J. Austral. Math. Soc. (Series A) 29 (1980), 431–438

ADAMS COMPLETION FOR COHOMOLOGY THEORIES ARISING FROM KAN EXTENSIONS

SRIBATSA NANDA

(Received 6 June; revised 21 November 1979)

Communicated by J. H. Rubinstein

Abstract

It is shown that a cohomology theory over an admissible category, which is obtained from an additive cohomology theory over a smaller admissible category, through the Kan extension process, always admits global Adams completion.

1980 Mathematics subject classification (Amer. Math. Soc.): 55 P 60.

1. Introduction

Let $\mathscr{C}$ be an arbitrary category and S a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of $\mathscr{C}$ with respect to S; then for a given object Y of $\mathscr{C}$,

$$\mathscr{C}[S^{-1}](-, Y) : \mathscr{C} \to \text{Sets}$$

defines a contravariant functor. If this functor is representable by an object Y_S of $\mathscr{C}$, that is,

$$\mathscr{C}[S^{-1}](-, Y) \cong \mathscr{C}(-, Y_S)$$

then Y_S is called the (generalized) Adams completion of Y with respect to the set of morphisms S or simply the S-completion of Y (see Deleanu and others (1972), Adams (1973)). We shall often refer to Y_S simply as the completion of Y.

In particular, let $\mathscr{C}$ be a full subcategory of the category of based topological spaces and based continuous maps such that (a) it contains singletons (b) it contains entire based homotopy types and which moreover has the following property. Let $f : X \to Y$ be in $\mathscr{C}$; let $i_f : X \to M_f$ denote the canonical inclusion of X into M_f, the (reduced) mapping cylinder of the map f. Clearly, i_f is in $\mathscr{C}$. Given a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd}$$

in $\mathscr{C}$, we form the push-out of g and i_f in the category of all spaces:

$$\begin{tikzcd} X \arrow[r, "i_f"] \arrow[d, "g"'] & M_f \arrow[d, "k"] \\ Z \arrow[r, "u"'] & W \end{tikzcd}$$

We assume that (c) $\mathscr{C}$ contains the push-out diagram above. Such a category will be called admissible. It is clear that an admissible category is closed under the formation of mapping cones.

Let $\mathscr{C}$ be an admissible category as above and $\tilde{\mathscr{C}}$ its homotopy category. Let h be a generalized homology (cohomology) theory defined on $\tilde{\mathscr{C}}$. Let S be the set of morphisms of $\tilde{\mathscr{C}}$ which are carried into isomorphisms by h. If every object of $\tilde{\mathscr{C}}$ admits a completion with respect to S, then we say that the homology (cohomology) theory h admits global Adams completion. Deleanu (1974) has shown that any additive theory h on the homotopy category of based CW-complexes and based continuous maps admits global Adams completion. In this note, we show that every cohomology theory on an admissible category arising from an additive cohomology theory on a smaller admissible category through the Kan extension process (see Deleanu and Hilton (1968, 1971), Hilton (1968)) always admits global Adams completion. More precisely, let $\mathscr{C}_0$ and $\mathscr{C}_1$ be admissible cocomplete categories with $\mathscr{C}_0 \subset \mathscr{C}_1$. Let h_0 be an additive cohomology theory on $\tilde{\mathscr{C}}_0$ such that its Kan extension h_1 over $\tilde{\mathscr{C}}_1$ is also a cohomology theory; then h_1 admits global Adams completion. Deleanu and Hilton (1968, 1971) have given a number of examples of cohomology theories arising through the Kan extension process. An important consequence of this result is that all those theories admit global Adams completion. The proof of this result depends mainly on the particularly nice description of the cohomology group $h_1^n(X)$ as given by Hilton (1968) and additivity of the functor h_0. In the next section, we briefly review this description and in Section 3, we present the proof of the main result.

2. Description of the groups $h_1^n(X)$

Let h_0 be a given additive cohomology theory on $\tilde{\mathscr{C}}_0$. For a given object X of $\tilde{\mathscr{C}}_1$, consider the category of $\tilde{\mathscr{C}}_0$-objects under X: an object in this category is a

https://doi.org/10.1017/S1446788700021595 Published online by Cambridge University Press

morphism $f: X \to Y$ in $\tilde{\mathscr{C}}_1$ where Y is an object of $\tilde{\mathscr{C}}_0$. A morphism u in this category $u: f_0 \to f_1$ is a commutative diagram

$$\begin{tikzcd} & Y_0 \arrow[dd, "u"] \\ X \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & \\ & Y_1 \end{tikzcd}$$

The Kan extension h_1 of the cohomology theory h_0 is now defined as follows:

$$h_1^n(X) = \varinjlim_{f} (h_0^n(Y), h_0^n(u)).$$

Obviously, h_1^n is an extension of h_0^n and defines a contravariant functor on $\tilde{\mathscr{C}}_1$ with values in the category of abelian groups. We suppose that this extension defines a cohomology theory on $\tilde{\mathscr{C}}_1$. For various conditions under which this happens, see Deleanu and Hilton (1968, 1971), Hilton (1968).

We now drop the superscript n for convenience and give an alternative description of the groups $h_1(X)$ (see, for example, Hilton (1968)). Consider the set of all pairs (α, f) where $f: X \to Y$ is in $\tilde{\mathscr{C}}_1$, Y an object of $\tilde{\mathscr{C}}_0$ and $\alpha \in h_0(Y)$. Introduce a relation $(\alpha_1, f_1) \sim (\alpha_2, f_2)$ if there is a diagram

$$\begin{tikzcd} & Y_1 \\ X \arrow[ur, "f_1"] \arrow[r, "f"] \arrow[dr, "f_2"'] & Y \arrow[u, "u_1"'] \arrow[d, "u_2"] \\ & Y_2 \end{tikzcd}$$

with $u_1, u_2 \in \tilde{\mathscr{C}}_0$, $u_1 f = f_1$, $u_2 f = f_2$ and $u_1^*(\alpha_1) = u_2^*(\alpha_2)$ where $u_i^* = h_0(u_i)$. This is an equivalence relation; we denote by $[\alpha, f]$ the equivalence class of (α, f). Now define addition of these classes by the rule

$$[\alpha_1, f_1] + [\alpha_2, f_2] = [p_1^*(\alpha_1) + p_2^*(\alpha_2), \{f_1, f_2\}]$$

$$\begin{tikzcd} & Y_1 \\ X \arrow[ur, "f_1"] \arrow[r, "\{f_1, f_2\}"] \arrow[dr, "f_2"'] & Y_1 \times Y_2 \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & Y_2 \end{tikzcd}$$

https://doi.org/10.1017/S1446788700021595 Published online by Cambridge University Press

where p_1, p_2 are the respective projections. That this rule of addition is independent of the representatives chosen, can be easily checked. With respect to this addition, the set $E(X)$ of all equivalence classes forms an abelian group. We note, in particular, that a class $[\alpha, f] = 0$ if and only if there is a commutative diagram

$$\begin{tikzcd} & Y_1 \\ X \arrow[ur, "f"] \arrow[dr, "f_0"'] & \\ & Y_0 \arrow[uu, "u"'] \end{tikzcd}$$

in $\tilde{\mathscr{C}}_1$ with $u \in \tilde{\mathscr{C}}_0$ such that $u^*(\alpha) = 0$. Now define $i_f : h_0(Y) \to E(X)$ by the rule

$$i_f(\alpha) = [\alpha, f].$$

The group $E(X)$ together with the homorphisms $\{i_f\}$ enjoys the universal mapping property by which a direct limit is defined. We therefore identify $E(X)$ with

$$\varinjlim_{f} (h_0(Y), \quad h_0(u)) = h_1(X).$$

Note also that if $g : X \to X'$ is a morphism in $\tilde{\mathscr{C}}_1$, then $h_1(g) = g^* : h_1(X') \to h_1(X)$ is defined by the rule:

$$g^*[\alpha, f] = [\alpha, fg]$$

3. Adams completion for the cohomology theory h_1

We assume throughout that both $\tilde{\mathscr{C}}_0$ and $\tilde{\mathscr{C}}_1$ are admissible and cocomplete and $\tilde{\mathscr{C}}_1$ is a small $\mathscr{U}$-category, where $\mathscr{U}$ is a fixed Grothendieck universe, that is (i) the set of objects of $\tilde{\mathscr{C}}_1$ form a subset of $\mathscr{U}$ and (ii) for any pair of objects X and Y of $\tilde{\mathscr{C}}_1$, the set $\mathrm{Hom}_{\tilde{\mathscr{C}}_1}(X, Y)$ of all morphisms of $\tilde{\mathscr{C}}_1$ from X to Y is a set in $\mathscr{U}$. Recall that Deleanu (1975) has given two conditions for a cocomplete small v-category $\mathscr{C}$ to admit global Adams completion with respect to a set of morphisms T that admits a calculus of left fractions. One of these conditions (condition (*) of Deleanu's main theorem (Deleanu (1975)) is imposed only to ensure that the category of fractions $\mathscr{C}[T^{-1}]$ remains within the given initial universe. We have shown that this condition is automatically satisfied (see Nanda (1979)). Thus we have to check only one condition which shall be referred to as the *compatibility axiom* : If $\{t_\alpha : X_\alpha \to X'_\alpha\}_{\alpha \in L}$ is a set of morphisms of $\mathscr{C}$ such that each $t_\alpha \in T$ and $L \in v$, then

$$\bigvee_{\alpha \in L} t_\alpha : \bigvee_{\alpha \in L} X_\alpha \to \bigvee_{\alpha \in L} X'_\alpha \text{ is in } T.$$

We now apply this result to the category $\tilde{\mathscr{C}}_1$. Let S be the set of morphisms of $\tilde{\mathscr{C}}_1$ which are carried into isomorphisms by the cohomology functor h_1. It follows easily

https://doi.org/10.1017/S1446788700021595 Published online by Cambridge University Press

from Theorem 3.1 of Deleanu and others (1972) (by replacing homology by cohomology, throughout the proof) that S admits a calculus of left fractions. We start by proving the following

Proposition 3.1. *The cohomology theory h_1 satisfies the wedge axiom.*

Proof. We denote by i_α the homotopy class of the inclusion map $X_\alpha \to \bigvee_{\beta\in L} X_\beta$, $\alpha \in L$ The map i_α induces homorphisms

$$i_\alpha^* = h_1(i_\alpha) : h_1(\bigvee_{\beta\in L} X_\beta) \to h_1(X_\alpha).$$

We have to show that the homorphism

$$\{i_\alpha^*\} : h_1(\bigvee_{\beta\in L} X_\beta) \to \prod_\alpha h_1(X_\alpha)$$

is an isomorphism.

Suppose that $[w, f]$ is a class in $h_1(\bigvee X_\alpha)$ such that $\{i_\alpha^*\}([w, f]) = 0$. We then have a map $f\colon \bigvee X_\alpha \to Y$ in $\tilde{\mathscr{C}}_1$ with $Y \in \tilde{\mathscr{C}}_0$ and $w \in h_0(Y)$ such that $[w, fi_\alpha] = 0$ for every $\alpha \in L$. This implies that for each $\alpha \in L$, there is a space Y_α in $\tilde{\mathscr{C}}_0$ and maps $u_\alpha : Y_\alpha \to Y$ in $\tilde{\mathscr{C}}_0$ and $v_\alpha : X_\alpha \to Y_\alpha$ in $\tilde{\mathscr{C}}_1$ such that $u_\alpha^*(w) = h_0(u_\alpha)(w) = 0$ and the following diagram is commutative:

$$\begin{tikzcd} & Y \\ X_\alpha \arrow[ur, "fi_\alpha"] \arrow[dr, "v_\alpha"'] & \\ & Y_\alpha \arrow[uu, "u_\alpha"'] \end{tikzcd}$$

Let $j_\alpha\colon Y_\alpha \to \bigvee_{\beta\in L} Y_\beta$ denote the homotopy class of the inclusion map. We can then factorise $u_\alpha = tj_\alpha$ with $t : \bigvee_{\alpha\in L} Y_\alpha \to Y$. Thus, for each $\alpha \in L$,

$$0 = u_\alpha^*(w) = j_\alpha^* t^*(w)$$

so that

$$\{j_\alpha^*\} : h_0(\bigvee_{\alpha\in L} Y_\alpha) \to \prod_{\alpha\in L} h_0(Y_\alpha)$$

maps the element $t^*(w)$ to zero. By assumption, the theory h_0 is additive, so that $\{j_\alpha^*\}$ is an isomorphism; thus $t^*(w) = 0$.

Now consider the maps $j_\alpha v_\alpha : X_\alpha \to \bigvee Y_\alpha$. By the universal property of the wedge, each $j_\alpha v_\alpha$ can be factorized as $j_\alpha v_\alpha = si_\alpha$ where $s: \bigvee X_\alpha \to \bigvee Y_\alpha$. Consider the diagram

$$\begin{tikzcd}
X_\alpha \arrow[r, "fi_\alpha"] \arrow[dr, "si_\alpha"'] & Y \\
& \bigvee_{\beta\in L} Y_\beta \arrow[u, "t"']
\end{tikzcd}$$

We have $tsi_\alpha = tj_\alpha v_\alpha = u_\alpha v_\alpha = fi_\alpha$; thus the diagram is commutative. It follows from the universal property of the wedge that $ts = f$. Thus we have a commutative diagram

$$\begin{tikzcd}
\bigvee_{\alpha\in L} X_\alpha \arrow[r, "f"] \arrow[dr, "s"'] & Y \\
& \bigvee_{\alpha\in L} Y_\alpha \arrow[u, "t"']
\end{tikzcd}$$

with $t\in\tilde{\mathscr{C}}_0$ and $t^*(w) = 0$; thus $[w, f] = 0$ implying that $\{i_\alpha^*\}$ is injective.

To show that $\{i_\alpha^*\}$ is surjective, let $\{[w_\alpha, f_\alpha]\}_{\alpha\in L}$ be an element of $\prod_{\alpha\in L} h_1(X_\alpha)$. For each $\alpha\in L$, let the class $[w_\alpha, f_\alpha]$ be represented by $f_\alpha: X_\alpha \to Y_\alpha$ with $Y_\alpha\in\tilde{\mathscr{C}}_0$ and $w_\alpha\in h_0(Y_\alpha)$. The element $\{w_\alpha\}\in\prod_{\alpha\in L} h_0(Y_\alpha)$ corresponds to some element $w\in h_0(\bigvee_{\alpha\in L} Y_\alpha)$ under the isomorphism $\{j_\alpha^*\}$ since h_0 satisfies the wedge axiom. Thus $j_\alpha^*(w) = w_\alpha$. Consider now the diagram

$$\begin{tikzcd}
& \bigvee_{\alpha\in L} Y_\alpha \\
X_\alpha \arrow[ur, "j_\alpha f_\alpha"] \arrow[r, "f_\alpha"] \arrow[dr, "f_\alpha"'] & Y_\alpha \arrow[u, "j_\alpha"'] \arrow[d, "\mathrm{id}_{Y_\alpha}"] \\
& Y_\alpha
\end{tikzcd}$$

Clearly, the two triangles are commutative. Thus for each $\alpha\in L$, $[w, j_\alpha f_\alpha] = [w_\alpha, f_\alpha]$. The map $j_\alpha f_\alpha: X_\alpha \to \bigvee_{\beta\in L} Y_\beta$ factors through $\bigvee_{\alpha\in L} X_\alpha$ giving rise to a commutative diagram

$$\begin{tikzcd}
\bigvee_{\beta\in L} X_\beta \arrow[dr, "g"] & \\
X_\alpha \arrow[u, "i_\alpha"] \arrow[r, "j_\alpha f_\alpha"'] & \bigvee_{\beta\in L} Y_\beta
\end{tikzcd}$$

Consider the class $[w, g]\in h_1(\bigvee_{\alpha\in L} X_\alpha)$. We have

$$i_\alpha^*[w, g] = [w, gi_\alpha] = [w, j_\alpha f_\alpha] = [w_\alpha, f_\alpha]$$

https://doi.org/10.1017/S1446788700021595 Published online by Cambridge University Press

for each $\alpha \in L$. Thus

$$\{i_\alpha^*\}[w, g] = \{[w_\alpha, f_\alpha]\}_{\alpha \in L}$$

showing that $\{i_\alpha^*\}$ is an epimorphism.

Theorem 3.2. *The cohomology theory h_1 admits global Adams completion.*

Proof. It suffices to prove that h_1 satisfies the compatibility axiom. Let

$$\{s_\alpha : X_\alpha \to X'_\alpha, \alpha \in L\} \subset S$$

such that the index $L \in \mathscr{U}$. We have to show that

$$\bigvee s_\alpha : \bigvee_{\alpha \in L} X_\alpha \to \bigvee_{\alpha \in L} X'_\alpha$$

is in S. Consider the diagram

$$\begin{tikzcd}
h_1\left(\bigvee_{\alpha \in L} X_\alpha\right) \arrow[r, "\{i_\alpha^*\}"] & \prod_{\alpha \in L} h_1(X_\alpha) \\
h_1\left(\bigvee_{\alpha \in L} X'_\alpha\right) \arrow[r, "\{i'^*_\alpha\}"] \arrow[u, "(\bigvee s_\alpha)^*"] & \prod_{\alpha \in L} h_1(X'_\alpha) \arrow[u, "\prod_{\alpha \in L} s_\alpha^*"']
\end{tikzcd}$$

Since each s_α^* is an isomorphism, so is $\prod_{\alpha \in L} s_\alpha^*$; Proposition 3.1 implies that the horizontal arrows are isomorphisms too. It follows, therefore, that $(\bigvee s_\alpha)^*$ is an isomorphism and the proof is complete.

4. Some final remarks

We conclude this note with some general remarks on the existence of the Adams completion. Suppose that $\mathscr{C}$ is a cocomplete admissible small $\mathscr{U}$-category and h a cohomology theory defined on $\tilde{\mathscr{C}}$. Let (A) and (B) and (C) describe the following conditions.

(A) h satisfies the wedge axiom.

(B) h satisfies the compatibility axiom.

(C) h admits global Adams completion.

It is clear from the diagram of Theorem 3.2 that (A) $\Rightarrow$ (B). Deleanu's result (1975, Theorem 1) implies that (B) $\Rightarrow$ (C). It is also proved in Theorem 2 that (C) $\Rightarrow$ (B). We thus have

$$(A) \Rightarrow (B) \Leftrightarrow (C).$$

https://doi.org/10.1017/S1446788700021595 Published online by Cambridge University Press

We do not know whether (B) $\Rightarrow$ (A). It is our conjecture that this is true. Were it true, it would imply that only additive cohomology theories would admit global Adams completion; thus, it would settle the question 'which cohomology theories on an admissible category admit global Adams completion'.

Similar considerations also apply if h is a homology theory on $\tilde{\mathscr{C}}$. Suppose that h satisfies the wedge axiom : For every collection $\{X_\alpha\}_{\alpha \in L}$ of spaces in $\mathscr{C}$, the inclusions $i_\alpha : X_\alpha \to \bigvee_{\beta \in L} X_\beta$ induce an isomorphism

$$\{i_{\alpha *}\} : \bigoplus_{\alpha \in L} h(X_\alpha) \to h(\bigvee_{\alpha \in L} X_\alpha).$$

Let $s_\alpha : X_\alpha \to X'_\alpha$, $\alpha \in L$ be a family of morphisms of $\tilde{\mathscr{C}}$ such that each $s_{\alpha *}$ is an isomorphism. The commutative diagram

$$\begin{tikzcd} \bigoplus_\alpha h(X_\alpha) \arrow[r, "\{i_{\alpha *}\}"] \arrow[d, "\bigoplus_\alpha s_{\alpha *}"'] & h(\bigvee_{\alpha \in L} X_\alpha) \arrow[d, "(\bigvee_{\alpha \in L} s_\alpha)^*"] \\ \bigoplus_\alpha h(X'_\alpha) \arrow[r, "\{i'_{\alpha *}\}"] & h(\bigvee_{\alpha \in L} X'_\alpha) \end{tikzcd}$$

shows that $(\bigvee_{\alpha \in L} s_\alpha)^*$ is an isomorphism, so that the wedge axiom implies the compatibility axiom. Thus, in case of homology we also have

$$(A) \Rightarrow (B) \Leftrightarrow (C).$$

References

J. F. Adams (1973), 'Idempotent functors in homotopy theory', *Proc. Geometry Conference*, Tokyo.

A. Deleanu (1974), 'Existence of the Adams completion for CW-complexes', *J. Pure Appl. Algebra* **4**, 299–308.

A. Deleanu (1975), 'Existence of the Adams completion for objects of cocomplete categories', *J. Pure Appl. Algebra* **6**, 31–39.

A. Deleanu, A. Frei and P. Hilton (1973), 'Generalised Adams completion', *Cahiers Topologie Géom. Différentielle*, XV-1, pp. 61–82.

A. Deleanu and P. Hilton (1968), 'Some remarks on general cohomology theories', *Math. Scand.* **22**. 227–240.

A. Deleanu and P. Hilton (1971), 'On Kan extension of cohomology theories and Serre class of groups'. *Fund. Math.* **73**, 143–165.

P. Hilton (1968), 'On the construction of cohomology theories', *Rend. Mat.* (6) **1**, 1–14.

S. Nanda (1979), 'A note on the universe of a category of fractions', *Canad. Math. Bull.*, to appear.

Mathematics Department
Regional Engineering College
Rourkela 769008 (Orissa), India