THE SPACE OF DIRICHLET-FINITE SOLUTIONS OF THE EQUATION $\Delta u = Pu$ ON A RIEMANN SURFACE

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Introduction and preliminaries

1. Let R be an open Riemann surface. By a *density* P on R we mean a non-negative and continuously differentiable functions $P(z)$ of local parameters $z = x + iy$ such that the expression $P(z)dxdy$ is invariant under the change of local parameters z. In this paper we always assume that $P \not\equiv 0$ unless the contrary is explicitly mentioned. We consider an elliptic partial differential equation

$$(1) \qquad \Delta u = Pu, \qquad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

which is invariantly defined on R. For absolutely continuous functions f in the sense of Tonelli defined on R, we denote *Dirichlet integrals* and *energy integrals* of f taken over R by

$$D_R[f] = \iint_R (|\partial f/\partial x|^2 + |\partial f/\partial y|^2) dx\,dy$$

and

$$E_R[f] = \iint_R (|\partial f/\partial x|^2 + |\partial f/\partial y|^2 + P|f|^2) dx\,dy$$

respectively. By a solution of (1) on R we mean a twice continuously differentiable function which satisfies the relation (1) on R. We denote by PB (or PD or PE) the totality of bounded (or Dirichlet-finite or energy-finite) solutions of (1) on R. We also denote by $PBD = PB \cap PD$ and $PBE = PB \cap PE$. If the class X contains no non-constant function, then we denote the fact by $R \in O_X$, where X stands for one of classes PB, PD, PE, PBD or PBE. Here we remark that a constant solution of (1) is necessarily zero, since we have assumed that $P \not\equiv 0$ on R. We also use the notation $R \in O_G$ to denote the fact that R is a

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parabolic Riemann surface. Ozawa [5], [6] proved that

$$O_G \subset O_{PB} \subset O_{PE} = O_{PBE}$$

and under the condition $\iint_R P\,dx\,dy < \infty$, $O_{PB} = O_{PE} = O_{PBE}$.

Functions considered in this paper are always assumed to be real-valued. For a class $\mathfrak{X}$ of functions, we denote by $\mathfrak{X}^+$ the totality of non-negative functions in $\mathfrak{X}$.

A subdomain of R is said to be analytic if its relative boundary consists of a finite number of analytic closed Jordan curves.

2. As far as the author knows a little is published about the class PD or O_{PD} (cf. Royden [8]). The aim of the present paper is to show that the class *PD shares in many properties of the class HD*, the totality of Dirichlet-finite harmonic functions on R. First we show $O_{PB} \subset O_{PD}$ (Theorem 1). With the classification scheme of Ozawa we then get

$$O_G \subset O_{PB} \subset O_{PD} \subset O_{PBD} \subset O_{PE} = O_{PBE}$$

and under the condition $\iint_R P\,dx\,dy < \infty$, $O_{PB} = O_{PD} = O_{PBD} = O_{PE} = O_{PBE}$. It is an interesting open question to settle whether the above inclusions are proper or not when $\iint_R P\,dx\,dy = \infty$. Next we prove that the class PD forms a vector lattice (Theorem 2). Hence in particular any Dirichlet-finite solution is represented as a difference of two non-negative Dirichlet-finite solutions. We believe that this will make further investigations of the class PD much easier. We then prove that the vector space structure of PD is completely determined by the behaviour of P at the ideal boundary of R. In other words, if R_0 is an analytic compact subdomain of R such that $R - \overline{R}_0$ is connected and if we denote by P_0D the class of all Dirichlet-finite solutions on $R - \overline{R}_0$ which vanish continuously on ∂R_0, then the classes PD and P_0D are isomorphic as vector spaces (Theorem 3). Here P_0D forms a Hilbert space with reproducing kernel with respect to Dirichlet-norm (Theorem 4). *Finally we characterize the property* O_{PD} by a maximum principle (Theorem 5). A similar consideration as our Theorem 5 for the property O_{PE} is found in the recent work of Ozawa [6].

3. For convenience we state some fundamental facts for solutions of (1) which we shall use later. In this section we admit the case $P \equiv 0$. A non-negative (or non-positive) solution on R does not take its maximum (or minimum) in R unless it is a constant. A solution on R which takes both of positive and negative values does not take its maximum and minimum in R. If R is a bordered compact surface and u is a non-negative solution of (1) and h is a harmonic function such that u and h are continuous on $R \cup \partial R$ and $h \geq u$ on ∂R, then $h \geq u$ on R. We shall quote these facts as ***maximum principle.***

For a compact subset K of R, there exists a positive constant k such that it holds the inequality

$$k^{-1}u(p) \leq u(q) \leq ku(p)$$

for any non-negative solution u on R and for any two points p and q in K. We shall call this inequality as ***Harnack type inequality.***

A monotone sequence of solutions on R which is bounded at a point of R converges to a solution uniformly on any compact subset of R. A bounded sequence of solutions on R contains a subsequence converging to a solution on R uniformly on any compact subset of R. We shall quote these facts as ***Harnack type theorem.***

If a sequence of solutions on R converges to a function uniformly on each compact subset of R, then the limiting function is a solution and the sequence of differentials of these solutions converges to the differential of this limiting solution.

A bounded solution on R except a compact subset of logarithmic capacity zero can be continued to a solution defined on R.

Other important facts for the equation $\Delta u = Pu$ is the solvability of Dirichlet problem on any analytic compact subdomain with continuous boundary value and the existence of Green's function of $\Delta u = Pu$ with respect to an arbitrary Riemann surface R unless $P \equiv 0$ on R.

For proofs of these facts, refer to Myrberg's fundamental work [2] and [3].

4. For an analytic compact subdomain D of R and a continuous function f defined on a set S containing D, we denote by f_D the continuous function on S defined by $f_D = f$ on $S - D$ and $\Delta f_D = Pf_D$ on D.

A continuous function f defined on a subdomain U of R is said to be a subsolution (or supersolution) if for any point p_0 in U there exists an analytic compact subdomain D_0 of R such that $p_0 \in D_0 \subset \overline{D}_0 \subset U$ and $f_D \geq f$ (or $f_D \leq f$) on U for any analytic subdomain D of D_0 with $p_0 \in D \subset \overline{D} \subset D_0$. A non-positive (or non-negative) constant is a subsolution (or supersolution). The functions $cf+g$, where c is a non-negative constant, and $\max(f, g)$ (or $\min(f, g)$) are subsolutions (or supersolutions) along with f and g. A solution is a subsolution and at the same time supersolution. Although the notions of subsolutions and supersolutions are of local character, we can derive the following global properties.

LEMMA 1. *Let f be a subsolution (or supersolution) defined on a subdomain U of R such that $\sup_U f \geq 0$ (or $\inf_U f \leq 0$). Then f does not take its maximum (or minimum) in U unless f is a constant.*

Proof. We only treat the case when f is a subsolution, since the situation for supersolution is quite parallel to that of subsolution. Contrary to the assertion, assume that u takes its maximum in U. Then we can find a point p_0 in U which lies in the boundary of the set $\{p;\ u(p) = \max_U u\}$, since f is not constant in U. Now we can find an analytic compact domain D_0 such that $p_0 \in D_0 \subset \overline{D}_0 \subset U$ and $f_D \geq f$ for any analytic subdomain D of D_0 with $p_0 \in D \subset \overline{D} \subset D$. At any point p in ∂D,

$$f_D(p_0) \geq f(p_0) \geq f(p) = f_D(p).$$

This shows that the solution f_D in D takes its maximum in D. Hence by the maximum principle f_D is a constant and so $f = f(p_0)$ on ∂D. By arbitrariness of D in D_0, we conclude that $f = f(p_0)$ in D_0, which contradicts the definition of p_0. Q.E.D.

LEMMA 2. *A continuous function f defined on a subdomain U of R is a subsolution (or supersolution) if and only if $f_D \geq f$ (or $f_D \leq f$) for any analytic compact subdomain D such that $D \subset U$.*

Proof. Consider the function $\varphi = f - f_D$ on D. This is a subsolution in D and so from Lemma 1 $\sup_D \varphi = \sup_{\partial D} \varphi = 0$. Thus $\varphi \leq 0$ on D or $f_D \geq f$. Q.E.D.

From this lemma we can conclude that a function which is a subsolution and at the same time a supersolution is a solution.

LEMMA 3. *Suppose that f is a twice continuously differentiable function* *defined on a subdomain U of R. Then f is a subsolution (or supersolution) in U if and only if $\Delta f-Pf\geq 0$ (or $\Delta f-Pf\leq 0$) on U.*

Proof. First we show the sufficiency of our condition. Let D be an arbitrary analytic compact subdomain such that $D\subset U$. Put $\varphi=f-f_D$ on D. We denote by G the Green's function with respect to D with the pole p, which is an arbitrary point in D. By Green's formula,

$$\varphi(p)=(2\pi)^{-1}\iint_D(\Delta\varphi-P\varphi)G\,dxdy.$$

As $\Delta\varphi-P\varphi\geq 0$ (or $\Delta\varphi-P\varphi\leq 0$), so $\varphi(p)\geq 0$ (or $\varphi(p)\leq 0$). Thus $\varphi\geq 0$ (or $\varphi\leq 0$) on D or $f_D\geq f$ (or $f_D\leq f$). Hence f is a subsolution (or supersolution).

Next we show the necessity of our condition. Contrary to the assertion, assume the existence of a point in D and hence a subdomain D of U such that $\Delta f-Pf<0$ (or $\Delta f-Pf>0$) in D. Then from the sufficiency of our condition we conclude that f is a supersolution (or subsolution) in D. As f is a sub- and supersolution in D, so f is a solution in D. Then $\Delta f-Pf=0$ in D. This is a contradiction. Thus we have shown that $\Delta f-Pf\geq 0$ (or $\Delta f-Pf\leq 0$) in U.

Q.E.D.

5. Let U be an analytic compact subdomain of R. We denote by $M(\overline{U})$ the totality of continuous functions on $\overline{U}$ which are absolutely continuous in the sense of Tonelli in U with finite Dirichlet integral taken over U. We also denote by $M^{\varphi}(\overline{U})$ the totality of functions f in $M(\overline{U})$ with $f=\varphi$ on ∂U, where φ is a fixed element in $M(\overline{U})$. Then we have

DIRICHLET PRINCIPLE: *if u satisfies $\Delta u=0$ on U and $u=\varphi$ on ∂U, then $D_U[u]\leq D_U[f]$ for all f in $M^{\varphi}(\overline{U})$, where the equality holds only for $f=u$.*

This simple fact plays an important and almost essential role in the study of the class HD in the theory of harmonic functions. This Dirichlet principle is a special case, i.e. $P\equiv 0$ on U, of the following

ENERGY PRINCIPLE: *if u satisfies $\Delta u=Pu$ on U and $u=\varphi$ on ∂U, then $E_U[u]\leq E_U[f]$ for all f in $M^{\varphi}(\overline{U})$, where the equality holds only for $f=u$.*

The proof of this is an immediate consequence of the identity $E_U[f]=E_U[u]+E_U[f-u]$ which follows from Green's formula. From the standpoint that we are asking to what extent the theory of the class HD can be extended to the class PD, we desire to get the validity of Dirichlet principle for solutions of $\Delta u = Pu$. Needless to say, this cannot be expected in general unless $P \equiv 0$. Hence to get the validity of Dirichlet principle for solutions of $\Delta u = Pu$, we have to impose some restrictions on the class $M^{\varphi}(\overline{U})$. For the aim, we denote by $S(\overline{U})$ the class of all non-negative functions in $M(\overline{U})$ which are subsolutions in U and by $S^{\varphi}(\overline{U})$ the totality of functions in $S(\overline{U})$ such that $f = \varphi$ on ∂U, where φ is a fixed function in $S(\overline{U})$. Then we have the following almost trivial but very useful fact, which we shall quote as weak Dirichlet principle.

LEMMA 4 [WEAK DIRICHLET PRINCIPLE]. *If u satisfies $\Delta u = Pu$ on U and $u = \varphi$ on ∂U, then $D_U[u] \leq D_U[f]$ for all f in $S^{\varphi}(\overline{U})$, where the equality holds only for $f = u$.*

Proof. As $f - u$ is a subsolution in U with $f - u = 0$ on ∂U, so by maximum principle (Lemma 1) $f - u \leq 0$ in U. Since f is non-negative, $u^2 - f^2 \geq 0$ on U. From this and the energy principle

$$D_U[f] - D_U[u] \geq \iint_U P(u^2 - f^2)\, dx dy \geq 0.$$

Next suppose that $D_U[f] = D_U[u]$. Then from above we get $E_U[f] = E_U[u]$. By the energy principle, we finally get $f = u$. Q.E.D.

Existence of bounded solutions

6. Virtanen [9] proved that the existence of a non-constant Dirichlet-finite harmonic function implies the existence of a non-constant bounded harmonic function. First we prove such a Virtanen type theorem for the equation $\Delta u = Pu$.

THEOREM 1. *The existence of a non-constant Dirichlet-finite solution of $\Delta u = Pu$ implies the existence of a non-constant bounded solution of $\Delta u = Pu$.*

Proof. Let u be a non-constant PD-function on R. Contrary to the assertion, assume that there exists no non-constant bounded solution of $\Delta u = Pu$ on R. Hence, in particular, u is not bounded. We take an exhaustion $\{R_n\}_1^{\infty}$ of R consisting of analytic compact subdomains R_n such that $R - \overline{R}_1$ is connected. Without loss of generality, we may assume $u > 0$ on ∂R_1. Let u_n^* be

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the continuous function defined on $\overline{R}_n-R_1$ such that u_n^* is the solution in $R_n-\overline{R}_1$ and $u_n^*=0$ on ∂R_n and $u_n^*=u$ on ∂R_1. By the maximum principle

$$m\geq n_{n+1}^*\geq u_n^*\geq 0$$

on $\overline{R}_n-R_1$, where $m=\sup_{\partial R_1}u$. Hence by the Harnack type theorem $\{u_n^*\}$ converges to a solution u^* on $R-\overline{R}_1$ which is continuous on $R-R_1$ with boundary value $u^*=u$ on ∂R_1 and $m\geq u^*\geq 0$ on $R-R_1$. By energy principle

$$E_{R_n-\overline{R}_1}[u_n^*]\geq E_{R_{n+1}-\overline{R}_1}[u_{n+1}^*].$$

By Fatou's lemma $E_{R-\overline{R}_1}[u^*]<\infty$ and a fortiori $D_{R-\overline{R}_1}[u^*]<\infty$. We put $u^{**}=u-u^*$, which is not identically zero, since $u^{**}\equiv 0$ implies the boundedness of u. Then u^{**} is a non-constant Dirichlet-finite solution in $R-\overline{R}_1$ vanishing on ∂R_1.

Next we put $f=|u^{**}|$. This is a Dirichlet-finite subsolution in $R-\overline{R}_1$ vanishing on ∂R_1. We denote by v_n (resp. h_n) the continuous function on $\overline{R}_n-R_1$ which is a solution (resp. harmonic) in $R_n-\overline{R}_1$ with boundary value f on $\partial(R_n-\overline{R}_1)$. By maximum principle

$$f\leq v_n\leq h_n$$

and the sequences $\{v_n\}$ and $\{h_n\}$ are non-decreasing. By using Dirichlet principle.

$$D_{R_n-\overline{R}_1}[h_n]\leq D_{R_n-\overline{R}_1}[f].$$

As ∂R_1 consists of analytic curves and $h_n=0$ on ∂R_1 and Dirichlet integral of h_n is bounded by $D[f]$, so $\{h_n\}$ converges to a harmonic function h with finite Dirichlet integral on $R-\overline{R}_1$. Thus $f\leq v_n\leq h$ and so by the Harnack type theorem, $\{v_n\}$ converges to a solution v such that

$$f\leq v\leq h,$$

which shows $v=0$ on ∂R_1 and $v>0$ in $R-\overline{R}_1$. As $D[h]<\infty$, so h^2 admits a harmonic majorant h^* (cf. Parreau [7]). Hence v^2 admits a harmonic majorant h^*. Here we notice that v^2 is a subsolution. In fact,

$$\Delta v^2-Pv^2=Pv^2+2|\operatorname{grad}v|^2\geq 0.$$

Hence by Lemma 3, v^2 is a subsolution. We denote by v_n^* the continuous function on $\overline{R}_n-R_1$ such that v_n^* is the solution in $R_n-\overline{R}_1$ with boundary value v^2

on $\partial(R_n - \overline{R}_1)$. Then by the maximum principle

$$v^2 \leq v_n^* \leq v_{n+1}^* \leq h^*$$

on $\overline{R}_n - R_1$. Hence by the Harnack type theorem $\{v_n^*\}$ converges to a solution v^* such that $v^2 \leq v^*$.

Let w_n be the continuous function on $\overline{R}_n - R_1$ and the solution in $R_n - R_1$ with boundary values $w_n = 0$ on ∂R_1 and $w_n = 1$ on ∂R_n. By the maximum principle

$$1 \geq w_n \geq w_{n+1} \geq 0$$

on $\overline{R}_n - R_1$ and so by the Harnack type theorem $\{w_n\}$ converges to a solution $w(p;\ \partial R,\ R - \overline{R}_1)$ on $R - \overline{R}_1$. As we have assumed $R \in O_{PB}$, so by a theorem of Ozawa [5], $w(p;\ \partial R,\ R - \overline{R}_1) \equiv 0$.

Fix an arbitrary point p in $R - \overline{R}_1$. For integers n such that $R_n - \overline{R}_1$ contains p, we denote by $G_n(q,\ p)$ the Green's function of the equation $\Delta u - Pu = 0$ with respect to $R_n - \overline{R}_1$ with pole p. We put

$$d\mu_n(q) = \frac{1}{2\pi} \frac{\partial G_n(q,\ p)}{\partial \nu_q} ds_q$$

on $\partial(R_n - \overline{R}_1)$, where $\partial/\partial\nu$ denotes the inner normal derivative on $\partial(R_n - \overline{R}_1)$ and ds denotes the line element of $\partial(R_n - \overline{R}_1)$. By Green's formula

$$v(p) = \int_{\partial R_n} v\, d\mu_n$$

and

$$v^*(p) = \int_{\partial R_n} v^*\, d\mu_n$$

and

$$w_n(p) = \int_{\partial R_n} w_n\, d\mu_n.$$

By Schwarz's inequality

$$(v(p))^2 = \left(\int_{\partial R_n} v\, d\mu_n\right)^2 \leq \int_{\partial R_n} d\mu_n \cdot \int_{\partial R_n} v^2\, d\mu_n.$$

Using $v^2 \leq v^*$, we get $(v(p))^2 \leq w_n(p) v^*(p)$. Hence by making $n \nearrow \infty$,

$$(v(p))^2 \leq w(p;\ \partial R,\ R - \overline{R}_1) v^*(p).$$

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Since p is arbitrary in $R-\overline{R}_1$ and $w(p;\ \partial R,\ R-\overline{R}_1)\equiv 0$, we get $v\equiv 0$. This is a contradiction. Thus $R\notin O_{PB}$. Q.E.D.

7. REMARK TO THEOREM 1. *Let R_0 be an analytic compact subdomain of R such that $R-\overline{R}_0$ is connected. Assume that there exists a non-constant Dirichlet-finite solution u of $\Delta u=Pu$ on $R-\overline{R}_0$ which vanishes continuously on ∂R_0. Then there exists a non-constant bounded solution of $\Delta u=Pu$.*

The proof of this fact is contained in the proof of Theorem 1. We must notice that this fact is not true in general in the case when $P\equiv 0$ on R, i.e. in the harmonic case. In fact, if $R\in O_{HB}-O_G$, then the harmonic measure of the ideal boundary of R relative to the domain $R-\overline{R}_0$ is a non-constant and Dirichlet-finite harmonic function on $R-\overline{R}_0$ which vanishes on ∂R_0 but there exists no non-constant bounded harmonic function on R. (O_{HB} denotes the class of all Riemann surfaces on which no non-constant bounded harmonic function exist. For the existence of R in $O_{HB}-O_G$, confer Tôki [9].)

Lattice property of the class *PD*

8. It is known that the class *HD* forms a vector lattice (cf. [4]). Here the lattice operations in *HD* are induced one from the usual function ordering in the class of all harmonic functions. Corresponding to this fact, we prove that the class *PD* forms a vector lattice with lattice operations induced by the function ordering in the class of all solutions. More precisely, for two solutions u and v we denote by $u\vee v$ (or $u\wedge v$) the solution w such that $w\geq u$ and v (or $w\leq u$ and v) and $w\leq w'$ (or $w\geq w'$) for any solution w' such that $w'\geq u$ and v (or $w'\leq u$ and v). The function $u\vee v$ does not exist in general. Clearly the necessary and sufficient condition for the existence of $u\vee v$ is that there exists a solution which is less smaller than u and v.

THEOREM 2. *The class PD forms a vector lattice with lattice operations $\vee$ and $\wedge$. In particular any Dirichlet-finite solution of $\Delta u=Pu$ can be represented as a difference of two non-negative Dirichlet-finite solutions of $\Delta u=Pu$.*

Proof. If *PD* does not contains no non-constant function, our assertion is obvious. Hence we may assume $R\notin O_{PD}$ and so by Theorem 1 $R\notin O_{PB}$. It is known that $O_G\subset O_{PB}$ (cf. Ozawa [5]). Thus $R\notin O_G$.

First we prove that $u \vee 0$ exists and belongs to PD for any u in PD. For the aim we put $f = \max(u, 0)$, which is a subsolution on R. We take an exhaustion $\{R_n\}_0^\infty$ of R consisting of analytic subdomains. Let v_n be the solution in R_n with boundary value u on ∂R_n $(n \geq 1)$. By the maximum principle

$$f \leq v_n \leq v_{n+1}$$

on R_n and by the weak Dirichlet principle

$$D_{R_n}[v_n] \leq D_{R_n}[f]. \tag{1}$$

We denote by h_n the harmonic function in $R_n - \bar{R}_0$ with boundary value 1 on ∂R_0 and 0 on ∂R_n. Then by the maximum principle and Diriclet principle

$$0 \leq h_n \leq h_{n+1} \leq 1$$

and

$$D_{R_n-\bar{R}_0}[h_n] \geq D_{R_{n+1}-\bar{R}_0}[h_{n+1}]$$

and $\{h_n\}$ converges to a harmonic function h on $R - \bar{R}_0$ and

$$D_{R-\bar{R}_0}[h] = \lim_n D_{R_n-\bar{R}_0}[h_n].$$

By $R \notin O_G$, h is non-constant and $D_{R-\bar{R}_0}[h] > 0$. By Green's formula

$$\int_{\partial R_0} (v_n - f)^* dh_n = \int_{\partial(R_n - \bar{R}_0)} (v_n - f)^* dh_n \tag{2}$$
$$= \iint_{R_n - \bar{R}_0} d(v_n - f) \wedge {}^* dh_n.$$

By Schwarz's inequality and by (1)

$$\left| \iint_{R_n - \bar{R}_0} d(v_n - f) \wedge {}^* dh_n \right|^2 \leq D_{R_n-\bar{R}_0}[v_n - f] D_{R_n-\bar{R}_0}[h_n] \tag{3}$$
$$\leq 4 D[f] D_{R_n-\bar{R}_0}[h_n].$$

As we have from (2) and (3)

$$\inf_{\partial R_0} |v_n - f| \int_{\partial R_0} {}^* dh_n \leq 2(D[f] D_{R_n-\bar{R}_0}[h_n])^{1/2}$$

and by Green's formula

$$\int_{\partial R_n} {}^* dh_n = D_{R_n-\bar{R}_0}[h_n],$$

so we get

$$\inf_{\partial R_0} |v_n - f| \leq 2(D[f]/D_{R_n - \bar{R}_0}[h_n])^{1/2} \leq 2(D[f]/D_{R-\bar{R}_0}[h])^{1/2}.$$

We fix a point p_0 in R_0. By the Harnack type inequality there exists a finite and positive constant k such that

$$v_n(p_0) \leq k \inf_{\partial R_0} v_n.$$

Hence by putting $m = \sup_{\partial R_0} |f|$,

$$v_n(p_0) \leq km + 2k(D[f]/D_{R-\bar{R}_0}[h])^{1/2}.$$

Thus by the Harnack type theorem the non-decreasin sequence $\{v_n\}$ of solutions converges to a solution v on R and by Fatou's lemma

$$D[v] \leq \underline{\lim}_n D_{R_n}[v_n] \leq D[f] < \infty.$$

Hence v belongs to the class PD and $v \geq f$ or $v \geq u$ and 0. To conclude $v = u \vee 0$, we have to show that $v' \geq v$ for any solution v' such that $v' \geq u$ and 0. This follows from the inequality $v' \geq v_n \geq f$, which is a consequence of the maximum principle.

For two arbitrary elements u and v in PD, $(u-v) \vee 0$ exists and belongs to the class PD as we have seen above. Clearly the element $w = (u-v) \vee 0 + v$ belongs to the class PD and $w = u \vee v$. The existence $u \wedge v$ in PD is immediate if we notice the relation $u \wedge v = -((-u) \vee (-v))$.

Hence we have proved that the class PD forms a lattice with respect to the operations $\vee$ and $\wedge$. These operations are easily seen to be compatible with the vector space structure of PD. Thus the class PD forms a vector lattice with lattice operations $\vee$ and $\wedge$.

The last part is nothing but the Jordan decomposition of the element u in PD: $u = u \vee 0 - (-(u \wedge 0))$. Q.E.D.

9. Remark 1 to Theorem 2. *Suppose that R is embedded into a Riemann surface R' as its subsurface and Γ consists of a finite number of analytic closed Jordon curves which are contained in the boundary of R relative to R'. Moreover suppose that the density P on R is the restriction of a density P' on R' to R. Assume that two functions u and v in $PD(R)$ are continuously extended to $R \cup \Gamma$. Then $u \vee v$ and $u \wedge v$ are continuously extended to $R \cup \Gamma$ and $u \vee v = \max(u, v)$ and $u \wedge v = \min(u, v)$ on Γ.*

Proof. As we have identities $u \vee v = (u-v) \vee 0 + v$, $\max(u, v) = \max(u$

$-v, 0)+v$, $u \wedge v = -((-u) \vee (-v))$ and $\min(u, v) = -\max-(u, -v)$, so we have only to prove the above assertion for $u-v$ and 0 in $PD(R)$ and for the operation $\vee$. Hence we have to prove that $u \vee 0$ is continuously extended on $R \cup \Gamma$ and $u \vee 0 = \max(u, 0)$ on Γ if u is continuous on $R \cup \Gamma$.

Let $\{R_n^*\}_1$ be a sequence of analytic compact subdomains R_n^* of R' such that $R_n^* \subset R_{n+1}^*$ and ∂R_n^* (relative to R') contains Γ and $\partial R_n^* - \Gamma$ is contained in R and $R = \cup_n R_n^*$. We put $f = \max(u, 0)$, which is continuous on $R \cup \Gamma$ and a subsolution in R. We denote by v_n^* the solution of $\Delta u = Pu$ in R_n^* with boundary value f on ∂R_n^*. By the maximum principle,

$$f \leq v_n^* \leq v_{n+1}^* \leq u \vee 0$$

on R_n^*. Hence by the definition of $u \vee 0$, $u \vee 0 = \lim_n v_n^*$. Now we denote by w the solution of $\Delta u = Pu$ on R_1^* with boundary value f on Γ and $u \vee 0$ on $\partial R_1^* - \Gamma$. Again by the maximum principle, $f \leq v_n^* \leq w$ on R_1^* and by making $n \nearrow \infty$, $f \leq u \vee 0 \leq w$, which shows that $u \vee 0 = w$ on R_1^* and so $u \vee 0$ is continuous on $R_1^* \cup \partial R_1^*$ and a fortiori on $R \cup \Gamma$ and $u \vee 0 = f = \max(u, 0)$.

Remark 2 to Theorem 2. *From the proof of Theorem 2, we can easily see that the following inequality holds:*

$$D_R[u \vee 0],\ D_R[u \wedge 0] \leq D_R[u].$$

The class *PD* and the ideal boundary of *R*

10. It is well known that O_{HD}-property of a Riemann surface is determined by its ideal boundary. The corresponding facts are also valid for the equation $\Delta u = Pu$. In our case, more strong facts hold, i.e. the vector space structure of the class PD is completely determined by the behaviour of P at the ideal boundary of R. This means that vector spaces PD and $P'D$ are isomorphic if P and P' are two densities on R which are not identically zero on R and $P \equiv P'$ except a compact subset of R. This fact is also formulated as follows. Let R_0 be an analytic compact subdomain of R such that $R - \overline{R}_0$ is connected. We denote by P_0D the class of all Dirichlet-finite solutions of $\Delta u = Pu$ on $R - \overline{R}_0$ vanishing continuously at ∂R_0. Then we have

Theorem 3. *The class PD is isomorphic to the class P_0D as vector spaces.*

Proof. We take an exhaustion $\{R_n\}_0^\infty$ of R such that R_n is an analytic compact subdomain of R. With each u in PD^+, we associate a function u^* as follows. We denote by u_n^* the solution in $R_n - \bar{R}_0$ $(n \geq 1)$ with boundary values $u_n^* = u$ on ∂R_0 and $u_n^* = 0$ on ∂R_n. By the maximum principle,

$$u_n^* \leq u_{n+1}^* \leq u$$

and

$$0 \leq u_n^* \leq \sup_{\partial R_0} u.$$

By the Harnack type theorem, $\{u_n^*\}$ converges to a solution u^* in $R - \bar{R}_0$ such that $u^* = u$ on ∂R_0 and $0 \leq u^* \leq u$ and $0 \leq u^* \leq \sup_{\partial R_0} u$ on $R - \bar{R}_0$. By the energy principle,

$$E_{R_{n+1} - \bar{R}_0}[u_{n+1}^*] \leq E_{R_n - R_0}[u_n^*] < \infty.$$

Hence by Fatou's lemma

$$E_{R - \bar{R}_0}[u^*] \leq \underline{\lim}_n\, E_{R_n - \bar{R}_0}[u_n^*] \leq E_{R_1 - \bar{R}_0}[u_1^*] < \infty$$

and a fortiori

$$D_{R - \bar{R}_0}[u^*] < \infty.$$

Clearly the mapping $u \to u^*$ satisfies

(1) $$(u_1 + u_2)^* = u_1^* + u_2^*$$

and

(2) $$(cu)^* = cu^*,$$

where c is a non-negative constant. Now we put $\pi u = u - u^*$, which is an element in P_0D^+. Hence we get a mapping π of PD^+ into P_0D^+ such that

(3) $$\pi(u_1 + u_2) = \pi u_1 + \pi u_2$$

and

(4) $$\pi(cu) = c\pi u,$$

where c is a non-negative constant. These follow from (1) and (2).

We first show that π is onto, i.e. there exists a w in PD^+ such that $\pi w = v$ for any v in P_0D^+. If $v \equiv 0$, then we have only to take $w \equiv 0$. So we may suppose that $v \not\equiv 0$. We fix a point p_0 in R_0 and a sequence $\{R_{-m}\}_1^\infty$ of analytic subdomains of R such that $R_0 \supset \bar{R}_{-m} \supset R_{-m} \supset \bar{R}_{-(m+1)}$ and $\cap_1^\infty \bar{R}_{-m} = \{p_0\}$. We denote by $w_{m,n}$ the solution in $R_n - \bar{R}_{-m}$ with boundary values v on ∂R_n and 0 on ∂R_{-m}. By the maximum principle

$$0 \leq w_{m,n} \leq w_{m+1,n} \leq \sup_{\partial R_n} v.$$

Hence by the Harnack type theorem $\{w_{m,n}\}_{m=1}^{\infty}$ converges to a solution w_n on $R_n - \{p_0\}$ such that

$$0 \leq w_n \leq \sup_{\partial R_n} v$$

and

$$w_n \geq v$$

on $R_n - R_0$. Hence w_n can be continued to R_n so as to be a solution on R_n and if we set $w_{m,n} = 0$ (resp. $v = 0$) in R_{-m} (resp. R_0), then $w_{m,n}$ (resp. v) is a subsolution in R_n. By the weak Dirichlet principle,

$$D_{R-\bar{R}_0}[v] \geq D_{R_n-\bar{R}_{-m}}[w_{m,n}] \geq D_{R_n-\bar{R}_{-(m+1)}}[w_{m+1,n}].$$

Hence by Fatou's lemma, we get

$$D_{R_n}[w_n] \leq D[v].$$

As $v \leq w_n$, so by the maximum principle

$$v \leq w_n \leq w_{n+1}.$$

Now we take an analytic compact subdomain V in $R_1 - \bar{R}_0$ such that $(R_1 - \bar{R}_0) - \bar{V}$ is a domain. We denote by h_n the harmonic function with boundary values 1 on ∂V and 0 on ∂R_n. Then by Green's formula

$$\int_{\partial V} (w_n - v)\, {}^*dh_n = \int_{\partial(R_n - \bar{V})} (w_n - v)\, {}^*dh_n = \iint_{R_n - \bar{V}} d(w_n - v) \wedge {}^*dh_n. \tag{5}$$

By Schwarz's inequality

$$\left| \iint_{R_n - V} d(w_n - v) \wedge {}^*dh_n \right|^2 \leq D_{R_n}[w_n - v]\, D_{R_n - \bar{V}}[h_n]. \tag{6}$$

Hence by (5) and (6)

$$\inf_{\partial V} (w_n - v) \int_{\partial V} {}^*dh_n \leq 2(D_{R_n}[v]\, D_{R_n - \bar{V}}[h_n])^{1/2}.$$

As we have by Green's formula

$$\int_{\partial V} {}^*dh_n = D_{R_n - \bar{V}}[h_n],$$

so we get

$$\inf_{\partial V}(w_n - v) \leq 2(D[v]/D_{R_n - \bar{V}}[h_n])^{1/2}.$$

By the remark to Theorem 1, R does not belong to O_{PB} and so by Ozawa's lemma R does not belong to O_G, since $v \not\equiv 0$. Hence $\{h_n\}$ converges increasingly to a non-constant harmonic function h on $R - \bar{V}$ and the sequence $\{D_{R_n - \bar{V}}[h_n]\}$ converges decreasingly to $D_{R-\bar{V}}[h]$, which is strictly positive. By the Harnack type inequality there exists a constant k for a fixed point p in V such that $w_n(p) \leq k \inf_{\partial V} w_n$ for all n. Hence by putting $a = \sup_{\partial V} v$,

$$w_n(p) \leq ka + 2k(D[v]/D_{R-\bar{V}}[h])^{1/2} < \infty.$$

So, by the Harnack type theorem, the non-decreasing sequence $\{w_n\}$ converges to a solution w on R such that $w \geq v$ and by Fatou's lemma

$$D[w] \leq \varliminf_n D_{R_n}[w_n] \leq D[v] < \infty.$$

Thus w belongs to the class PD^+. To conclude $\pi w = v$, we have to show $w^* = w - v$. For the aim we put $w'_n = w_n - v \geq 0$. As $w_n - v = w_n$ converges to $w - v = w$ uniformly on ∂R_0, so we can find for an arbitrary given positive number ε an N such that for any $n \geq N$

$$0 \leq w_n^* - w'_n \leq \varepsilon$$

on ∂R_0, where w_n^* is, by the definition of the operation *, the solution in $R_n - \bar{R}_0$ with boundary values w on ∂R_0 and 0 on ∂R_n. As $w'_n = 0$ on ∂R_n, so we get by the maximum principle

$$0 \leq w_n^* - w'_n \leq \varepsilon$$

on $R_n - \bar{R}_0$. Notice that $\lim_n w_n^* = w^*$ and $\lim_n w'_n = w - v$. Hence by making $n \nearrow \infty$ in the above inequality, $0 \leq w^* - (w - v) \leq \varepsilon$. This shows that $w^* = w - v$. Thus we have proved that $\pi w = v$, $w \in PD^+$, or that π is a mapping of PD^+ onto P_0D^+.

Now we extend π to the mapping of PD onto P_0D. By Theorem 2, any function u in PD can be expressed as

$$u = u' - u'',$$

where u' and u'' are in PD^+. We define πu by

$$\pi u = \pi u' - \pi u'',$$

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which belongs to P_0D. This definition does not depend on the special choice of the decomposition $u = u' - u''$. In fact, assume that $u = \tilde{u}' - \tilde{u}''$ is another such decomposition. Then $u' + \tilde{u}'' = \tilde{u}' + u''$. Hence by (3), $\pi u' + \pi \tilde{u}'' = \pi \tilde{u}' + \pi u''$ or $\pi u' - \pi u'' = \pi \tilde{u}' - \pi \tilde{u}''$. It is easily seen from (3) and (4) that π is a linear mapping of PD into P_0D.

By Remark 1 to Theorem 2, any v in P_0D can be expressed as $v = v' - v''$, where v' and v'' are in P_0D^+. As π maps PD^+ onto P_0D^+, so the extended π maps PD onto P_0D.

Finally we show that π is one-to-one. Assume that $\pi u = 0$ for a u in PD. We have to show that $u = 0$. Let $u = u' - u''$ be a decomposition such that u' and u'' are in PD^+. Then $\pi u' = \pi u''$ or $u' - u'^* = u'' - u''^*$ or

$$u' - u'' = u'^* - u''^*.$$

From this we have to conclude that $u' - u'' \equiv 0$. Contrary to the assertion assume that $u' - u'' \not\equiv 0$. Let $\{u_n'^*\}$ and $\{u_n''^*\}$ be defining sequences of u'^* and u''^* respectively. Then $u_n'^* - u_n''^*$ vanishes on ∂R_n and $v_n'^* - v_n''^* = v_n' - v_n''$ on ∂R_0. By the maximum principle,

$$|u_n'^* - u_n''^*| \leq \sup_{\partial R_0} |u' - u''|.$$

As $u_n'^* - u_n''^*$ converges to $u'^* - u''^*$, so

$$|u'^* - u''^*| \leq \sup_{\partial R_0} |u' - u''|.$$

Thus we have

$$\begin{aligned} \sup_R |u' - u''| &= \max(\sup_{R - \bar{R}_0} |u' - u''|, \sup_{R_0} |u' - u''|) \\ &= \max(\sup_{R - \bar{R}_0} |u'^* - u''^*|, \sup_{\partial R_0} |u' - u''|) \\ &= \sup_{\partial R_0} |u' - u''|. \end{aligned}$$

There exists a point p in ∂R_0 such that

$$\sup_{\partial R_0} |u' - u''| = |u'(p) - u''(p)|.$$

Replacing $u' - u''$ by $u'' - u'$, if necessary, we may assume that

$$u'(p) - u''(p) > 0.$$

Then there exists a compact neighborhood U of such that $u' - u'' > 0$ in U. As $u' - u''$ does not take its maximum in U unless it is a constant, so $u' - u'' \equiv c$ in U, where c is a positive constant. Now we put

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$$R_c = \{q \in R;\ u'(q) - u''(q) = c\}.$$

Then R_c contains p and so it is not empty. By the similar argument as above, R_c is open in R. Clearly R_c is closed. Thus by the connectedness of R, $R_c = R$, or $u' - u'' \equiv c > 0$. This is a contradiction, since any non-zero constant is not a solution. Thus we have proved $u' - u'' \equiv 0$ or $u \equiv 0$. This shows that π is one-to-one.

Hence π is a one-to-one linear mapping of PD onto P_0D and so the class PD is isomorphic to P_0D as vector spaces. Q.E.D.

11. Remark to Theorem 3. In the proof of Theorem 3, we have constructed an isomorphism π of PD onto P_0D. From the proof we can easily see that

$$\sup_{R-\bar{R}_0} |\pi u| = \sup_R |u|.$$

Hence π and π^{-1} preserve boundedness. If we denote by P_0BD the totality of bounded functions in P_0D, then we can state that the normed spaces PBD and P_0BD are isometrically isomorphic, where the norms in PBD and P_0BD are sup. norms.

Thus the condition $R \notin O_{PBD}$ is equivalent to the fact $P_0BD \neq \{0\}$.

Hilbert space P_0D and its reproducing kernel

12. Only in this section we admit the case $P \equiv 0$. Let R_0 be analytic compact subdomain of R such that $R - \bar{R}_0$ is connected. We denote by P_0D the totality of Dirichlet-finite solutions on $R - \bar{R}_0$ vanishing continuously on ∂R_0. We define the inner product of elements u and v in P_0D by the following

$$(u, v) = \iint_{R-\bar{R}_0} \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx\, dy.$$

Hence $\|u\| = (u, u)^{1/2} = (D_{R-\bar{R}_0}[u])^{1/2}$. First we prove the following

Lemma 5. *There exist a finite-valued function $c(p)$ and a neighborhood $U(p)$ of p in $R - \bar{R}_0$ such that for any function u in the class P_0D*

$$|u(q)| \leq c(p)\|u\|$$

holds for any q in $U(p)$.

Proof. First assume that u belongs to P_0D^{+}. We take a subarc γ of ∂R_0

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with two end points a and b. Let Γ be analytic arc in R with end points a and b such that Γ is contained in $R-\overline{R}_0$ except its end points. Moreover we assume that $\gamma+\Gamma$ is the boundary ∂U of a simply connected subdomain U of $R-\overline{R}_0$ such that U contains p. We denote by h the harmonic function on U with boundary value u on ∂U. By the maximum principle and Dirichlet principle

$$u \leq h$$

on U and

$$D_U[h] \leq D_U[u] \leq \|u\|^2.$$

Let $z=\varphi(q)$ be the direct conformal mapping of U onto $\Pi^+=(z;\ \mathrm{Im}\, z>0)$ such that $\varphi(b)=\varepsilon>0$ and $\varphi(a)=-\varepsilon$ and $\varphi(p)=y_0 i\ (y_0>0)$. We put $H(z)=h(\varphi^{-1}(z))$ on Π^+. As $H(x)=0$ on $-\varepsilon<x<\varepsilon$, so $H(z)$ can be harmonically continued to $\Pi=(z;\ |z|<\infty)-(x;\ x\geq\varepsilon$ or $x\leq-\varepsilon)$. We denote by $\hat{H}(z)$ this extended function. We set $W=\{z;\ |z-y_0 i|<y_0\}$ and $r=(y_0^2+\varepsilon^2)^{1/2}-y_0$. We denote by $\hat{H}^*(z)$ the conjugate harmonic function of $\hat{H}(z)$ on Π such that $\hat{H}^*(0)=0$ and put $F(z)=\hat{H}(z)+i\hat{H}^*(z)$. Then $F(0)=0$ and

$$\iint_{\Pi} |F'(z)|^2 dx\,dy = D_{\Pi}[\hat{H}] = 2\,D_{\Pi^+}[H] = 2\,D_U[h] \leq 2\|u\|^2.$$

As $|F'(z)|^2$ is subharmonic on Π, so for z_0 in W

$$|F'(z_0)|^2 \leq (1/\pi r^2)\iint_{|z-z_0|<r} |F'(z)|^2 dx\,dy.$$

Thus we have for z_1 in W

$$H(z_1) \leq |F(z_1)| \leq \left|\int_0^{|z_1|} F'(te^{\arg z_1})e^{\arg z_1}dt\right| \leq (2/\pi r^2)^{1/2}\|u\|.$$

Hence if we set $U(p)=\varphi^{-1}(W)$, then

$$u(q) \leq h(q) = H(\varphi(q)) \leq (2/\pi r^2)^{1/2}\|u\|.$$

For an arbitrary u in P_0D, we can apply Jordan decomposition $u=u\vee 0+u\wedge 0$, since P_0D is a vector lattice (cf. Theorem 2 and Remark 1 to Theorem 2). By Remark 2 to Theorem 2, $D_{R-\overline{R}_0}[u\vee 0]$, $D_{R-\overline{R}_0}[u\wedge 0] \leq D_{R-\overline{R}_0}[u]$. Then from the above

$$|u(q)| \leq |(u\vee 0)(q)| + |(-u\vee 0)(q)| \leq c(p)\|u\|,$$

where $c(p)=2(2/\pi r^2)^{1/2}$. Q.E.D.

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THEOREM 4. *The class P_0D forms a Hilbert space with respect to the inner product* $(u, v) = (D_{R-\bar{R}_0}[u+v] - D_{R-\bar{R}_0}[u-v])/2$ *and this Hilbert space posesses the reproducing kernel* $k(p, q)$, i.e. *the symmetric function on* $(R-\bar{R}_0) \times (R-\bar{R}_0)$ *such that* $k(p, q)$ *belongs to the space* P_0D *as the function of* p *and for any* u *in* P_0D

$$u(q) = (u(p),\ k(p, b)).$$

Proof. To show that P_0D is a Hilbert space, we have only to prove that P_0D is complete. Let $\{u_n\}$ be a Cauchy sequence in the inner product space P_0D. By Lemma 5, u_n converges to a function u on $R-\bar{R}_0$ uniformly on each compact subset of $R-\bar{R}_0$. Hence u is a solution on $R-\bar{R}_0$. It is easy to see that u vanishes continuously on ∂R_0. By Fatou's lemma

$$\|u-u_n\| \leq \underline{\lim}_m \|u_m - u_n\|.$$

Hence u belongs to the class P_0D and $\lim_n \|u-u_n\| = 0$.

To prove the second part we notice that by Lemma 5 the linear functional $u \to u(p)$ is bounded. Thus by Riesz's theorem there exists an element u_p in P_0D such that

$$u(p) = (u,\ u_p).$$

As $u_p(q) = (u_p,\ u_q) = (u_q,\ u_p) = u_q(p)$, so by putting $u_q(p) = k(p, q)$ we get the required kernel $k(p, q)$ of P_0D. Q.E.D.

The property O_{PD} and a maximum principle

13. A. Mori [1] proved that R belongs to O_{HD} if and only if one of the following holds; $\sup_{R-\bar{R}_0} u = \sup_{\partial R_0} u$ and $\inf_{R-\bar{R}_0} u = \inf_{\partial R_0} u$, where R_0 is an analytic compact subdomain of R such that $R-\bar{R}_0$ is connected and u is an arbitrary function in $HD(R-\bar{R}_0)$ such that u is continuous on $R-R_0$. We shall show that the corresponding fact also holds for O_{PD}. In this case, the above two inequalitsies can be replaced by $\sup_{R-\bar{R}_0}|u| = \sup_{\partial R_0}|u|$.

THEOREM 5. *The following three statements are mutually equivalent.*

(a) *R belongs to O_{PD};*

(b) *for any analytic compact subdomain R_0 such that $R-\bar{R}_0$ is connected, it holds that*

$$\sup_{R-\bar{R}_0}|u| = \sup_{\partial R_0}|u|$$

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for any u in $PD(R-\bar{R}_0)$ such that u is continuous on $R-R_0$;

(c) *there exists an analytic compact subdomain R_0 such that $R-\bar{R}_0$ is connected and*

$$\sup_{R-\bar{R}_0}|u| = \sup_{\partial R_0}|u|$$

for any u in $PD(R-\bar{R}_0)$ such that u is continuous on $R-R_0$.

Proof. (a) implies (b). To prove this, take an arbitrary u in $PD(R-\bar{R}_0)$ which is continuous on $R-R_0$. By the remark 1 to Theorem 2, u can be decomposed as

$$u = u_1 - u_2,$$

where u_1 and u_2 are in $PD^+(R-\bar{R}_0)$ which are continuous on $R-R_0$ and

$$u_1 = \max(u, 0)$$

and

$$u_2 = -\min(u, 0)$$

on ∂R_0. We take an exhaustion $\{R_n\}_0^\infty$ of R such that R_n is an analytic compact subdomain of R. Let $v_{i,n}$ be the solution in $R_n-\bar{R}_0$ with boundary value u_i on ∂R_0 and 0 on ∂R_n. By the maximum principle,

$$0 \le v_{i,n} \le v_{i,n+1} \le \sup_{\partial R_0} u_i$$

and by the energy principle, $E_{R_n-\bar{R}_0}[v_{i,n}] \ge E_{R_{n+1}-\bar{R}_0}[v_{i,n+1}]$. By the Harnack type theorem and by Fatou's lemma, $\{v_{i,n}\}_{n=1}^\infty$ converges to a solution v_i in $R-\bar{R}_0$ such that

$$0 \le v_i \le \sup_{\partial R_0} u_i$$

and $v_i = u_i$ on ∂R_0 and

$$E_{R-\bar{R}_0}[v_i] \le \varliminf_n E_{R_n-\bar{R}_0}[v_{i,n}] \le E_{R_1-\bar{R}_0}[v_{i,1}]$$

and a fortiori

$$D_{R-\bar{R}_0}[v_i] < \infty.$$

Hence by using $\sup_{\partial R_0}|u| = \max(\sup_{\partial R_0} u_i;\ i=1,2)$,

$$\begin{aligned}\sup_{R-R_0} u &\le \max(\sup_{R-\bar{R}_0} u_i;\ i=1,2)\\ &= \max(\sup_{\partial R_0} u_i;\ i=1,2) = \sup_{\partial R_0}|u|.\end{aligned}$$

From this we get (b).

The implication (b) → (c) is clear. Finally we show that (c) implies (a). Contrary to the assertion, assume that R does not belong to O_{PD}. Then by Theorem 3, P_0D contains a function u which is not identically zero. By Remark to Theorem 2, we may assume $u>0$ in $R-\overline{R}_0$ and $u=0$ on ∂R_0. Then

$$\sup_{R-\overline{R}_0}|u|>\sup_{\partial R_0}u=0,$$

which contradicts the assumption (c). Q.E.D.

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