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POLYNOMIAL APPROXIMATION AND GROWTH OF GENERALIZED AXISYMMETRIC POTENTIALS

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1. Introduction. *Generalized axisymmetric potentials* F^α (GASP) are regular solutions to the generalized axisymmetric potential equation

$$(1.1)\qquad \frac{\partial^2 F^\alpha}{\partial x^2} + \frac{\partial^2 F^\alpha}{\partial y^2} + \frac{2\alpha+1}{y}\frac{\partial F^\alpha}{\partial y} = 0, \quad \alpha > -1/2$$

in some neighborhood Ω of the origin where they are subject to the initial data

$$(1.2)\qquad F^\alpha(x, 0) = f(x), \quad \frac{\partial F^\alpha}{\partial y}(x, 0) = 0$$

along the singular line $y = 0$. In Ω, these potentials may be uniquely expanded in terms of the complete set of normalized ultraspherical polynomials

$$(1.3)\qquad F_n^\alpha(x, y) = r^n P_n^{(\alpha,\alpha)}(xr^{-1})/P_n^{(\alpha,\alpha)}(1), \quad n = 0, 1, 2, \ldots$$
$$r = (x^2 + y^2)^{1/2}$$

defined from the symmetric Jacobi polynomials $P_n^{(\alpha,\alpha)}(\xi)$ of degree n with parameter α as Fourier series

$$(1.4)\qquad F^\alpha(x, y) = \sum_{n=0}^{\infty} a_n F_n^\alpha(x, y).$$

Series terminating with zero coefficients for index $n \geqq m + 1$ are referred to here as *harmonic polynomials of degree m*.

For integers $2\alpha = k - 3$, $k = 3, 4, \ldots$ equation (1.1) reduces to the axisymmetric LaPlace equation in R^k whose solutions, axisymmetric potentials, serve as fundamental models for higher dimensional generalizations of the theory of harmonic or analytic functions in the complex **C**-plane and also provide models for the study of solutions of more general elliptic equations by the Method of Ascent (see [**8**]). The common idea unifying many techniques used to analyse GASP stems from the Bergman [**2**; **23**] and Gilbert [**7**] Integral Operator Methods whereby classes of *associated* analytic functions are mapped onto GASP by LaPlace type integral representations. Conversely, generating function expansions are used to define operators which map GASP onto their associates. Properties of the operators determine those of the associate which are "transplanted" to the GASP.

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R. P. Gilbert [7, p. 167 ff.] developed the A_μ operator (and A_μ^{-1}) mapping associated analytic functions of one complex variable onto Fourier series expansions of GASP in terms of the complete set of harmonics $\{r^n C_n^{(\alpha-1/2)}(\xi)\}_{n=0}^{\infty}$ defined from the Gegenbauer polynomials $C_n^{(\alpha-1/2)}(\xi)$ of degree n with parameter $\alpha - 1/2$. The coefficients were shown to identify properties of the GASP such as the singularities (Gilbert [7, p. 182]), zeros (Marden [13], McCoy [16]) and extrema (McCoy [7]) by transformation of the corresponding known properties of the associates due to Hadamard-Mandelbrojt [4, p. 335], Carathéodory-Toeplitz [21, p. 157] and Carathéodory-Fejer [10, p. 147] respectively. These are essentially local characteristics.

The singularities theorem identified GASP which are entire functions. This led Gilbert [7; 9] to consider global properties. Defining order and type of an entire function GASP as in complex function theory [11, p. 182] he developed coefficient theorems computing their values (when finite and positive) in the classical sense of Dienes [4, p. 293]. Subsequent application of these [5; 6] produced complete sequences of axisymmetric harmonic polynomials to approximate GASP uniformly on simply connected compact sets about the origin in the sense of Gel′fond-Markuševič [11, p. 217]. Thus global properties of GASP are deduced from local behavior, viz. the Fourier coefficients may be computed from derivatives of GASP regular about the origin.

Recent application of Integral Operator Methods by Marden [14] produced simultaneous interpolation and local uniform approximation of axisymmetric potentials by harmonic polynomials. A principal goal then, in analogy with the coefficient problem, is to establish global existence and characterize the growth of GASP asymptotically as a function of uniform local approximation by harmonic polynomials.

In this paper, the error in the local uniform harmonic polynomial approximation is determined as a function of the degree of the extremal approximating polynomial in the sense of Chebyshev. Suitable limits defined from the error identify real GASP without finite singularities and measure their growth, viz. order and type. We accomplish this by developing integral operators which "transplant" the classical theorems of S. N. Bernstein [3; 22, p. 176] and R. S. Varga [22] describing related properties for polynomial approximation of associated real entire analytic functions of one complex variable.

2. Definitions and preliminary results. The vehicle for this investigation is an invertible integral operator W_α, an alternate to A_μ, mapping *W_α-associated* analytic functions

$$(2.1)\qquad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z = x + iy \in \mathbf{C}$$

onto GASP (1.4). We now construct the operator, the inverse operator and establish basic properties as summarized in Theorem 1.

To define the W_α operator, the ultraspherical harmonics are written by the LaPlace type integral for the symmetric Jacobi polynomials [**1**, p. 32] as the transform

$$(2.2)\qquad F_n^\alpha(x, y) = \int_0^\pi \zeta^n d\mu_\alpha(t), \quad n = 0, 1, 2, \ldots$$

of the Bergman-Whittaker variable

$$\zeta = x + iy \cos t$$

relative to the non-negative measure

$$d\mu_\alpha(t) = [\sqrt{\pi}\Gamma(\alpha + 1/2)/\Gamma(\alpha + 1)] (\sin t)^{2\alpha} dt.$$

Defining $F^\alpha \equiv W_\alpha[f]$ from the associated function f analytic in the disk $D_R \equiv \{z \in \mathbf{C} : |z| < R\}$ by the operator

$$(2.3)\qquad F^\alpha(x, y) = \int_0^\pi f(\zeta) d\mu_\alpha(t)$$

generates GASP regular on compacta in the open set $\Sigma_R^\alpha = \{(x, y): (x^2 + y^2)^{1/2} < R\}$ which is referred to as a *hypersphere* in view of the geometric interpretation of the GASP equation for integers 2α. In the **C**-plane $(\alpha\downarrow -1/2)$, the hypersphere $\Sigma_R^{-1/2}$ corresponds to the disk D_R. The subscript is dropped for hyperspheres of unit radius.

The transform inverse to (2.3) employs orthogonality of the symmetric Jacobi polynomials [**1**, p. 8] to write

$$(2.4)\qquad z^n = \int_{\xi=-1}^{\xi=+1} K_\alpha(zr^{-1}, \xi) F_n^\alpha(r\xi, r\sqrt{1 - \xi^2}) d\nu_\alpha(\xi), \quad n = 0, 1, 2, \ldots$$

for the kernel

$$K_\alpha(zr^{-1}, \xi) = \sum_{n=0}^\infty z^n P_n^{(\alpha,\alpha)}(\xi) P_n^{(\alpha,\alpha)}(1)/r^n h_n^{(\alpha,\alpha)},$$

$$h_n^{(\alpha,\alpha)} = \frac{2^{2\alpha+1}[\Gamma(n + \alpha + 1)]^2}{(2n + 2\alpha + 1)\Gamma(n + 1)\Gamma(n + 2\alpha + 1)}, \quad n = 0, 1, 2, \ldots$$

relative to the non-negative measure

$$d\nu_\alpha(\xi) = (1 - \xi^2)^\alpha d\xi.$$

Evidently for each $R > 0$ and all $-1 \leqq \xi \leqq +1$, the kernel defines an analytic function of z on compacta in the disk D_R upon which the kernel is summed in closed form by application of the binomial theorem to the Poisson formula [**1**, p. 12] as

$$K_\alpha(\eta, \xi) = \frac{(1 - \eta)}{(1 + \eta)^{2(\alpha+1)}} [1 - 2\eta(1 + \xi)/(1 + \eta)]^{-\alpha-1/2}$$

$$\gamma_\alpha = (\alpha + 1)\Gamma(\alpha + 1/2)^2/\Gamma(\alpha + 1).$$

The complex η-plane contains no branch points of K_α for $|\eta| < 1$, e.g. $|z| < R$. This justifies definition of the radicals by their principle values on the segment $0 < \eta < 1$ (see [**1**, p. 128]). Consequently, the operator inverse to (2.3) represents the function f analytic on compacta in the disk D_R as the transform $f = W_\alpha^{-1}[F^\alpha]$,

$$(2.5)\qquad f(z) = \int_{\xi=-1}^{\xi=+1} K_\alpha(zr^{-1}, \xi) F^\alpha(r\xi, r\sqrt{1-\xi^2}) d\nu_\alpha(\xi),$$

of the unique GASP F^α regular in the hypersphere Σ_R^α. We summarize the above properties of the operators in the following theorem.

THEOREM 1. *For each* GASP F^α *regular in the hypersphere* Σ_R^α, *there is a unique* W_α *associated function* f *analytic in disk* D_R *and conversely.*

Moving from the useful information concerning the operators, we begin a preliminary study of growth with the definition of the maximum moduli for the entire function GASP F^α and the W_α associate f as

$$M(r, F^\alpha) \equiv \sup \{|F^\alpha(x, y)| : (x^2 + y^2)^{1/2} \leqq r\} \quad \text{and}$$
$$m(r, f) \equiv \sup \{|f(z)|: |z| \leqq r\}.$$

The *order* $P = P(F^\alpha)$ and *type* $T = T(F^\alpha)$ of GASP are defined as in [**6**; **9**] by

$$(2.6)\qquad P(F^\alpha) \equiv \limsup_{r\to\infty} \{\log\log M(r, F^\alpha)/\log r\} \quad \text{and}$$

$$(2.7)\qquad T(F^\alpha) \equiv \limsup_{r\to\infty} \{\log M(r, F^\alpha)/r^P\}$$

following the function theory definitions [**11**, p. 182] of order and type of the associate which are respectively

$$(2.8)\qquad \rho(f) \equiv \limsup_{r\to\infty} \{\log\log m(r, f)/\log r\} \quad \text{and}$$

$$(2.9)\qquad \tau(f) \equiv \limsup_{r\to\infty} \{\log m(r, f)/r^\rho\}.$$

We now assert the respective growth measures are identical through the following.

THEOREM 2. *The real entire function* GASP F^α *has finite positive order and type if, and only if, the entire function* W_α *associate* f *has positive order and type. Then the orders and types are respectively equal.*

Proof. We shall prove the statement for orders. Let f, an entire analytic function of finite positive order $\rho(f)$, generate the GASP $F^\alpha \equiv W_\alpha[f]$. From the normalization $W_\alpha[1] = 1$ and non-negativity of the measure of W_α, we see that

$$M(r, F^\alpha) \leqq m(r, f), \quad r > 0.$$

Monotonicity of the logarithm with definitions (2.8) and (2.6) gives

$$P(F^\alpha) \leqq \rho(f).$$

Then the order of F^α is finite. For the reverse inequality we set $zr^{-1} = \lambda e^{i\theta}$, $0 < \lambda < 1$, and reason with $f = W_\alpha^{-1}[F^\alpha]$ as in [**6**] to obtain the appraisal

$$m(r, f) \leqq K_\alpha(\lambda) M(r\lambda^{-1}, F^\alpha),$$
$$K_\alpha(\lambda) \equiv \sup \{|K_\alpha(\lambda e^{i\theta}, \xi)| : 0 \leqq \theta < 2\pi, -1 \leqq \xi \leqq +1\}$$

leading to

$$\rho(f) \leqq \limsup_{r\to\infty} \{\log \log M(r\lambda^{-1}, F^\alpha)/\log r\lambda^{-1}\} = P(F^\alpha).$$

It follows that the order of F^α is positive. Combination of the inequalities above verifies that

$$\rho(f) = P(F^\alpha).$$

The converse statement for orders is obtained by reversing the argument. Similarly, verification of equality of types is accomplished.

Henceforth, we adopt the notations $\rho \equiv \rho(f) = P(F^\alpha)$ and $\tau \equiv \tau(f) = T(F^\alpha)$. The order and type are computed not from the Fourier coefficients as in the Gilbert formula [**7**, p. 188] but rather from approximations in the uniform norm

$$\text{(2.10)} \quad |||F^\alpha - P^\alpha||| \equiv \sup \{|F^\alpha(x, y) - P^\alpha(x, y)| : x^2 + y^2 = 1\}$$

of the GASP F^α regular in the hypersphere Σ^α and continuous on $\bar{\Sigma}^\alpha$ with P^α in the set $\tilde{H}_n^\alpha$ of all real harmonic polynomials of degree at most n. Specifically, the essential measure is the error in the approximation defined by the Chebyshev norms

$$\text{(2.11)} \quad E_n(F^\alpha) \equiv \inf \{|||F^\alpha - P^\alpha||| : P^\alpha \in \tilde{H}_n^\alpha\}, \quad n = 0, 1, 2, \ldots .$$

This error is analyzed by "transplanting" characterizations of the error in the Chebyshev norms

$$\text{(2.12)} \quad e_n(f) \equiv \inf \{||f - p|| : p \in \tilde{h}_n\}, n = 0, 1, 2, \ldots$$

over the sets $\tilde{h}_n \equiv \{W_\alpha^{-1}[P^\alpha] : P^\alpha \in \tilde{H}_n^\alpha\}$, $n = 0, 1, 2, \ldots$ found by Bernstein [**3**; **22**] and Varga [**22**] for the entire function associates in the uniform norm

$$\text{(2.14)} \quad ||f - p|| \equiv \sup \{|f(x) - p(x)| : -1 \leqq x \leqq +1\}.$$

This brings us to the main objective which we treat in the next section.

3. Growth and approximation. We establish a global existence criterion for GASP before computing order and type; this is achieved by analogy with S. N. Bernstein.

THEOREM 3. *Let the real* GASP F^α *be regular in the hypersphere* Σ^α *and continuous on* $\bar{\Sigma}^\alpha$. *Then a necessary and sufficient condition for* F^α *to have an analytic continuation as an entire function is that*

$$\lim_{n\to\infty} E_n^{1/n}(F^\alpha) = 0.$$

Proof. Let the real GASP F^α be regular in the hypersphere Σ^α and continuous on $\bar{\Sigma}^\alpha$ and let

$$\lim_{n\to\infty} E_n^{1/n}(F^\alpha) = 0. \tag{3.1}$$

Then by Theorem 1, the associate $f = W_\alpha^{-1}[F^\alpha]$ is regular in the disk D. Moreover, by the normalization $W_\alpha[1] = W_\alpha^{-1}[1] = 1$, the identity

$$f(x) - p(x) = F^\alpha(x, 0) - P^\alpha(x, 0), \quad -1 \leqq x \leqq +1$$

is valid on the symmetry axis for each positive integer n and all $P^\alpha \in \tilde{H}_n^\alpha$ with $p = W_\alpha^{-1}[P^\alpha]$. From the initial data, the GASP F^α may be continued analytically as an even function in y about the symmetry axis so that application of the maximum principle for GASP [**19**, p. 26] to $F^\alpha - P^\alpha$ gives

$$|F^\alpha(x, 0) - P^\alpha(x, 0)| \leqq |||F^\alpha - P^\alpha|||, \quad -1 \leqq x \leqq +1. \tag{3.2}$$

Therefore

$$e_n(f) \leqq |||F^\alpha - P^\alpha|||, \quad P^\alpha \in \tilde{H}_n^\alpha \tag{3.3}$$

for $n = 0, 1, 2, \ldots$. Let $\epsilon > 0$ be given; then for each n there exists a subset $\{P_{n,1}^\alpha, P_{n,2}^\alpha, \ldots\} \subseteq \tilde{H}_n^\alpha$ such that for each positive integer k,

$$|||F^\alpha - P_{n,k}^\alpha||| \leqq E_n(F^\alpha) + \epsilon. \tag{3.4}$$

Combining (3.3) and (3.4) gives

$$e_n(f) \leqq E_n(F^\alpha) + \epsilon, \quad n = 1, 2, \ldots$$

and

$$\lim_{n\to\infty} e_n^{1/n}(f) \leqq \lim_{n\to\infty} E_n^{1/n}(F^\alpha). \tag{3.5}$$

Consequently, by hypothesis (3.1) the W_α associate f of F^α satisfies the Bernstein limit

$$\lim_{n\to\infty} e_n^{1/n}(f) = 0$$

and is necessarily an entire function $f = f(z)$, $z \in \mathbf{C}$ (see [**22**]). By Theorem 1, the GASP F^α is an entire function.

Now let the real GASP F^α be regular in the hypersphere Σ^α, continuous on $\bar{\Sigma}^\alpha$ and have analytic continuation as an entire function. By Theorem 1 the associate $f = W_\alpha^{-1}[F^\alpha]$ is entire. We select for each n, any $p \in \tilde{h}_n$, define $P^\alpha = W_\alpha[p]$ and write the global relation

$$F^\alpha(x, y) - P^\alpha(x, y) = W_\alpha[f - p]. \tag{3.6}$$

The measure of the W_α transform is non-negative and $W_\alpha[1] = 1$, so that from (3.6)

$$|||F^\alpha - P^\alpha||| \leqq \sup \{|f(z) - p(z)| : |z| \leqq 1\}.$$

We arrive at the estimate

$$(3.7) \qquad E_n(F^\alpha) \leqq \sup \{|f(z) - p(z)| : |z| \leqq 1\}, p \in \tilde{h}_n$$

which we shall use to obtain an upper bound on $E_n(F^\alpha)$ for each positive integer n. To do this, we expand the entire function W_α associate f in a series $[-1, +1]$ in terms of the Chebyshev polynomials T_n[**20**; **22**, p. 32] defined by

$$T_n(x) = 1/2 \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k}, \quad |x| \leqq 1$$

and continue analytically as

$$f(z) = \alpha_0 + 2 \sum_{n=1}^{\infty} \alpha_n T_n(z)$$

to the ellipse $\tilde{E}_\beta \equiv \{z \in \mathbf{C} : |z - 1| + |z + 1| \leqq 2\beta\}$ for $\beta > 2$. The Chebyshev coefficients $\alpha_n = \alpha_n(f)$, defined as contour integrals of f over $\partial \tilde{E}_\beta$ (see [**20**; **22**, p. 91]), are bounded as

$$|\alpha_n| \leqq \tilde{M}(\beta)/\beta^n, \quad n = 0, 1, 2, \ldots$$
$$\tilde{M}(\beta) \equiv \sup \{|f(z)| : z \in \tilde{E}_\beta\}.$$

We now choose the unique extremal polynomial $p_* \in \tilde{h}_n$ (see [**22**, p. 91]),

$$p_*(z) = \alpha_0 + 2 \sum_{k=1}^{n} \alpha_k T_k(z),$$

relative to the norm (2.12). Inequality (3.7) together with the fact that $\tilde{E}_\beta \supseteq D$ for the specified range of β gives the estimates

$$(3.8) \qquad \begin{aligned} E_n(F^\alpha) &\leqq \sup \left\{ \left| \sum_{k=n+1}^{\infty} \alpha_k T_k(z) \right| : |z| \leqq 1 \right\} \leqq \sum_{k=n+1}^{\infty} \left\{ |\alpha_k| \sup_{|z| \leqq 1} |T_k(z)| \right\} \\ &\leqq \frac{2\tilde{M}(\beta)}{\beta - 1} \sum_{k=n+1}^{\infty} (5/4\beta)^k < 2^n R_n(\beta, f), \quad n = 0, 1, 2, \ldots, \beta > 2 \end{aligned}$$

for

$$(3.9) \qquad R_n(\beta, f) \equiv 2\tilde{M}(\beta)/(\beta - 1)\beta^n.$$

Now

$$E_n^{1/n}(F^\alpha) < 2R_n^{1/n}(\beta, f)/\beta, \quad n = 1, 2, \ldots .$$

and

$$(3.10) \qquad \lim_{n \to \infty} E_n^{1/n}(F^\alpha) \leqq 2/\beta, \quad \beta > 2.$$

Thus the Bernstein limit holds when $\beta \to +\infty$.

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Having identified the characteristic approximation of entire function GASP, we show that a measure of the convergence rate of the sequence $E_n^{1/n}(F^\alpha)$ defines the order of F^α. We now consider this analogy of the classical Varga result.

THEOREM 2. *Let the real* GASP F^α *be regular in the hypersphere* Σ^α *and continuous on* $\bar{\Sigma}^\alpha$. *Then*

$$\rho = \limsup_{n\to\infty} \{n \log n / -\log E_n(F^\alpha)\}$$

is non-negative and finite if, and only if, the GASP F^α *has analytic continuation as an entire function of finite order* $\rho = \rho(F^\alpha)$.

Proof. Let the real GASP F^α be regular in the specified hypersphere and continuous on its boundary. Let us assume that F^α has an analytic continuation as an entire function of finite positive order ρ. The same is necessarily true of the W_α associate f by Theorems 1 and 2. The reasoning of the previous theorem has shown that

$$(3.11)\quad e_n(f) \leqq E_n(F^\alpha) \leqq 2^n R_n(\beta, f), \quad n = 1, 2, \ldots, \beta > 2.$$

Then monotonocity of the logarithm gives

$$\frac{-\log e_n(f)}{n \log n} \geqq \frac{-\log E_n(F^\alpha)}{n \log n} \geqq \frac{-\log 2}{n \log n} - \frac{\log R_n(\beta, f)}{n \log n} \quad n = 2, 3, \ldots .$$

Each term in this appraisal is positive for all n sufficiently large because the largest member of inequality (3.11) is ultimately less than one. It follows by the Varga's theorem [**22**] and limit properties of sequences that

$$(3.12)\quad \begin{aligned} \rho^{-1} = \liminf_{n\to\infty} \frac{-\log e_n(f)}{n \log n} &\geqq \liminf_{n\to\infty} \frac{-\log E_n(F^\alpha)}{n \log n} \\ &\geqq \liminf_{n\to\infty} \frac{-\log R_n(\beta, f)}{n \log n}, \quad \beta > 2. \end{aligned}$$

Moreover, by Varga's proof [**22**, p. 177] we see that it was shown that for every $\epsilon > 0$,

$$(3.13)\quad \liminf_{n\to\infty} \frac{-\log R_n(\beta, f)}{n \log n} \geqq (\rho + \epsilon)^{-1}.$$

Then from (3.12), (3.13) and ultimate positivity of the sequences

$$(3.14)\quad \rho \leqq \limsup_{n\to\infty} \frac{n \log n}{-\log E_n(F^\alpha)} \leqq \rho + \epsilon, \quad \epsilon > 0$$

which establishes the first assertion. At this point, we complete the proof by considering a real GASP F^α regular in Σ^α and continuous on $\bar{\Sigma}^\alpha$ for which

$$(3.15)\quad \limsup_{n\to\infty} \frac{n \log n}{-\log E_n(F^\alpha)} = \sigma$$

is finite and positive. Appealing to the previous proof we find that for the W_α associate f,

(3.16) $\quad e_n(f) \leqq E_n(F^\alpha), \quad n = 1, 2, \ldots .$

From (3.8) and (3.9) we see that as in Varga [see **22**, p. 177; eqn. (8) ff.] for $\epsilon > 0$ and $N = N(\epsilon)$

$$E_n(F^\alpha) \leqq 1/n^{n/\sigma+\epsilon}, \quad n \geqq N(\epsilon).$$

Combining with (3.16) gives

(3.17) $\quad e_n^{1/n}(f) \leqq 1/n^{n/\sigma+\epsilon}, \quad n \geqq N(\epsilon)$

and $\lim_{n\to\infty} e_n^{1/n}(f) = 0$. Therefore f continues analytically as an entire function of finite order by the classical Bernstein theorem. The same is true of the GASP F^α. It remains to show that σ is the order of F^α.

Inequality (3.12) permits the order of the entire function f to be computed by Varga's formula. Moreover, inequality (3.12) is valid so that

$$\rho^{-1} = \liminf_{n\to\infty} \frac{-\log e_n(f)}{n \log n} \geqq \sigma^{-1} \geqq \liminf_{n\to\infty} \frac{-\log R_n(\beta, f)}{n \log n} .$$

Earlier in our discussion we established (3.13). Therefore for $\epsilon > 0$ given

(3.18) $\quad \rho^{-1} = \liminf_{n\to\infty} \dfrac{-\log e_n(f)}{n \log n} \geqq \sigma^{-1} \geqq (\rho + \epsilon)^{-1}.$

Thus, $\rho = \sigma$ and the order of the associate is also given by

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{-\log E_n(F^\alpha)} .$$

Because this is also the order of the entire function GASP F^α, the proof is completed.

As a final application, we cite the inequalities (3.14) and (3.18) along with essential relations between orders and types of entire function GASP and associate to state a characterization of finite type. It is the "transplant" of the Varga result [**22**] which we now find.

THEOREM 3. *Let the real* GASP F^α *be regular in the hypersphere* Σ^α *and continuous on* $\bar{\Sigma}^\alpha$. *Then* F^α *has an extension as an entire* GASP *of order* ρ *and some finite type* τ *if, and only if,*

$$\limsup_{n\to\infty} n^{1/\rho} E_n(F^\alpha)$$

is finite.

4. Generalizations. We find that the GASP equation is invariant under homothetic transformations and translations along the singular line. The ultraspherical polynomials F_k^α are homogeneous functions of degree k in x and y

and are invariant under translations along $y = 0$. Therefore, the preceding development can be taken up directly for hyperspheres of arbitrary positive radius center on the singular line.

The Methods of Ascent and Descent (see Gilbert [**8**]) transform GASP onto solutions of more general elliptic partial differential equations and conversely. For second or higher order equations where the coefficients of the lower order derivatives permit construction of ascending and descending operators from GASP which preserve norms and map entire functions onto entire functions, direct generalization of the preceding is suggested by composition of operators. For equations whose coefficients do not have this property, viz. the operators map entire functions onto entire functions in one direction, upper or lower estimates on the order and type are possible as are necessary or sufficient conditions for the existence of solutions which are entire functions.

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