform. Eval.* **77** 7–17.
[22] Møller, J. and O'Reilly, E. (2021) Couplings for determinantal point processes and their reduced Palm distributions with a view to quantifying repulsiveness. *J. Appl. Probab.* **58** 469–483.
[23] Otto, M. (2023) Extremal behavior of large cells in the Poisson hyperplane mosaic. *Electron. J. Probab.* **162** 1–29. paper no.
[24] Otto, M. (2025) Poisson approximation of Poisson-driven point processes and extreme values in stochastic geometry. *Bernoulli* **31** 30–54.
[25] Otto, M. and Thäle, C. (2023) Large nearest neighbour balls in hyperbolic stochastic geometry. *Extremes* **26**(3) 13–31.
[26] Poinas, A. (2019) A bound of the β-mixing coefficient for point processes in terms of their intensity functions. *Statist. Probab. Lett.* **148** 88–93.
[27] Shirai, T. and Takahashi, Y. (2003) Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes. *J. Funct. Anal.* **205**(2) 414–463.
[28] Soshnikov, A. (2000) Determinantal random point fields. *Russian Math. Surveys* **55**(5) 923–975.

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