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THE QUOTIENT SET OF k-GENERALISED FIBONACCI NUMBERS IS DENSE IN $\mathbb{Q}_p$

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Abstract

The quotient set of $A \subseteq \mathbb{N}$ is defined as $R(A) := \{a/b : a, b \in A, b \neq 0\}$. Using algebraic number theory in $\mathbb{Q}(\sqrt{5})$, Garcia and Luca ['Quotients of Fibonacci numbers', *Amer. Math. Monthly*, to appear] proved that the quotient set of Fibonacci numbers is dense in the p-adic numbers $\mathbb{Q}_p$ for all prime numbers p. For any integer $k \geq 2$, let $(F_n^{(k)})_{n \geq -(k-2)}$ be the sequence of k-generalised Fibonacci numbers, defined by the initial values $0, 0, \ldots, 0, 1$ (k terms) and such that each successive term is the sum of the k preceding terms. We use p-adic analysis to generalise the result of Garcia and Luca, by proving that the quotient set of k-generalised Fibonacci numbers is dense in $\mathbb{Q}_p$ for any integer $k \geq 2$ and any prime number p.

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1. Introduction

Given a set of nonnegative integers A, the *quotient set* of A is defined as

$$R(A) := \{a/b : a, b \in A, b \neq 0\}.$$

The question of when $R(A)$ is dense in $\mathbb{R}^+$ is a classical topic. Strauch and Tóth [15] proved that if A has lower asymptotic density at least equal to $1/2$, then $R(A)$ is dense in $\mathbb{R}^+$ (see also [1]). Bukor *et al.* [3] showed that if $A \cup B$ is a partition of $\mathbb{N}$, then at least one of $R(A)$ or $R(B)$ is dense in $\mathbb{R}^+$. Moreover, the density of $R(\mathbb{P})$ in $\mathbb{R}^+$, where $\mathbb{P}$ is the set of prime numbers, is a well-known consequence of the prime number theorem [10].

On the other hand, the analogous question of when $R(A)$ is dense in the p-adic numbers $\mathbb{Q}_p$, for some prime number p, has been studied only recently [7, 8]. Let $(F_n)_{n \geq 0}$ be the sequence of Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 1$. Using algebraic number theory in the field $\mathbb{Q}(\sqrt{5})$, Garcia and Luca [8] proved the following result.

THEOREM 1.1. *For any prime p, the quotient set of Fibonacci numbers is dense in $\mathbb{Q}_p$.*

One of the many generalisations of the Fibonacci numbers is the sequence of *k-generalised Fibonacci numbers* $(F_n^{(k)})_{n\geq -(k-2)}$, also called the *Fibonacci k-step sequence*, *Fibonacci k-sequence* or *k-bonacci sequence*. For any integer $k \geq 2$, the sequence $(F_n^{(k)})_{n\geq -(k-2)}$ is defined by

$$F_{-(k-2)}^{(k)} = \cdots = F_0^{(k)} = 0, \quad F_1^{(k)} = 1$$

and

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$$

for all integers $n > 1$.

Usually, the study of the arithmetic properties of the k-generalised Fibonacci numbers is more difficult than that of Fibonacci numbers. Indeed, for $k \geq 3$, the sequence of k-generalised Fibonacci numbers lacks several nice properties of the sequence of Fibonacci numbers, which is a strong divisibility sequence [13, page 9], has a primitive divisor theorem [17] and has a simple formula for its p-adic valuation [11, 14].

We prove the following generalisation of Theorem 1.1.

Theorem 1.2. *For any integer $k \geq 2$ and any prime number p, the quotient set of the k-generalised Fibonacci numbers is dense in $\mathbb{Q}_p$.*

It seems likely that Theorem 1.2 could be extended to other linear recurrences over the integers. However, in our proof we use some specific features of the k-generalised Fibonacci numbers. Therefore, we state the following open question.

Question 1.3. *Let $(S_n)_{n\geq 0}$ be a linear recurrence of order $k \geq 2$ satisfying*

$$S_n = a_1 S_{n-1} + a_2 S_{n-2} + \cdots + a_k S_{n-k}$$

for all integers $n \geq k$, where $a_1, \ldots, a_k, S_0, \ldots, S_{k-1} \in \mathbb{Z}$, with $a_k \neq 0$. For which prime numbers p is the quotient set of $(S_n)_{n\geq 0}$ dense in $\mathbb{Q}_p$?

Clearly, without loss of generality, one can suppose that $\gcd(S_0, \ldots, S_{k-1}) = 1$. Also, it seems reasonable to assume that $(S_n)_{n\geq 0}$ is nondegenerate, which in turn implies that $(S_n)_{n\geq 0}$ is definitely nonzero [5, Section 2.1]. Finally, a necessary condition for $(S_n)_{n\geq 0}$ to be dense in $\mathbb{Q}_p$ is that $(\nu_p(S_n))_{n\geq 0, S_n\neq 0}$ is unbounded. This is certainly the case if $S_0 = 0$ and $p \nmid a_k$ (since $p \nmid a_k$ implies that $(S_n)_{n\geq 0}$ is periodic modulo p^h for any positive integer h [5, Section 3.1]), so this could be a useful additional hypothesis.

2. Proof of Theorem 1.2

From now on, fix an integer $k \geq 2$ and a prime number p. In light of Theorem 1.1, we can suppose that $k \geq 3$. Let

$$f_k(X) = X^k - X^{k-1} - \cdots - X - 1$$

be the characteristic polynomial of the k-generalised Fibonacci numbers.

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It is known [16, Corollary 3.4] that f_k is separable. Let K be the splitting field of f_k over $\mathbb{Q}_p$ and let $\alpha_1, \ldots, \alpha_k \in K$ be the k distinct roots of f_k. By [4, Theorem 1],

$$F_n^{(k)} = \sum_{i=1}^{k} c_i \alpha_i^n \tag{2.1}$$

for all integers $n \geq 0$, where

$$c_i := \frac{\alpha_i - 1}{(k+1)\alpha_i^2 - 2k\alpha_i} \tag{2.2}$$

for $i = 1, \ldots, k$.

Now we shall interpolate a subsequence of $(F_n^{(k)})_{n\geq 0}$ by an analytic function over $\mathbb{Z}_p$. This is a classical method in the study of linear recurrences, which goes back at least to the proof of the Skolem–Mahler–Lech theorem [5, Theorem 2.1].

We refer the reader to [9, Chs. 4–6] for the p-adic analysis used hereafter. Let $\mathcal{O}_K$ be the valuation ring of K, e and f the ramification index and the inertial degree of K over $\mathbb{Q}_p$, respectively, and π a uniformiser of K.

Since $f_k(0) = -1$, each α_i $(i = 1, \ldots, k)$ is a unit of $\mathcal{O}_K$, so that $|\alpha_i|_p = 1$. In particular, $\alpha_i \not\equiv 0 \pmod{\pi}$. Since $\mathcal{O}_K/\pi\mathcal{O}_K$ is a finite field of p^f elements, it follows that $\alpha_i^{p^f-1} \equiv 1 \pmod{\pi}$. Now pick any positive integer s such that $p^s \geq e + 1$. Since $|\pi|_p = p^{-1/e}$, we have $\pi^{p^s} \equiv 0 \pmod{p\pi}$ and, in turn, it follows that $\alpha_i^t \equiv 1 \pmod{p\pi}$, where $t := p^s(p^f - 1)$. At this point,

$$|\alpha_i^t - 1|_p \leq |p\pi|_p = p^{-1-1/e} < p^{-1/(p-1)} \tag{2.3}$$

for $i = 1, \ldots, k$.

Now let $\log_p$ and $\exp_p$ denote the p-adic logarithm and the p-adic exponential functions, respectively. From (2.3),

$$\alpha_i^t = \exp_p(\log_p(\alpha_i^t))$$

for $i = 1, \ldots, k$, which together with (2.1) implies that $F_{nt}^{(k)} = G(n)$ for all integers $n \geq 0$, where

$$G(z) := \sum_{i=1}^{k} c_i \exp_p(z \log_p(\alpha_i^t))$$

is an analytic function over $\mathbb{Z}_p$.

Let $r > 0$ be the radius of convergence of the Taylor series of $G(z)$ at $z = 0$ and let $\ell \geq 0$ be an integer. On the one hand, the radius of convergence of the Taylor series of $G(p^\ell z)$ at $z = 0$ is $p^\ell r$. On the other hand,

$$G(p^\ell z) = \sum_{i=1}^{k} c_i \exp_p(p^\ell z \log_p(\alpha_i^t)) = \sum_{i=1}^{k} c_i \exp_p(z \log_p(\alpha_i^{p^\ell t})).$$

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Therefore, taking s sufficiently large, we can assume that $r > 1$. In particular,

$$G(z) = \sum_{j=0}^{\infty} \frac{G^{(j)}(0)}{j!} z^j \tag{2.4}$$

for all $z \in \mathbb{Z}_p$.

Now we shall prove that $G'(0) \neq 0$. For the sake of contradiction, assume that

$$G'(0) = \sum_{i=1}^{k} c_i \log_p(\alpha_i^t) = 0.$$

Since $f_k(0) = -1$ and t is even, we have $\alpha_1^t \cdots \alpha_k^t = 1$, so that

$$\log_p(\alpha_k^t) = -\log_p(\alpha_1^t) - \cdots - \log_p(\alpha_{k-1}^t)$$

and consequently

$$\sum_{i=1}^{k-1} (c_i - c_k) \log_p(\alpha_i^t) = 0. \tag{2.5}$$

We need the following lemma, which is a special case of a general result of Mignotte [12] on Pisot numbers.

Lemma 2.1 [6, Lemma 1]. *The roots $\alpha_1, \ldots, \alpha_{k-1}$ are multiplicatively independent, that is, $\alpha_1^{e_1} \cdots \alpha_{k-1}^{e_{k-1}} = 1$ for some integers $e_1, \ldots, e_{k-1}$ if and only if $e_1 = \cdots = e_{k-1} = 0$.*

Thanks to Lemma 2.1, $\alpha_1^t, \ldots, \alpha_{k-1}^t$ are multiplicatively independent. Hence, $\log_p(\alpha_1^t), \ldots, \log_p(\alpha_{k-1}^t)$ are linearly independent over $\mathbb{Z}$. Then, by [2, Theorem 1], $\log_p(\alpha_1^t), \ldots, \log_p(\alpha_{k-1}^t)$ are linearly independent over the algebraic numbers; hence, (2.5) implies that

$$c_1 = c_2 = \cdots = c_k. \tag{2.6}$$

At this point, from (2.2) and (2.6), it follows that $\alpha_1, \ldots, \alpha_k$ are all roots of the polynomial

$$c_1(k+1)X^2 - (2c_1 k + 1)X + 1,$$

but that is clearly impossible, since $k \geq 3$. Hence, we have proved that $G'(0) \neq 0$.

Taking $z = 1$ in (2.4), we find that $\nu_p(G^{(j)}(0)/j!) \to +\infty$ as $j \to +\infty$. In particular, there exists an integer $\ell \geq 0$ such that $\nu_p(G^{(j)}(0)/j!) \geq -\ell$ for all integers $j \geq 0$. As a consequence of this, and since $G(0) = F_0^{(k)} = 0$, taking $z = mp^h$ in (2.4) gives

$$G(mp^h) = G'(0)mp^h + O(p^{2h-\ell})$$

for all integers $m, h \geq 0$. Therefore, for $h > h_0 := \ell + \nu_p(G'(0))$,

$$\frac{G(mp^h)}{G(p^h)} - m = \frac{G'(0)mp^h + O(p^{2h-\ell})}{G'(0)p^h + O(p^{2h-\ell})} - m = \frac{O(p^{h-\ell})}{G'(0) + O(p^{h-\ell})} = O(p^{h-h_0}),$$

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that is,

$$\lim_{h\to+\infty}\left|\frac{G(mp^h)}{G(p^h)} - m\right|_p = 0.$$

In conclusion, we have proved that

$$\lim_{v\to+\infty}\left|\frac{F^{(k)}_{mp^v(p^f-1)}}{F^{(k)}_{p^v(p^f-1)}} - m\right|_p = 0$$

for all integers $m \geq 0$. In other words, the closure (with respect to the p-adic topology) of the quotient set of k-generalised Fibonacci numbers contains the nonnegative integers $\mathbb{N}$.

The next easy lemma is enough to conclude the proof.

Lemma 2.2. *Let $A \subseteq \mathbb{N}$. If the closure of $R(A)$ contains $\mathbb{N}$, then $R(A)$ is dense in $\mathbb{Q}_p$.*

Proof. Let C be the closure of $R(A)$ as a subspace of $\mathbb{Q}_p$. Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, we have $\mathbb{Z}_p \subseteq C$. Moreover, the inversion $\iota : \mathbb{Z}_p^\times \to \mathbb{Q}_p : x \to x^{-1}$ is continuous and, obviously, sends nonzero elements of $R(A)$ to $R(A)$ and hence $\iota(\mathbb{Z}_p) \subseteq C$. Finally, $\mathbb{Q}_p = \mathbb{Z}_p \cup \iota(\mathbb{Z}_p)$ and thus $C = \mathbb{Q}_p$ and $R(A)$ is dense in $\mathbb{Q}_p$. □

The proof of Theorem 1.2 is complete.

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