TAUBERIAN THEOREMS FOR INTEGRALS

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When, for the generalized summation of series, we use A and B methods, giving A and B *sums*, respectively, we say that *the A method is included in the B method*, $A \subset B$, if the B sum exists and is equal to the A sum whenever the latter exists. A theorem proving such a result is called an *Abelian theorem*. For example, there is an Abelian theorem stating that if the A and B sums are the first Cesàro mean and the Abel mean, respectively, then $A \subset B$. If $A \subset B$ and $B \subset A$, we say that *A and B are equivalent*, $A \equiv B$. For example, the nth Hölder and nth Cesàro means are equivalent. When $A \subset B$, $B \not\subset A$, then a *Tauberian theorem* is one in which we infer the existence of the A sum from the existence of the B sum if a specified restriction is put on the series. For example, if the A and B methods give ordinary convergence and the Abel sum, respectively, if the B sum exists, and if the series consists of non-negative terms, then we have the Tauberian theorem that the A sum exists.

Let us now suppose that the A and B methods apply to integrals. Then many Abelian theorems occur in the literature. For example,

$$(\text{Riemann}) \subset (\text{Lebesgue}) \subset (\text{Perron}),\ (\text{Perron}) \not\subset (\text{Lebesgue}) \not\subset (\text{Riemann}),$$

the names denoting integrals in an obvious notation. Further, I conjectured in **(4**, pp. 110, 132**)** that there is a close connection between the general Denjoy integral and Burkill's approximate Perron integral. But G. Tolstoff **(8)** had already proved the conjecture false, and gave the generalized Perron integral that is equivalent to the general Denjoy integral. It is likely that these will also be equivalent to the corresponding variational integral.

Tauberian theorems also occur in the literature.

THEOREM 1. *Let $f \geqslant 0$, where f is Perron-integrable in the finite interval $[a, b]$. Then f is Lebesgue-integrable in $[a, b]$.*

See, for example, Saks **(7**, p. 203, Theorem (6.5)**)**. Further, S. Foglio **(1)** gives the following result.

THEOREM 2. *If $h \geqslant 0$ is a finitely additive interval function, if $f \geqslant 0$ is a finite Baire function, and if the N-integral of* **(2)**,

$$(N) \int_a^b f dh,$$

exists, then the corresponding Ward integral exists.

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As (Ward) ≡ (Perron) when $h(x, y) = y - x$, we can combine Theorems 1 and 2 to arrive at the Lebesgue integrability of f in this case. More generally, we suppose that there exists an A-integral using a continuous monotonically increasing integrator $m(x)$, and obeying the following properties:

(1) *It is distributive in the integrand.*

(2) *When the A-integral exists, the integrand is m-measurable.*

(3) (Radon) $\subset$ (A), *using* $m(x)$.

(4) *If the integrand* (f) *is non-negative, so is the A-integral.*

Most integrals stronger than Lebesgue's and Radon's satisfy (1) to (4).

THEOREM 3. *Let $f \geqslant g$, where f is A-integrable and g is Radon-integrable, both with respect to $m(x)$ in a finite interval $[a, b]$. If the A-integral satisfies* (1) *to* (4), *then f is Radon-integrable with respect to $m(x)$ in $[a, b]$.*

By (1), (3), $f - g \geqslant 0$ is A-integrable with respect to $m(x)$ in $[a, b]$. From (2), $f - g$ is m-measurable, so that if E_s is the set of x where

$$s \leqslant f(x) - g(x) < s + 1,$$

with characteristic function $\mathrm{ch}(E_s; x)$, and m-measure $m(E_s)$, then

$$f - g \geqslant \sum_{s=0}^{n} s \cdot \mathrm{ch}(E_s; \cdot) = f_n, \tag{5}$$

and f_n is Radon-integrable with respect to $m(x)$. Using (1), (3), (4), and (5),

$$(A)\int_a^b (f - g)dm \geqslant (A)\int_a^b f_n dm = \sum_{s=0}^{n} s \cdot m(E_s).$$

Letting $n \to \infty$, we prove that

$$\sum_{s=0}^{\infty} s \cdot m(E_s) < \infty,$$

which is sufficient to show that $f - g$ is Radon-integrable with respect to $m(x)$ in the finite interval $[a, b]$.

We now suppose that the A-integral also satisfies:

(6) *If f is A-integrable with respect to $m(x)$ on a bounded perfect set P, there is a portion $(a', b') \cap P$ on which f is Radon-integrable with respect to $m(x)$.*

Then we can prove a further Tauberian theorem. Note that properties (1), (2), (3), (4,) and (6) are true for the Perron, special and general Denjoy, and variational integrals, and the special Denjoy N-integral of (**3**, p. 285), which is equivalent to the N-variational integral if the convergence factors N satisfy (27), (28), and (29) of (**3**, p. 289). The proof of the latter is given in (**3**, Theorem 4, p. 290).

THEOREM 4. *Let the A-integral satisfy* (1), (2), (3), (4), *and* (6). *Let f be A-integrable with respect to $m(x)$ in the bounded perfect set P. Let g be Radon-integrable with respect to $m(x)$ in P. Further, let there be an everywhere dense set of the ends of intervals I in P for which*

$$\liminf_{s\to\infty} m(H_s \cap P \cap I)/m(H_s \cap P) > 0, \tag{7}$$

where H_s is the set where $g - f \geqslant s$. (In particular, then, P and $P \cap I$ have positive m-measure.) Then f is Radon-integrable with respect to $m(x)$ in P.

For proof we take in the (a', b') of (6) for $f - g$ an interval I satisfying (7). From the Radon-integrability with respect to $m(x)$ of $f - g$ in $P \cap I$ we infer that

$$\sum_{s=1}^{\infty} m(H_s \cap P \cap I) = \sum_{s=1}^{\infty} s.m\{(H_s - H_{s+1}) \cap P \cap I\} < \infty.$$

Hence from (7) we obtain

$$\sum_{s=1}^{\infty} s.m\{(H_s - H_{s+1}) \cap P\} = \sum_{s=1}^{\infty} m(H_s \cap P) < \infty,$$

so that there is a Radon-integrable function $k \geqslant 0$ with respect to $m(x)$ in the bounded P, such that $f \geqslant g - k$. The result now follows from Theorem 3, using $(f - g + k)\text{ch}(P; .) \geqslant 0$, in place of $f - g$.

Theorem 4 shows that if a function is A-integrable but not Radon-integrable in $[a, b]$, with respect to $m(x)$, where the A-integral satisfies (1), (2), (3), (4), and (6), then the function tends to infinity in an unsymmetrical way in $[a, b]$. More particularly, taking $g = 0$, and given a perfect set $P \subset [a, b]$, then either f is not A-integrable on P with respect to $m(x)$, or else in each portion of P there is a portion $P \cap (a', b')$ such that for each interval $J \subset (a', b')$,

$$\liminf_{s\to\infty} m(H_s \cap P \cap J)/m(H_s \cap P) = 0. \tag{8}$$

But as the union of two abutting intervals J satisfying (8) need not be another interval J satisfying (8), there seems no point in considering the perfect component of $[a, b] - G$, where G is the union of the interiors of intervals J for which (8) is true with $P = [a, b]$. Note that if in Theorem 4 we replace $f - g$ by $(f - g)\text{ch}(P; .)$, and P in (7) by a containing interval $[a, b]$, then (7) could not be true for I lying in the complement of P. Thus the use of P is an extension of the corresponding theorem with $[a, b]$ for P. Further, it is not possible to replace lim inf by lim sup in (7), as the conclusion would only be that a partial sum of

$$\sum_{s=1}^{\infty} m(H_s \cap P)$$

is convergent. It might be conjectured that Theorem 2 can be extended to a form using interval functions, as follows.

Let the interval function $j \geqslant 0$ be N-variationally integrable in $[a, b]$. Then j is variationally integrable in $[a, b]$.

To show that this is false we take $j(I) = 0$ except for $I = [a, h)$, where h lies in a set of measure zero in $h > a$ with a as one of its limit-points, and then $j(I) = 1$. Then for every absolutely continuous N, the N-variational integral is 0, while the variational integral cannot exist.

Using Tauberian theorems we can extend theorems on the interchange of limits and integrals, taking note of Pratt **(6)** on Lebesgue integration. Theorem 1 of **(5**, Chapter 5, Section 37**)** is as follows.

THEOREM 5. *Let $h_s \geqslant 0$ $(s = l, r)$ be a pair of interval functions, and let the point functions $f_1, f_2, f(., y)$ be variationally integrable with respect to $\mathbf{h} = \{h_l, h_r\}$ in an elementary set E, with*

(9) $$f(x, y) \geqslant f_1(x)$$

for each fixed y, and, in the case when y takes all values in $y \geqslant 0$, for each $0 \leqslant Y < Z$ let

(10) $$\inf_{Y \leqslant y \leqslant Z} f(x, y)$$

be variationally integrable with respect to $\mathbf{h}$ in E. Then

(11) $$(V) \int_E \liminf_{y \to \infty} f(., y) d\mathbf{h} \leqslant \liminf_{y \to \infty} (V) \int_E f(., y) d\mathbf{h}.$$

If instead of (9) *and* (10), *we have, for each fixed y,*

(12) $$f(x, y) \leqslant f_2(x),$$

and, in the case when y takes all values in $y \geqslant 0$, for each $0 \leqslant Y < Z$, if

(13) $$\sup_{Y \leqslant y \leqslant Z} f(x, y)$$

is variationally integrable with respect to $\mathbf{h}$ in E, then

(14) $$(V) \int_E \limsup_{y \to \infty} f(., y) d\mathbf{h} \geqslant \limsup_{y \to \infty} (V) \int_E f(., y) d\mathbf{h}.$$

(15) *If in* (11) *or* (14),

$$\lim_{y \to \infty} f(x, y)$$

exists, then we do not need the variational integrability of **(10)** *or* **(13)** *with respect to $\mathbf{h}$ in E.*

(16) $$\textit{If } f_1(x) \leqslant f(x, y) \leqslant f_2(x),\ f(x) = \lim_{y \to \infty} f(x, y),$$

except possibly in a set X of x with

(17) $$V(\mathbf{h}; E; X) = 0,$$

then if we put $f(x) = 0$ for x in X,

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(18) $$(V)\int_E f d\mathbf{h} = \lim_{y\to\infty} (V)\int_E f(.,y)d\mathbf{h}.$$

(19) *In particular, if* **h** *is variationally integrable in* E, *the results are true when* f_1, f_2 *are constants, so that* $f(x, y)$ *is bounded above, or below, or both, in* x *and* y.

In this theorem an elementary set is a union of a finite number of closed intervals, while a set of zero variation, (17), corresponds to a set of measure zero in Lebesgue theory. Further, we have supposed that the range of y is either all points in $y \geqslant 0$, or else all integers $1, 2, \ldots$.

Keeping these two ranges of y, we can consider the proofs of special cases of a general theorem of this nature, using an A-integration with **h** as integrator.

THEOREM 6. *Let* $h_s \geqslant 0$ $(s = l, r)$ *be a pair of interval functions, and let the point functions* $f(.,y), p(.,y), q(.,y)$ *be* A-*integrable with respect to* **h**, *with*

(20) $$\lim_{y\to\infty} p(x, y) = p(x), \qquad \lim_{y\to\infty} q(x, y) = q(x),$$

(21) $$\lim_{y\to\infty} (A)\int_E p(.,y)d\mathbf{h} = (A)\int_E p(x)d\mathbf{h},$$

$$\lim_{y\to\infty} (A)\int_E q(x,y)d\mathbf{h} = (A)\int_E q(x)d\mathbf{h}.$$

(22) *If* $f(x, y) \geqslant p(x, y)$ *for each fixed* y, *and, in the case when* y *takes all values in* $y \geqslant 0$, *if for each fixed* $0 \leqslant Y < Z$,

(23) $$\inf_{Y\leqslant y\leqslant Z} \{f(x, y) - p(x, y)\}$$

is A-*integrable with respect to* **h** *in* E, *then*

(24) $$(A)\int_E \liminf_{y\to\infty} f(.,y)d\mathbf{h} \leqslant \liminf_{y\to\infty} (A)\int_E f(.,y)d\mathbf{h}.$$

If instead of (22) *and* (23) *we have, for each fixed* y,

(25) $$f(x, y) \leqslant q(x, y),$$

and, in the case when y *takes all values in* $y \geqslant 0$, *for each* $0 \leqslant Y < Z$, *if*

(26) $$\sup_{Y\leqslant y\leqslant Z} \{f(x, y) - q(x, y)\}$$

is A-*integrable with respect to* **h** *in* E, *then*

(27) $$(A)\int_E \limsup_{y\to\infty} f(.,y)d\mathbf{h} \geqslant \limsup_{y\to\infty} (A)\int_E f(.,y)d\mathbf{h}.$$

(28) *If in* (24) *or* (27),

$$\lim_{y\to\infty} f(x, y)$$

exists except possibly in a set X of A-variation zero, relative to E, then we do not need the A-integrability of (23) *or* (26) *with respect to* $\mathbf{h}$ *in* E.

(29) *If* $p(x, y) \leqslant f(x, y) \leqslant q(x, y), \qquad f(x) = \lim_{y\to\infty} f(x, y),$

except possibly in a set X of x of A-variation zero, relative to E, then if we put $f(x) = 0$ for x in X,

$$(30) \qquad (A)\int_E f d\mathbf{h} = \lim_{y\to\infty} (A)\int_E f(.\,, y) d\mathbf{h}.$$

In particular, if $\mathbf{h}$ *is variationally A-integrable in E, the results are true when p, q are constants, so that $f(x, y)$ is bounded above, or below, or both, in x and y.*

Here, a set X *of A-variation zero relative to E* is such that if r is a point function equal to zero except possibly in X, then r is A-integrable relative to $\mathbf{h}$ in E, with A-integral zero.

If (A) is the N-variational integral, then a set X of A-variation zero is such that $NV(\mathbf{h}; E; X) = 0$. Then we can prove Theorem 6 when $\mathbf{h}$ is N-variationally integrable in E, with indefinite integral H. For by (**4**, Theorem 6(41), Theorem 13(59)), we can replace $\mathbf{h}$ by $H \geqslant 0$ in all the integrals. When (22) is true, with $f - p$ a finite Baire function, then by Theorem 2, and the equivalence of the N-integral and the N-variational integral, and the equivalence of the Ward and variational integrals, we prove (24) from (11). The rest of Theorem 6 follows by elementary arguments in this case.

If (A) is the Lebesgue-Stieltjes (Radon) integral, we take

$$h_l(a, b) = h_r(a, b) = m(b) - m(a),$$

where m is some continuous monotonically increasing point function. We assume continuity to avoid trouble at discontinuities of the integrand. Then Theorem 6 is an easy extension of (24), in this case, which is a slight extension of Fatou's lemma. If, now, (A) satisfies (1), (2), (3), and (4), we use Theorem 3 to reduce the A-integral to the Radon integral, proving the result for $f - p$ or $q - f$, as the case may be, with Radon integrals. The last step is to use (3), followed by (1), and we prove Theorem 6 with m for $\mathbf{h}$, and A-integrals. More generally, instead of (22), we can assume that for each fixed $y \geqslant 0$, $f(., y)$ and $p(., y)$ are connected in the same way as f and g in Theorem 4, with $P = E$, and then we can again go from (A) to (Radon), by using Theorem 4 if (A) satisfies (6) as well.

The above results show how to reduce the proof of Theorem 6 in special cases to the consideration of special integrals, by using Tauberian theorems. It is an unsolved problem to prove Theorem 6 in its full generality when, say, $(A) = (NV)$.

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