Invariant matrices and S-functions

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1. It is known that an induced matrix of an induced matrix is expressible as the direct sum of invariant matrices, or more generally that an invariant matrix of an invariant matrix can be expressed as a direct sum[1] of invariant matrices. The spurs of the irreducible invariant matrices of a given matrix $A = [a_{st}]$, are the S-functions[2] of the latent roots of A.

If $A^{[\lambda]} = A^{[\lambda_1, \lambda_2, \ldots, \lambda_p]}$ denotes the invariant matrix with spur $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$, then

$$[A^{[\lambda]}]^{[\mu]} = \Sigma\, K_{\lambda\mu\nu}\, A^{[\nu]}, \tag{1}$$

where Σ stands for the direct sum.

2. From (1) D. E. Littlewood defines a new type of multiplication[3] for S-functions, namely

$$\{\lambda\} \otimes \{\mu\} = \Sigma\, K_{\lambda\mu\nu}\, \{\nu\}. \tag{2}$$

He has also shown that a problem involving the concomitants of polynomials may be solved by expressing the induced matrix of an induced matrix as the direct sum of invariant matrices. Hence the multiplication in (2) has especial importance when $\{\lambda\}$ and $\{\mu\}$ are the aleph symmetric functions, h_r's, and we therefore proceed to obtain a multiplication table for all cases when the weight of $\{\nu\}$ is $\leqq 12$, with the help of the tables of characters of the symmetric group.

2.1. Let $a, b, c, \ldots$ be the latent roots of A, and h_r and s_r be their aleph and power-sum symmetric functions. The spur of A is clearly h_1.

If the matrix is in canonical form, the diagonal elements will be $a, b, c, \ldots$, the spur of the mth induced matrix is $h_m = \Sigma\, a^r\, b^s\, c^t \ldots$

[1] Schur, "Ueber eine Klasse von Matrizen," *Diss. Berlin* (1901).

[2] See D. E. Littlewood and Richardson, *Phil. Trans. Roy. Soc.* (A) 233 (1934), 107-115 for definition, and also Schur, *loc. cit.*

[3] D. E. Littlewood, *Journal London Math. Soc.* 11 (1936), 49-54. I am indebted to Mr Littlewood for suggesting the problem.

Hence the canonical form contains elements $a^r\, b^s\, c^t \ldots.$ in the leading diagonal[1].

If H_r and Z_r are the aleph and the power-sum symmetric functions of the latent roots of this matrix, then H_r is the spur of the rth induced matrix, and hence we may write

$$h_m \otimes h_r = H_r.$$

3. The problem now reduces to the expression of H_r as the sum of S-functions of $a, b, c \ldots.$

Now[2]

$$Z_1 = H_1 = h_m = \frac{1}{m!}\, \Sigma\, h_\rho\, s_\rho,$$

where h_ρ denotes the order of the class ρ and $s_\rho = s_1^p\, s_2^q \ldots.$, the class ρ having p cycles on 1 symbol, q cycles on 2 and so on, and

$$Z_2 = \Sigma\, a^{2r}\, b^{2s} \ldots. = \Sigma\, (a^2)^r\, (b^2)^s \ldots.;$$

hence Z_2 can be obtained from Z_1 by replacing s_m by s_{2m}. Similarly Z_r can be obtained from Z_1 by replacing s_m by s_{rm}.

H_r is expressible as a function of Z_r's, which are known as functions of s_r's, and so H_r is known in terms of s_r's.

To express H_r as a linear function of S-functions, we read down the columns in the tables of characters to obtain the coefficients. Thus the multiplication $\{m\} \otimes \{r\}$ is obtained and the table constructed.

As a check for the calculations we use

$$\frac{(mr)!}{(m!)^r\, r!} = \Sigma\, K_{mr\nu}\, \chi_0^{(\nu)}.$$

The check may be derived as follows. Putting $s_1 = 1$, $s_2 = s_3 = \ldots. = 0$, we have

$$Z_1 = \frac{1}{m!}, \quad Z_2 = Z_3 = \ldots. = 0,$$

$$H_r = \frac{1}{r!}\, Z_1^r = \frac{1}{r!\,(m!)^r}. \qquad (3)$$

Also from[3]

$$n!\,\{\lambda\} = \Sigma\, h_\rho\, \chi_\rho^{(\lambda)}\, S_\rho,$$

we get

$$\{\lambda\} = \frac{1}{n!}\, \chi_0^{(\lambda)}.$$

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[1] Schur, *op. cit.*, pp. 10, 11.

[2] D. E. Littlewood and A. R. Richardson, *op. cit.*, p. 109.

[3] Littlewood and Richardson, *Phil. Trans. Roy. Soc.* (A) 233 (1934), 109.

Since H_r is expressible as a sum of S-functions, therefore

$$H_r = \frac{1}{(mr)!} \Sigma K_{rm\nu} \chi_0^{(\nu)}$$

which combined with (3) gives the check.

4. The actual working will be shown by obtaining $\{2\} \otimes \{3\}$.

$$Z_1 = H_1 = h_2 = \tfrac{1}{2}(s_1^2 + s_2)$$
$$Z_2 = \tfrac{1}{2}(s_2^2 + s_4)$$
$$Z_3 = \tfrac{1}{2}(s_3^2 + s_6)$$
$$H_3 = \tfrac{1}{6}(Z_1^3 + 3Z_1 Z_2 + 2Z_3).$$
$$48H_3 = s_1^6 + 3s_1^4 s_2 + 6s_1^2 s_4 + 9s_1^2 s_2^2 + 8s_6 + 6s_2 s_4 + 7s_2^3 + 8s_3^2.$$

From the table of degree 6, H_3 turns out to be

$$\{6\} + \{42\} + \{2^3\}.$$

Therefore $$\{2\} \otimes \{3\} = \{6\} + \{42\} + \{2^3\}.$$

5. With the help of the tables[1] of the group characters the following *multiplication table* is constructed.

$$\{2\} \otimes \{2\} = \{4\} + \{2^2\}.$$
$$\{2\} \otimes \{3\} = \{6\} + \{42\} + \{2^3\}.$$
$$\{2\} \otimes \{4\} = \{8\} + \{62\} + \{4^2\} + \{42^2\} + \{2^4\}.$$
$$\{2\} \otimes \{5\} = \{10\} + \{82\} + \{64\} + \{62^2\} + \{2^5\} + \{4^2\,2\} + \{42^3\}.$$
$$\{2\} \otimes \{6\} = \{12\} + \{10, 2\} + \{84\} + \{82^2\} + \{6^2\} + \{642\} + \{62^3\} + \{4^3\} + \{4^2\,2^2\} + \{42^4\} + \{2^6\}.$$
$$\{3\} \otimes \{2\} = \{6\} + \{42\}.$$
$$\{3\} \otimes \{3\} = \{9\} + \{72\} + \{63\} + \{52^2\} + \{4^2\,1\}.$$
$$\{3\} \otimes \{4\} = \{12\} + \{10, 2\} + \{93\} + \{84\} + \{82^2\} + \{741\} + \{732\} + \{6^2\} + \{642\} + \{62^3\} + \{5421\} + \{4^3\}.$$
$$\{4\} \otimes \{2\} = \{8\} + \{62\} + \{4^2\}.$$
$$\{4\} \otimes \{3\} = \{12\} + \{10, 2\} + \{93\} + \{84\} + \{82^2\} + \{741\} + \{6^2\} + \{642\} + \{4^3\}.$$
$$\{5\} \otimes \{2\} = \{10\} + \{82\} + \{64\}.$$
$$\{6\} \otimes \{2\} = \{12\} + \{10, 2\} + \{84\} + \{6^2\}.$$

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[1] Tables as far as 10th degree will be found in D. E. Littlewood's paper, *Proc. London Math. Soc.* (2) 39 (1935), 177-183. The 12th degree table is to appear in *Proc. London Math. Soc.*