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A ZERO-$\sqrt{5}/2$ LAW FOR COSINE FAMILIES

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Abstract

Let $a \in \mathbb{R}$, and let $k(a)$ be the largest constant such that $\sup |\cos(na) - \cos(nb)| < k(a)$ for $b \in \mathbb{R}$ implies that $b \in \pm a + 2\pi\mathbb{Z}$. We show that if a cosine sequence $(C(n))_{n\in\mathbb{Z}}$ with values in a Banach algebra A satisfies $\sup_{n\geq 1} \|C(n) - \cos(na).1_A\| < k(a)$, then $C(n) = \cos(na).1_A$ for $n \in \mathbb{Z}$. Since $\sqrt{5}/2 \leq k(a) \leq 8/3\sqrt{3}$ for every $a \in \mathbb{R}$, this shows that if some cosine family $(C(g))_{g\in G}$ over an abelian group G in a Banach algebra satisfies $\sup_{g\in G} \|C(g) - c(g)\| < \sqrt{5}/2$ for some scalar cosine family $(c(g))_{g\in G}$, then $C(g) = c(g)$ for $g \in G$, and the constant $\sqrt{5}/2$ is optimal. We also describe the set of all real numbers $a \in [0, \pi]$ satisfying $k(a) \leq \frac{3}{2}$.

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1. Introduction

Let G be an abelian group. Recall that a G-cosine family of elements of a unital normed algebra A with unit element 1_A is a family $(C(g))_{g\in G}$ of elements of A satisfying the so-called d'Alembert equation

$$C_0 = 1_A, C(g + h) + C(g - h) = 2C(g)C(h), \quad (g \in G, h \in G).$$

A $\mathbb{R}$-cosine family is called a cosine function, and a $\mathbb{Z}$-cosine family is called a cosine sequence.

A cosine family $C = (C(g))_{g\in G}$ is said to be bounded if there exists $M > 0$ such that $\|C(g)\| \leq M$ for every $g \in G$. In this case, we set

$$\|C\|_\infty = \sup_{g\in G} \|C(g)\|, \quad \operatorname{dist}(C_1, C_2) = \|C_1 - C_2\|_\infty.$$

A cosine family is said to be scalar if $C(g) \in \mathbb{C}.1_A$ for every $g \in G$. It is easy to see and well known that a bounded complex-valued cosine sequence satisfies $C(n) = \cos(an)$ for some $a \in \mathbb{R}$.

Strongly continuous operator valued cosine functions are a classical tool in the study of differential equations (see, for example, [1, 3, 14, 18]) and a functional calculus approach to these objects was developed recently in [10, 11].

Bobrowski and Chojnacki proved in [4] that if a strongly continuous operator valued cosine function on a Banach space $(C(t))_{t\in\mathbb{R}}$ satisfies $\sup_{t\geq 0}\|C(t)-c(t)\| < 1/2$ for some scalar bounded continuous cosine function $c(t)$, then $C(t)=c(t)$ for $t\in\mathbb{R}$, and Zwart and Schwenninger showed in [16] that this result remains valid under the condition $\sup_{t\geq 0}\|C(t)-c(t)\| < 1$. The proofs were based on rather involved arguments from operator theory and semigroup theory. Very recently, Bobrowski *et al.* [5] showed more precisely that if a cosine function $C=C(t)$ satisfies $\sup_{t\in\mathbb{R}}\|C(t)-c(t)\| < 8/3\sqrt{3}$ for some scalar bounded continuous cosine function $c(t)$, then $C(t)=c(t)$ for $t\in\mathbb{R}$, without any continuity assumption on C, and the same result was obtained independently by the author in [9]. The constant $8/3\sqrt{3}$ is obviously optimal, since $\sup_{t\in\mathbb{R}}|\cos(at)-\cos(3at)| = 8/3\sqrt{3}$ for every $a\in\mathbb{R}\setminus\{0\}$.

The author also proved, in [9], that if a cosine sequence $(C(t))_{t\in\mathbb{R}}$ satisfies $\sup_{t\in\mathbb{R}}\|C(t)-\cos(at)1_A\| = m < 2$ for some $a\neq 0$, then the closed algebra generated by $(C(t))_{t\in\mathbb{R}}$ is isomorphic to $\mathbb{C}^k$ for some $k\geq 1$, and there exists a finite family $p_1,\ldots,p_k$ of pairwise orthogonal idempotents of A and a family $(b_1,\ldots,b_k)$ of distinct elements of the finite set $\Delta(a,m) := \{b\geq 0 : \sup_{t\in\mathbb{R}}|\cos(bt)-\cos(at)|\leq m\}$ such that $C(t)=\sum_{j=1}^{k}\cos(b_jt)p_j\ \ (t\in\mathbb{R})$.

Also, Chojnacki developed, in [6], an elementary argument to show that if $(C(n))_{n\in\mathbb{Z}}$ is a cosine sequence in a unital normed algebra A satisfying $\sup_{n\geq 1}\|C(n)-c(n)\| < 1$ for some scalar cosine sequence $(c(n))_{n\in Z}$, then $c(n)=C(n)$ for every n, which obviously implies the result of Zwart and Schwenninger. His approach is based on an elaborated adaptation of a very short elementary argument used by Wallen in [19] to prove an improvement of the classical Cox–Nakamura–Yoshida–Hirschfeld–Wallen theorem [7, 12, 15] which shows that if an element a of a unital normed algebra A satisfies $\sup_{n\geq 1}\|a^n-1\| < 1$, then $a=1$.

Applying this result to the cosine sequences $C(ng)$ and $c(ng)$ for $g\in G$, Chonajcki observed, in [6], that if a cosine family $C(g)$ satisfies $\sup_{g\in G}\|C(g)-c(g)\| < 1$ for some scalar cosine family $c(g)$, then $C(g)=c(g)$ for every $g\in G$.

In the same direction, Schwenninger and Zwart showed, in [17], that if a cosine sequence $(C(n))_{n\in\mathbb{Z}}$ in a Banach algebra A satisfies $\sup_{n\geq 1}\|C(n)-1_A\| < \frac{3}{2}$, then $C(n)=1_A$ for every n.

The purpose of this paper is to obtain optimal results of this type. We prove a 'zero-$\sqrt{5}/2$' law: if a cosine family $(C(g))_{g\in G}$ satisfies $\sup_{g\in G}\|C(g)-c(g)\| < \sqrt{5}/2$ for some scalar cosine family $(c(g))_{g\in G}$, then $C(g)=c(g)$ for every $g\in G$. Since $\sup_{n\geq 1}|\cos(2n\pi/5)-\cos(4n\pi/5)| = \cos(2\pi/5)+\cos(\pi/5) = \sqrt{5}/2$, the constant $\sqrt{5}/2$ is optimal.

In fact, for every $a\in\mathbb{R}$, there exists a largest constant $k(a)$ such that $\sup_{n\geq 1}|\cos(nb)-\cos(na)| < k(a)$ implies that $\cos(nb)=\cos(na)$ for $n\geq 1$, and there exists $b\in\mathbb{R}$ such that $\sup_{n\geq 1}|\cos(na)-\cos(nb)| = k(a)$ (see the remark following

Proposition 2.2). We prove that if a cosine sequence $(C(n))_{n\in\mathbb{Z}}$ in a Banach algebra A satisfies $\sup_{n\geq 1}|C(n) - \cos(na)1_A| < k(a)$, then $C(n) = \cos(na).1_A$ for $n \geq 1$. This follows from the following result, which was proved by the author in [9].

THEOREM 1.1. *Let $(C(n))_{n\in\mathbb{Z}}$ be a bounded cosine sequence in a Banach algebra A. If* $\operatorname{spec}(C(1))$ *is a singleton, then the sequence $(C(n))_{n\in\mathbb{Z}}$ is scalar, and so there exists $a \in \mathbb{R}$ such that $C(n) = \cos(na).1_A$ for $n \geq 1$.*

The second part of the paper is devoted to a discussion of the values of the constant $k(a)$. As mentioned above, it follows from [17] that $k(0) = \frac{3}{2}$, and it is obvious that $k(a) \leq \sup_{n\geq 1}|\cos(na) - \cos(3na)| \leq 8/3\sqrt{3}$ if $a \notin (\pi/2)\mathbb{Z}$. We observe that $k(a) = 8/3\sqrt{3}$ if a/π is irrational, and we prove, using basic results about cyclotomic fields, that $k(a) < 8/3\sqrt{3}$ if a/π is rational.

We also show that the set $\Omega(m) := \{a \in [0, \pi] : k(a) \leq m\}$ is finite for every $m < 8/3\sqrt{3}$. We describe in detail the set $\Omega(\frac{3}{2})$: it contains 43 elements, and the only values for $k(a)$ for which $k(a) < \frac{3}{2}$ are $\sqrt{2}/5 = \cos(\pi/5) + \cos(2\pi/5) \approx 1.1180$, $\sqrt{2} = \cos(\pi/4) + \cos(3\pi/4) \approx 1.4142$ and $\cos(2\pi/11) + \cos(3\pi/11) \approx 1.4961$.

The zero-$\sqrt{5}/2$ law follows from the fact that $k(a) \geq \cos(\pi/5) + \cos(2\pi/5) = \sqrt{5}/2$ for every $a \in \mathbb{R}$.

We also show that, given $a \in \mathbb{R}$ and $m < 2$, the set $\Gamma(a, m)$ of scalar cosine sequences $(c(n))_{n\in\mathbb{Z}}$ satisfying $\sup_{n\in\mathbb{Z}}|c(n) - \cos(na)| \leq m$ is finite. This implies that if a cosine sequence $(C(n))_{n\in\mathbb{Z}}$ satisfies $\sup_{n\in\mathbb{Z}}\|C(n) - \cos(an)1_A\| \leq m$, then there exists $k \leq \operatorname{card}(\Gamma(a, m))$ such that the closed algebra generated by $(C(n))_{n\in\mathbb{Z}}$ is isomorphic to $\mathbb{C}^k$ and there exists a finite family $p_1, \ldots, p_k$ of pairwise orthogonal idempotents of A and a finite family $c_1, \ldots, c_k$ of distinct elements of $\Gamma(a, m)$ such that

$$C(n) = \sum_{j=1}^{k} \cos(c_j n)p_j, \quad (n \in \mathbb{Z}).$$

This last result does not extend to cosine families over the general abelian group. Let $G = (\mathbb{Z}/3\mathbb{Z})^{\mathbb{N}}$; we give an easy example of a G-cosine family $(C(g))_{g\in G}$ with values in l^∞ such that the closed subalgebra generated by $(C(g))_{g\in G}$ equals l^∞, while $\sup_{g\in G}\|1_{l^\infty} - C(g)\| = \frac{3}{2}$.

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2. Distance between bounded scalar cosine sequences

We introduce the following notation, to be used throughout the paper.

DEFINITION 2.1. Let $a \in \pi\mathbb{Q}$. The order of a, denoted by $\operatorname{ord}(a)$, is the smallest integer $u \geq 1$ such that $e^{iua} = 1$.

Recall that a subset S of the unit circle $\mathbb{T}$ is said to be independent if $z_1^{n_1}\cdots z_k^{n_k}\neq 1$ for every finite family $(z_1,\ldots,z_k)$ of distinct elements of S and every family $(n_1,\ldots,n_k)\in\mathbb{Z}^k$ such that $n_j\neq 0$ for $1\le j\le k$. It follows from a classical theorem of Kronecker (see, for example, [13], page 21) that if $S=\{z_1,\ldots,z_k\}$ is a finite independent set, then the sequence $(z_1^n,\ldots,z_k^n)_{n\ge 1}$ is dense in $\mathbb{T}^k$. We deduce from Kronecker's theorem the following observation.

Proposition 2.2. *Let $a\in[0,\pi]$. For $m\ge 0$, set*

$$\Gamma(a,m)=\Big\{b\in[0,\pi]:\sup_{n\ge 1}|\cos(na)-\cos(nb)|\le m\Big\}.$$

Then $\Gamma(a,m)$ is finite for every $m<2$.

Proof. Fix $m\in[1,2)$. Notice that if $b\in\mathbb{R}$ and if the set $\{e^{ia},e^{ib}\}$ is independent, then it follows from Kronecker's theorem that the sequence $((e^{ina},e^{inb}))_{n\ge 1}$ is dense in $\mathbb{T}^2$, and so $\sup_{n\ge 1}|\cos(na)-\cos(nb)|=2$ and $b\notin\Gamma(a,m)$.

Suppose that $(a/\pi)\in\mathbb{Q}$, and denote by u the order of a, so that $e^{iua}=1$. If $(b/\pi)\notin\mathbb{Q}$, then the sequence $(e^{iunb})_{n\ge 1}$ is dense in $\mathbb{T}$, and so

$$2\ge\sup_{n\ge 1}|\cos(na)-\cos(nb)|\ge\sup_{n\ge 1}|1-\cos(nub)|=2,$$

which shows that $b\notin\Gamma(a,m)$.

The same argument shows that if $(a/\pi)\notin\mathbb{Q}$ and if $(b/\pi)\in\mathbb{Q}$, then $b\notin\Gamma(a,m)$. So we are left with two situations:

(1) $a/\pi\notin Q$, and there exists $p\neq 0$, $q\neq 0$ and $k\in\mathbb{Z}$ such that $bq=ap+2k\pi$; and
(2) $a/\pi\in\mathbb{Q}$ and $b/\pi\in\mathbb{Q}$.

We consider the first case. Replacing $b\in[0,\pi]$ by $-b\in[-\pi,0]$, if necessary, we can assume that $p\ge 1$ and $q\ge 1$, and we can assume that

$$qb=pa+\frac{2k\pi}{r},$$

with greatest common divisor (gcd)$(p,q)=1$, $r\ge 1$, $\gcd(r,k)=1$ if $k\neq 0$.

Since $(ra/\pi)\notin\mathbb{Q}$,

$$\begin{aligned}\sup_{n\ge 1}&|\cos(na)-\cos(nb)|\\ &\ge\sup_{n\ge 1}|\cos(nrqa)-\cos(nrqb)|\\ &=\sup_{n\ge 1}|\cos(nrqa)-\cos(nrpa)|=\sup_{t\in\mathbb{R}}|\cos(qt)-\cos(pt)|.\end{aligned}$$

Since $\gcd(p,q)=1$, $\sup_{t\in\mathbb{R}}|\cos(qt)-\cos(pt)|=2$ if p or q is even, so we can assume that p and q are odd. Set $s=(q-1)/2$.

It follows from Bezout's identity that there exist $n\ge 1$ such that $e^{2inp\pi/q}=e^{2is\pi/q}$, and setting $t=2n\pi/q$,

$$\sup_{t\in\mathbb{R}}|\cos(qt)-\cos(pt)|\ge 1-\cos\Big(\frac{2s\pi}{2s+1}\Big)=1+\cos\Big(\frac{\pi}{q}\Big).$$

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The same argument shows that

$$\sup_{t\in\mathbb{R}} |\cos(qt) - \cos(pt)| \geq 1 + \cos\Big(\frac{\pi}{p}\Big).$$

Hence

$$p \leq \frac{\pi}{\arccos(m-1)}, \quad q \leq \frac{\pi}{\arccos(m-1)}.$$

Also

$$\sup_{n\geq 1} |\cos(na) - \cos(nb)| \geq \sup_{n\geq 1} |\cos(nqa) - \cos(nqb)| = \sup_{n\geq 1}\Big|\cos(nqa) - \cos\Big(npa + \frac{2nk\pi}{r}\Big)\Big|.$$

Assume that $k \neq 0$. Since $\gcd(k, r) = 1$, there exists $u \geq 1$ such that $2uk\pi/r \in (2\pi/r) + 2\pi\mathbb{Z}$. This gives

$$\sup_{n\geq 1} |\cos(na) - \cos(nb)| \geq \sup_{n\geq 1}\Big|\cos(nuqa) - \cos\Big(npua + \frac{2n\pi}{r}\Big)\Big|.$$

If r is even, set $r_1 = r/2$.

$$\sup_{n\geq 1}\Big|\cos(nuqa) - \cos\Big(npua + \frac{2n\pi}{r}\Big)\Big| \geq \sup_{n\geq 0} |\cos((2n+1)r_1uqa) - \cos((2n+1)r_1upa) + \pi|.$$

Since $2r_1ua \notin \pi\mathbb{Q}$, there exists a sequence $(n_j)_{j\geq 1}$ of integers such that

$$\lim_{j\to+\infty} |e^{i2n_jr_1ua+ir_1ua}| = 1,$$

so that

$$\lim_{j\to+\infty} |\cos((2n_j+1)r_1uqa) - \cos((2n_j+1)r_1upa) + \pi| = 2,$$

and, in this situation, $\sup_{n\geq 1} |\cos(na) - \cos(nb)| = 2$.

So we can assume that r is odd. Set $r_1 = (r-1)/2$. The same calculation as above gives

$$\begin{aligned}\sup_{n\geq 1}\Big|\cos(nuqa) - \cos\Big(npua + \frac{2n\pi}{r}\Big)\Big| &\geq \sup_{n\geq 1}\Big|\cos((nr+r_1)uqa) - \cos\Big((nr+r_1)upa + \frac{2(nr+r_1)}{r}\pi\Big)\Big| \\ &\geq 1 + \cos\Big(\frac{2r_1}{r}\pi - \pi\Big) = 1 + \cos\Big(\frac{\pi}{r}\Big).\end{aligned}$$

Hence $r \leq \pi/\arccos(m-1)$.

This gives

$$|k| \leq \frac{r}{2\pi}|qb - pa| \leq \Big(\frac{\pi}{\arccos(m-1)}\Big)^2.$$

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We see that $\Gamma(a, m)$ is finite if $a/\pi \notin Q$ and that

$$\mathrm{card}(\Gamma(a, m)) \le \left(\frac{2\pi}{\arccos(m-1)}\right)^5.$$

Now consider the case where $a/\pi \in \mathbb{Q}$, $b/\pi \in \mathbb{Q}$. We first discuss the case where $a = 0, b \neq 0$. We know that $b = p\pi/q$, where $1 \le p \le q$, $\gcd(p, q) = 1$.

If $p = q = 1$, then $b = \pi$ and $\sup_{n\ge1} |1 - \cos(n\pi)| = 2$. So we may assume that $p \le q - 1$. If p is odd,

$$\sup_{n\ge1} |1 - \cos(nb)| \ge |1 - \cos(qb)| = 1 - \cos(p\pi) = 2.$$

So we can assume that p is even, so that q is odd. Set $r = (q - 1)/2$. There exists $n_0 \ge 1$ and $r \in \mathbb{Z}$ such that $n_0 p - r \in q\mathbb{Z}$ and

$$\sup_{n\in\mathbb{Z}} |1 - \cos(nb)| \ge |1 - \cos(2n_0 b)| = \left|1 - \cos\left(\frac{2r\pi}{2r+1}\right)\right| = 1 + \cos\left(\frac{\pi}{q}\right).$$

Again $q \le \pi/\arccos(m - 1)$ and $\mathrm{card}(\Gamma(0, m)) \le (\pi/\arccos(m - 1))^2$.

Now assume that $a \neq 0$ and let $u \ge 2$ be the order of a.

$$\sup_{n\ge1} |1 - \cos(nub)| = \sup_{n\ge1} |\cos(nua) - \cos(nub)| \le m,$$

and so there exists $c \in \Gamma(0, m)$ such that $\cos(nc) = \cos(nub)$ for $n \ge 1$. In particular, $\cos(c) = \cos(ub)$, and $b = \pm(c/u) + (2k\pi/u)$, where $k \in \mathbb{Z}$.

$$\mathrm{card}(\Gamma(a, m)) \le 2u\,\mathrm{card}(\Gamma(0, m)) \le 2u\left(\frac{\pi}{\arccos(m-1)}\right)^2. \qquad \square$$

We do not know whether it is possible to obtain a majorant for $\mathrm{card}(\Gamma(a, m))$ which depends only on m when $a \in \pi\mathbb{Q}$.

REMARK. It follows immediately from Proposition 2.2 that, for every $a \in \mathbb{R}$, there exists $b \in \mathbb{R}$ such that $k(a) = \sup_{n\ge1} |\cos(na) - \cos(nb)|$.

THEOREM 2.3. *Let $a \in \mathbb{R}$, let $m < 2$ and let $(C(n))_{n\in\mathbb{Z}}$ be a cosine sequence in a Banach algebra A such that $\sup_{n\ge1} \|C(n) - \cos(na)\| \le m$. Then there exists $k \le \mathrm{card}(\Gamma(a, m))$ such that the closed algebra generated by $(C(n))_{n\in\mathbb{Z}}$ is isomorphic to $\mathbb{C}^k$, and there exists a finite family $p_1, \ldots, p_k$ of pairwise orthogonal idempotents of A and a finite family $b_1, \ldots, b_k$ of distinct elements of $\Gamma(a, m)$ such that*

$$C(n) = \sum_{j=1}^{k} \cos(nb_j)p_j, \quad (n \in \mathbb{Z}).$$

Proof. Since $c_n = P_n(c_1)$, where P_n denotes the nth Tchebishev polynomial, A_1 is the closed unital subalgebra generated by c_1 and the map $\chi \to \chi(c_1)$ is a bijection from $\widehat{A_1}$ onto $\mathrm{spec}_{A_1}(c_1)$. Now let $\chi \in \widehat{A_1}$. The sequence $(\chi(c_n))_{n\geq 1}$ is a scalar cosine sequence and

$$\sup_{n\geq 1} |\cos(na) - \chi(c_n)| < 2.$$

It follows from Proposition 2.2 that $\mathrm{spec}_{A_1}(c_1) := \{\lambda = \chi(c_1) : \chi \in \widehat{A_1}\}$ is finite. Hence $\widehat{A_1}$ is finite. Let $\chi_1, \ldots, \chi_m$ be the elements of $\widehat{A_1}$. It follows from the standard one-variable holomorphic functional calculus (see, for example, [8]) that there exists, for every $j \leq m$, an idempotent p_j of A_1 such that $\chi_j(p_j) = 1$ and $\chi_k(p_j) = 0$ for $k \neq j$. Hence $p_j p_k = 0$ for $j \neq k$, and $\sum_{j=1}^m p_j$ is the unit element of A_1.

Let $x \in A_1$. Then $(p_j c_n)_{n\in\mathbb{Z}}$ is a cosine sequence in the commutative unital Banach algebra $p_j A_1$, and $\mathrm{spec}_{p_j A_1}(p_j c_1) = \{\chi_j(c_1)\}$.

Since $\sup_{n\geq 1} \|p_j \cos(na) - p_j c_n\| \leq 2\|p_j\|$, the sequence $(p_j c_n)_{n\geq 1}$ is bounded, and it follows from Theorem 2.3 that $(p_j c_n)_{n\geq 1}$ is a scalar sequence and there exists $\beta_j \in [0, \pi]$ such that $p_j c_n = \chi_j(c_n) p_j = \cos(n\beta_j) p_j$ for $n \in \mathbb{Z}$.

Hence $c_n = \sum_{j=1}^m \chi_j(c_n) p_j = \sum_{j=1}^m \cos(n\beta_j) p_j$ for $n \geq 1$. Since A_1 is the closed unital subalgebra of A generated by c_1, $x = \sum_{j=1}^m \chi_j(x) p_j$ for every $x \in A_1$, which shows that A_1 is isomorphic to $\mathbb{C}^m$. □

Corollary 2.4. *Let $a \geq 0 \in \mathbb{R}$ and let $k(a)$ be the largest positive real number m such that $\Gamma(a, m) = \{a\}$ for every $m < k(a)$. If $(C(n))_{n\in\mathbb{Z}}$ is a cosine sequence in a Banach algebra A such that $\sup_{n\geq 1}\|C(n) - \cos(na)1_A\| < k(a)$, then $C(n) = \cos(na)1_A$ for $n \in \mathbb{Z}$.*

Theorem 2.3 does not extend to cosine families over general abelian groups, as shown by the following easy result.

Proposition 2.5. *Let $G := (\mathbb{Z}/3\mathbb{Z})^{\mathbb{N}}$. Then there exists a G-cosine family $(C(g))_{g\in G}$ with values in l^∞ which satisfies the following two conditions.*

(i) $\sup_{g\in G} \|1_{l^\infty} - C(g)\| = \frac{3}{2}$.

(ii) *The algebra A generated by the family $(C(g))_{g\in G}$ is dense in l^∞.*

Proof. Elements g of G can be written in the form $g = (\overline{g}_m)_{m\geq 1}$, where $g_m \in \{0, 1, 2\}$. Set

$$C(g) := \left(\cos\left(\frac{2g_m\pi}{3}\right)\right)_{m\geq 1}.$$

Then $(C(g))_{g\in G}$ is a G-cosine family with values in l^∞ which obviously satisfies (i) since $\cos(2\pi/3) = \cos(4\pi/3) = -\frac{1}{2}$.

Now let $\phi = (\phi_m)_{m\in\mathbb{Z}}$ be an idempotent of l^∞ and let $S := \{m \geq 1 \mid \phi_m = 1\}$. Set $g_m = 1$ if $m \in S$, $g_m = 0$ if $m \geq 1, m \notin S$ and set $g = (\overline{g}_m)_{m\geq 1}$.

$$C(0_G) - C(g) = 1_{l^\infty} - C(g) = \tfrac{3}{2}\phi,$$

and so $\phi \in A$. We can identify l^∞ to $\mathscr{C}(\beta\mathbb{N})$, the algebra of continuous functions on the Stone–Cěch compactification of $\mathbb{N}$, and βN is an extremely disconnected compact set,

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which means that the closure of every open set is open (see, for example, [2], Ch. 6, Section 6). Since the characteristic function of every open and closed subset of $\beta\mathbb{N}$ is an idempotent of l^∞, the idempotents of l^∞ separate points of $\beta\mathbb{N}$, and it follows from the Stone–Weierstrass theorem that A is dense in l^∞, which proves (ii). □

3. The values of the constant $k(a)$

It was shown in [17] that $k(0) = \frac{3}{2}$. We also have the following result.

PROPOSITION 3.1. *$k(a) = 8/3\sqrt{3}$ if a/π is irrational and $k(a) < 8/3\sqrt{3}$ if a/π is rational.*

PROOF. Assume that $a/\pi \notin \mathbb{Q}$. Then $3a \notin \pm a + 2\pi\mathbb{Z}$ and

$$k(a) \le \sup_{n\ge 1} |\cos(na) - \cos(3na)| = \sup_{x\in\mathbb{R}} |\cos(x) - \cos(3x)| = \frac{8}{3\sqrt{3}}.$$

We saw above that if b/π in $\mathbb{Q}$, then $\sup_{n\ge1}|\cos(na) - \cos(nb)| = 2$, and we also know that $\sup_{n\ge1} |\cos(na) - \cos(nb)| = 2$ if $pa - qb \notin 2\pi\mathbb{Z}$ for $(p, q) \ne (0, 0)$. So if $\sup_{n\ge1} |\cos(na) - \cos(nb)| < 2$, there exists $p \in \mathbb{Z} \setminus \{0\}$, $q \in \mathbb{Z} \setminus \{0\}$ and $r \in \mathbb{Z}$ such that $pa - qb = 2r\pi$.

If $p \ne \pm q$, then it follows from [9, Lemma 3.5] that

$$\begin{aligned}
\sup_{n\ge1}|\cos(na) - \cos(nb)| &\ge \sup_{n\ge1}|\cos(nqa) - \cos(nqb)| \\
&= \sup_{n\ge1}|\cos(qna) - \cos(pna)| \\
&= \sup_{x\in\mathbb{R}}|\cos(qx) - \cos(px)| \\
&= \sup_{x\in\mathbb{R}}\left|\cos\left(\frac{p}{q}x\right) - \cos(x)\right| \ge \frac{8}{3\sqrt{3}}.
\end{aligned}$$

We are left with the case where $b = \pm a + (2s\pi/r)$, where $r \in \mathbb{Z} \setminus \{-1, 0, 1\}$, and we can restrict attention to the case where $b = a + (2s\pi/r)$, where $r \ge 2$, $1 \le s \le r - 1$, $\gcd(r, s) = 1$. It follows from Bezout's identity that there exists, for every $p \ge 1$, some positive integer u such that $ub - ua - (2p\pi/r) \in 2\pi\mathbb{Z}$. If r is even, set $p = r/2$. Since the set $\{e^{i(2n+1)a}\}_{n\ge1}$ is dense in the unit circle,

$$\begin{aligned}
\sup_{n\ge1} |\cos(nb) - \cos(na)| &\ge \sup_{n\ge1} |\cos((2n + 1)ub) - \cos((2n + 1)ua)| \\
&= 2 \sup_{n\ge1} |\cos((2n + 1)ua)| = 2.
\end{aligned}$$

Now assume that r is odd, and set $p = (r - 1)/2$.

$$\begin{aligned}
&\sup_{n\ge1} |\cos(nb) - \cos(na)| \\
&\quad\ge \sup_{n\ge1} |\cos((2n + 1)ub) - \cos((2n + 1)ua)| \\
&\quad\ge \sup_{n\ge1}\left|\cos\left((2nr + 1)ua + (2nr + 1)\left(\pi - \frac{\pi}{r}\right)\right) - \cos((2nr + 1)ua)\right| \\
&\quad\ge \sup_{x\in\mathbb{R}}\left|\cos(x) + \cos\left(x - \frac{\pi}{r}\right)\right| \ge 2\cos\left(\frac{\pi}{2r}\right) \ge \sqrt{3} > \frac{8}{3\sqrt{3}}.
\end{aligned}$$

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Now assume that a/π is rational. If the order of a is equal to one, then $k(a) = 1.5$, and we will see later that this is also true if the order of a equals two or four.

Otherwise,

$$k(a) \leq \sup_{n\geq 1} |\cos(na) - \cos(3na)| = \max_{1\leq n\leq u} |\cos(na) - \cos(3na)|.$$

We know that $|\cos(nx) - \cos(3nx)| < 8/(\pi\sqrt{3})$ if $x \notin \pm \arccos(1/\sqrt{3}) + \pi\mathbb{Z}$. If $na \in \pm\arccos(1/\sqrt{3}) + \pi\mathbb{Z}$ for some $n \geq 1$, then $\arccos(1/\sqrt{3})/\pi$ would be rational and $\alpha := 1/\sqrt{3} + (\sqrt{2}i)/\sqrt{3}$ would be a root of unity. So $\beta = \alpha^2 = -\frac{1}{3} + (2\sqrt{2}i)/3$ would have the form $\beta = e^{2ik\pi/n}$ for some $n \leq 1$ and some positive integer $k \geq n$ such that $\gcd(k, n) = 1$.

Let $\mathbb{Q}(\beta)$ be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q} \cup \beta$. Since $3\beta^2 + 2\beta + 3 = 0$, the degree of $\mathbb{Q}(\beta)$ over $\mathbb{Q}$ is equal to two. On the other hand, the Galois group $\mathrm{Gal}(\mathbb{Q}(\beta)/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$, the group of invertible elements of $\mathbb{Z}/n\mathbb{Z}$, and (see [20, Theorem 2.5])

$$H(n) = \deg(\mathbb{Q}(\beta)/\mathbb{Q}) = 2,$$

where $H(n) = \mathrm{card}((\mathbb{Z}/n\mathbb{Z})^\times)$ denotes the number of integers $p \in \{1, \ldots, n\}$ such that $\gcd(p, n) = 1$.

Let $P(n)$ be the set of prime divisors of n. It is well known that, writing $n = \Pi_{p\in P(n)} p^{\alpha_p}$ (see, for example, [20, Exercise 1.1]),

$$H(n) = \Pi_{p\in P(n)} p^{\alpha_p - 1}(p - 1).$$

It follows immediately from this identity that the only possibilities for getting $H(n) = 2$ are $n = 3$, $n = 4$ and $n = 6$. Since $\beta^3 \neq 1$, $\beta^4 \neq 1$ and $\beta^6 \neq 1$, we see that β/π is irrational, and so $k(a) < 8/3\sqrt{3}$ if a/π is rational. □

We know that if a/π is rational and if b/π is irrational, then $\sup_{n\geq 1} |\cos(na) - \cos(nb)| = 2$. We discuss now the case where a/π and b/π are both rational, with $b \notin \pm a + 2\pi\mathbb{Z}$.

Lemma 3.2. *Let $a, b \in (0, \pi]$.*

(i) *If $7a \leq b \leq \pi/2$ or if $\pi/2 \leq b \leq 5\pi/6$, with $|b - (2\pi/3)| \geq 7a$, then*

$$\sup_{n\geq 1} |\cos(na) - \cos(nb)| > 1.55.$$

(ii) *If $(5\pi/6) \leq b \leq \pi$ and if $b \geq 4a$, then*

$$\cos(a) - \cos(b) > 1.57.$$

Proof.

(i) Assume that $7a \leq b \leq \pi/2$, let p be the largest integer such that $pb < 3\pi/4$ and set $q = p + 1$. We know that $3\pi/4 \leq qb \leq 5\pi/4$, $0 \leq qa \leq 5\pi/28$, so

$$\sup_{n\geq 1} |\cos(na) - \cos(nb)| \geq \cos(qa) - \cos(qb) \geq \cos\left(\frac{5\pi}{28}\right) + \cos\left(\frac{\pi}{4}\right) > 1.55.$$

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Now assume that $\pi/2 \le b \le 5\pi/6$, with $|b - 2\pi/3| \ge 7a$, and set $c = |3b - 2\pi|$. Since $|b - (2\pi/3)| \le \pi/6$, $21a \le c \le \pi/2$, so

$$\begin{aligned}\sup_{n\ge1} &|\cos(na) - \cos(nb)| \\ &\ge \sup_{n\ge1} |\cos(3na) - \cos(3nb)| \\ &= \sup_{n\ge1} |\cos(3na) - \cos(nc)| \ge |\cos(3a) - \cos(c)| > 1.55.\end{aligned}$$

(ii) If $5\pi/6 \le b \le \pi$ and if $b \ge 4a$, then $0 < a \le \pi/4$ and

$$\cos(a) - \cos(b) \ge \cos\Big(\frac{\pi}{4}\Big) + \cos\Big(\frac{\pi}{6}\Big) > 1.57. \qquad \square$$

Lemma 3.3. *Let p, q be two positive integers such that $p < q$.*

(i) *If $q \ne 3p$, then there exists $u_{p,q} \ge 1$ such that, if $\operatorname{ord}(a) \ge u_{p,q}$,*

$$\sup_{n\ge1} |\cos(npa) - \cos(nqa)| > \frac{8}{\sqrt{3}}.$$

(ii) *If $q = 3p$, then, for, every $m < 8/3\sqrt{3}$, there exists $u_p(m) \ge 1$ such that, if $\operatorname{ord}(a) \ge u_p(m)$,*

$$\sup_{n\ge1} |\cos(npa) - \cos(3npa)| > m.$$

Proof. Set $\lambda = \sup_{x\in\mathbb{R}} |\cos(px) - \cos(qx)| = \sup_{x\ge0}|\cos(px) - \cos(qx)|$. An elementary verification shows that $\lambda > 8/3\sqrt{3}$ if $q \ne 3p$ and $\lambda = 8/3\sqrt{3}$ if $q = 3p$ (see, for example, [9]). Now let $\mu < \lambda$, and let $\eta < \delta$ be two real numbers such that $|\cos(px) - \cos(qx)| > \mu$ for $\eta \le x \le \delta$. Since $\{e^{ian}\}_{n\ge1} = \{e^{2ni\pi/u}\}_{1\le n\le u}$, we see that $\sup_{n\ge1} |\cos(npa) - \cos(nqa)| > \mu$ if $2\pi/u < \delta - \eta$, and the lemma follows. $\square$

Lemma 3.4. *Assume that a/π and b/π are rational, let $u \ge 1$ be the order of a and let v be the order of b.*

(i) *If $u \ne v$, $u \ne 3v$, $v \ne 3u$, then $\sup_{n\ge1} |\cos(na) - \cos(nb)| \ge 1 + \cos(\pi/5) > 1.8 > 8/3\sqrt{3}$.*

(ii) *If $u = v$ and if $b \notin \pm a + 2\pi\mathbb{Z}$, then there exists $w \in \mathbb{Z}$ such that $2 \le w \le u/2$ and $\gcd(u, w) = 1$ satisfying*

$$\sup_{n\ge1} |\cos(na) - \cos(nb)| = \sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{2nw\pi}{u}\Big)\Big|. \tag{3.1}$$

Conversely, if $a \in \pi\mathbb{Q}$ has order u, then, for every integer w such that $\gcd(w, u) = 1$, there exists $b \in \pi\mathbb{Q}$ of order u satisfying (3.1).

(iii) *If $v = 3u$, then there exists an integer w such that $1 \le w \le u/2$ and $\gcd(u, w) = 1$ satisfying*

$$\sup_{n\ge1} |\cos(na) - \cos(nb)| = \sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{3u}\Big) - \cos\Big(\frac{2nw\pi}{u}\Big)\Big|. \tag{3.2}$$

Conversely, if $a \in \pi\mathbb{Q}$ has order u, then, for every integer w such that $\gcd(w, u) = 1$, there exists $b \in \pi\mathbb{Q}$ of order $3u$ satisfying (3.2).

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(iv) *If $u = 3v$, then there exists an integer w such that $1 \le w \le u/6$ and $\gcd(u/3, w) = 1$ satisfying*

$$\sup_{n\ge1} |\cos(na) - \cos(nb)| = \sup_{n\ge1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{6nw\pi}{u}\right)\right|. \tag{3.3}$$

Conversely, if the order u of $a \in \pi\mathbb{Q}$ is divisible by three, then, for every integer w such that $\gcd(u/3, w) = 1$, there exists $b \in \pi\mathbb{Q}$ of order $u/3$ satisfying (3.3).

Proof.

(i) Assume that $u \neq v$, say, $u < v$, and let $w \neq 1$ be the order of ub, which is a divisor of v. We know that $ub = 2\pi\alpha/w$, with $\gcd(\alpha, w) = 1$, and there exists $\gamma \ge 1$ such that $\alpha\gamma - 1 \in w\mathbb{Z}$.

$$\sup_{n\ge1} |\cos(na) - \cos(nb)| \ge \sup_{n\ge1} |\cos(nu\gamma a) - \cos(nu\gamma b)| = \sup_{1\le n\le w} 1 - \cos\left(\frac{2n\pi}{w}\right).$$

If w is even, then $\sup_{n\ge1} |\cos(na) - \cos(nb)| = 2$. If w is odd, set $s = (w-1)/2$.

$$\sup_{n\ge1} |\cos(na) - \cos(nb)| \ge 1 - \cos\left(\frac{2s\pi}{w}\right) = 1 + \cos\left(\frac{\pi}{w}\right).$$

If $w \ge 5$,

$$\sup_{n\ge1} |\cos(na) - \cos(nb)| \ge 1 + \cos\left(\frac{\pi}{5}\right) > 1.8 > \frac{8}{3\sqrt{3}}.$$

If $w = 3$, let $d = \gcd(u, v)$ and set $r = (u/d)$. Then $w = 3 = (v/d) > r$. So either $r = 1$ or $r = 2$.

If $r = 2$, $u = 2d$, $v = 3d$, $a = (2p\pi/2d) = (p\pi/d)$ with p odd, $b = (2q\pi/3d)$ with $\gcd(q, 3d) = 1$, and so

$$\begin{aligned} 2 \ge \sup_{n\ge1} |\cos(na) - \cos(nb)| &\ge |\cos(3da) - \cos(3db)| \\ &\ge |\cos(3p\pi) - \cos(2q\pi)| = 2. \end{aligned}$$

If $r = 1$, then $u = d$ and $v = 3d = 3u$.

We thus see that if $v > u$ and $v \neq 3u$, then $\sup_{n\ge1} |\cos(na) - \cos(nb)| \ge 1 + \cos(\pi/5) > 1.8 > 3/\sqrt{3}$, which proves (i).

(ii) Assume that $u = v$ and that $b \notin \pm a + 2\pi\mathbb{Z}$. There exists $\alpha, \beta \in \{1, \ldots, u-1\}$, with $\alpha \neq \beta$, $\alpha \neq u - \beta$ such that $a \in \pm(2\alpha\pi/u) + 2\pi\mathbb{Z}$ and $b \in \pm(2\beta\pi/u) + 2\pi\mathbb{Z}$, and $\gcd(\alpha, u) = \gcd(\beta, u) = 1$. It follows from Bezout's identity that there exists $\gamma \in \mathbb{Z}$ such that $\alpha\gamma - 1 \in u\mathbb{Z}$. If $\beta\gamma \pm 1 \in u\mathbb{Z}$, then we would have $\alpha\beta\gamma \pm \alpha \in \alpha u\mathbb{Z} \subset u\mathbb{Z}$, and $\beta \pm \alpha \in u\mathbb{Z}$, which is impossible. Hence $\gamma\beta - w \in u\mathbb{Z}$ for some $w \in \{2, \ldots, u-2\}$, $\gcd(w, u) = 1$ since $\gcd(\gamma, u) = \gcd(\beta, u) = 1$, and

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$$\begin{aligned}
&\sup_{n\geq 1}|\cos(na)-\cos(nb)| \\
&\qquad \geq \sup_{n\geq 1}|\cos(n\gamma a)-\cos(n\gamma b)| \\
&\qquad = \sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right)-\cos\left(\frac{2nw\pi}{u}\right)\right| \geq \sup_{n\geq 1}\left|\cos\left(\frac{2n\alpha\pi}{u}\right)-\cos\left(\frac{2n\alpha w\pi}{u}\right)\right| \\
&\qquad = \sup_{n\geq 1}\left|\cos\left(\frac{2n\alpha\pi}{u}\right)-\cos\left(\frac{2n\beta\pi}{u}\right)\right| = \sup_{n\geq 1}|\cos(na)-\cos(nb)|.
\end{aligned}$$

By replacing w by $u-w$, if necessary, we can assume that $2 \leq w \leq u/2$.

Now let $w \in \mathbb{Z}$ such that $\gcd(u, w) = 1$. We know that $a = 2\alpha\pi/u$, with $\gcd(\alpha, u) = 1$. The same argument as above shows that

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right)-\cos\left(\frac{2nw\pi}{u}\right)\right| = \sup_{n\geq 1}|\cos(na)-\cos(nb)|,$$

with $b = 2w\alpha\pi/u$, which has order u.

(iii) Now assume that $v = 3u$. There exists $\alpha \in \{1, \ldots, u-1\}$ and $\beta \in \{1, \ldots, 3u-1\}$ such that $a \in \pm(2\alpha\pi/u) + 2\pi\mathbb{Z}$ and $b \in \pm(2\beta\pi/3u) + 2\pi\mathbb{Z}$, and $\gcd(\alpha, u) = \gcd(\beta, 3u) = 1$. Let $\gamma \in \mathbb{Z}$ such that $\beta\gamma - 1 \in 3u\mathbb{Z}$. Then $\gcd(\gamma, 3u) = 1$ and *a fortiori* $\gcd(\gamma, u) = 1$. There exists $w \in \mathbb{Z}$ such that $1 \leq w \leq u/2$ and $\alpha\gamma \in \pm w + u\mathbb{Z}$, and we see, as above, that

$$\begin{aligned}
&\sup_{n\geq 1}|\cos(na)-\cos(nb)| \\
&\qquad = \sup_{n\geq 1}\left|\cos\left(\frac{2n\alpha\pi}{u}\right)-\cos\left(\frac{2n\beta\pi}{3u}\right)\right| \\
&\qquad = \sup_{n\geq 1}\left|\cos\left(\frac{2n\alpha\gamma\pi}{u}\right)-\cos\left(\frac{2n\beta\gamma\pi}{3u}\right)\right| = \sup_{n\geq 1}\left|\cos\left(\frac{2nw\pi}{u}\right)-\cos\left(\frac{2n\pi}{3u}\right)\right|.
\end{aligned}$$

Conversely, let $a = 2\alpha\pi/u \in \pi\mathbb{Q}$ have order u, and let $w \in \mathbb{Z}$ be such that $\gcd(u, w) = 1$. If α is not divisible by three, then $\gcd(\alpha, 3u) = 1$. If α is divisible by three, then u is not divisible by three, and so $\alpha + u \in \alpha + u\mathbb{Z}$ is not divisible by three. So we can assume, without loss of generality, that α is not divisible by three, and there exists $\beta \geq 1$ such that $\alpha\beta - 1 \in 3u\pi\mathbb{Z}$. Similarly, we can assume, without loss of generality, that w is not divisible by three, and there exists $\gamma \geq 1$ such that $w\gamma - 1 \in 3u\pi\mathbb{Z}$. Set $b = (2\alpha\gamma\pi/3u)$. Then b has order $3u$, and we see, as above, that

$$\begin{aligned}
&\sup_{n\geq 1}\left|\cos\left(\frac{2nw\pi}{u}\right)-\cos\left(\frac{2n\pi}{3u}\right)\right| \\
&\qquad \geq \sup_{n\geq 1}\left|\cos\left(\frac{2n\alpha\gamma w\pi}{u}\right)-\cos\left(\frac{2n\alpha\gamma\pi}{3u}\right)\right| \\
&\qquad = \sup_{n\geq 1}|\cos(na)-\cos(nb)| \geq \sup_{n\geq 1}\left|\cos\left(\frac{2n\alpha\gamma w\beta w\pi}{u}\right)-\cos\left(\frac{2n\alpha\gamma\beta w\pi}{3u}\right)\right| \\
&\qquad = \sup_{n\geq 1}\left|\cos\left(\frac{2nw\pi}{u}\right)-\cos\left(\frac{2n\pi}{3u}\right)\right|,
\end{aligned}$$

which concludes the proof of (iii).

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(iv) Clearly, the first assertion of (iv) is a reformulation of the first assertion of (iii). Now assume that the order u of $a \in \pi\mathbb{Q}$ is divisible by three, set $v = u/3$, write $a = 2\alpha\pi/u$ and let $w \in \mathbb{Z}$ such that $\gcd(w, v) = 1$. We see, as above, that we can assume, without loss of generality, that $\gcd(u, w) = 1$.

Since $\gcd(\alpha, u) = 1$, *a fortiori* $\gcd(\alpha, v) = 1$, so that $\gcd(\alpha w, v) = 1$, so that $b := 6\alpha w\pi/u$ has order v and we see, as above, that a, b, u and w satisfy (3.3). □

In order to use Lemma 3.4, we introduce the following notions.

Definition 3.5. Let $u \geq 2$, denote by $\Delta(u)$ the set of all integers s satisfying $1 \leq s \leq u/2$, $\gcd(u, s) = 1$ and let $\Delta_1(u) = \Delta(u) \setminus \{1\}$. We set

$$\sigma(u) = \inf_{w \in \Delta(u)} \left[\sup_{n \geq 1} \left| \cos\left(\frac{2\pi}{3u}\right) - \cos\left(\frac{2w\pi}{u}\right) \right| \right],$$

$$\theta(u) = \inf_{w \in \Delta_1(u)} \left[\sup_{n \geq 1} \left| \cos\left(\frac{2\pi}{u}\right) - \cos\left(\frac{2w\pi}{u}\right) \right| \right],$$

with the convention $\theta(u) = 2$ if $\Delta_1(u) = \emptyset$.

Notice that $\Delta_1(u) = \emptyset$ if $u = 2, 3, 4$ or 6 and that $\Delta_1(u) \neq \emptyset$ otherwise, since, as we observed above, $H(n) = \operatorname{card}((\mathbb{Z}/n\mathbb{Z})^{\times}) \geq 3$ if $n \notin \{1, 2, 3, 4, 6\}$.

We obtain the following corollary, which shows, in particular, that the value of $k(a)$ depends only on the order of a.

Corollary 3.6. *Let $a \in \pi\mathbb{Q}$ and let $u \geq 1$ be the order of a.*

(i) *If u is not divisible by three, then $k(a) = \inf(\sigma(u), \theta(u))$.*

(ii) *If u is divisible by three, then $k(a) = \inf(\sigma(u/3), \sigma(u), \theta(u))$.*

Proof. Set:

- $\Lambda_1(a) = \{b \in \pi\mathbb{Q} | b \notin \pm a + 2\pi\mathbb{Z}, \operatorname{ord}(b) = \operatorname{ord}(a)\}$;
- $\Lambda_2(a) = \{b \in \pi\mathbb{Q} | \operatorname{ord}(b) = 3\operatorname{ord}(a)\}$;
- $\Lambda_3(a) = \{b \in \pi\mathbb{Q} | 3\operatorname{ord}(b) = \operatorname{ord}(a)\}$;
- $\Lambda_4(a) = \{b \in \pi\mathbb{Q} | \operatorname{ord}(b) \neq \operatorname{ord}(a) \neq 3\operatorname{ord}(b)\}$;

and, for $1 \leq i \leq 4$, set

$$\lambda_i(a) = \inf_{b \in \Lambda_i(a)} \sup_{n \geq 1} |\cos(na) - \cos(nb)|,$$

with the convention $\lambda_i(a) = 2$ if $\Lambda_i(a) = \emptyset$.

Since $b \notin \pm a + 2\pi\mathbb{Z}$ if $\operatorname{ord}(b) \neq \operatorname{ord}(a)$, $\lambda_2(a) \leq 8/3\sqrt{3}$, and it follows from Lemma 3.4(i) that

$$k(a) = \inf_{1 \leq i \leq 4} \lambda_i(a) = \inf_{1 \leq i \leq 3} \lambda_i(a)$$

and it follows from Lemma 3.4(ii), (iii) and (iv) that $\lambda_1(a) = \theta(u)$ if $\Delta_1(u) \neq \emptyset$, that $\lambda_2(a) = \sigma(u)$ and that $\lambda_3(a) = \sigma(u/3)$ if u is divisible by three. □

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We know that $\Delta_1(2) = \Delta_1(4) = \emptyset$, and so $k(a) = \sigma(2)$ if $\mathrm{ord}(a) = 2$ and $k(a) = \sigma(4)$ if $\mathrm{ord}(a) = 4$, and an immediate verification then shows that $k(a) = \frac{3}{2}$ if $\mathrm{ord}(a) \in \{2, 4\}$.

We have the following theorem.

THEOREM 3.7. *Let $m < 8/3\sqrt{3}$. Then the set $\Omega(m) := \{a \in [0, \pi] : k(a) \le m\}$ is finite.*

PROOF. It follows from Lemma 3.3 applied to $2\pi/u$ and $6\pi/u$ that there exists $u_0 \ge 1$ such that, for $u \ge u_0$,

$$\text{(i)}\ \sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{2wn\pi}{u}\Big)\Big| > m \quad \text{if } 2 \le w \le \inf\Big(\frac{u}{2}, 6\Big),$$

$$\text{(ii)}\ \sup_{n\ge1}\Big|\cos\Big(\frac{6n\pi}{u}\Big) - \cos\Big(\frac{2(3w+1)n\pi}{u}\Big)\Big| > m \quad \text{if } 0 \le w \le 6,$$

$$\text{(iii)}\ \sup_{n\ge1}\Big|\cos\Big(\frac{6n\pi}{u}\Big) - \cos\Big(\frac{2(3w+2)n\pi}{u}\Big)\Big| > m \quad \text{if } 0 \le w \le 6.$$

Let $u \ge u_0$, and let w be an integer such that $2 \le w \le u/2$. If $2w\pi/u \le \pi/2$ or if $2w\pi/u \ge 5\pi/6$, it follows from Lemma 3.2 and property (i) that

$$\sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{2wn\pi}{u}\Big)\Big| > m.$$

Now assume that $\pi/2 \le 2w\pi/u \le 5\pi/6$. If $|w - (u/3)| \ge 7$, it follows from Lemma 3.2 that

$$\sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{2wn\pi}{u}\Big)\Big| > 1.55 > m.$$

If $|w - (u/3)| < 7$, set $r = |3w - u|$. Then $0 \le r \le 20$ and

$$\sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{2wn\pi}{u}\Big)\Big| \ge \sup_{n\ge1}\Big|\cos\Big(\frac{6n\pi}{u}\Big) - \cos\Big(\frac{2nr\pi}{u}\Big)\Big|.$$

If u is not divisible by three, then either $r = 3s + 1$ or $r = 3s + 2$, with $0 \le s \le 6$, and it follows from (ii) and (iii) that

$$\sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{2wn\pi}{u}\Big)\Big| > m.$$

If u is divisible by three then r is also divisible by three. Set $v = u/3$ and $s = r/3$. Then $0 \le s \le 6$ and

$$\sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{2wn\pi}{u}\Big)\Big| \ge \sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{v}\Big) - \cos\Big(\frac{2ns\pi}{v}\Big)\Big|.$$

If $s \in \{2, 3, 4, 5, 6\}$, it follows from (i) that, if $u \ge 3u_0$,

$$\sup_{n\ge1}\Big|\cos\Big(\frac{2n\pi}{v}\Big) - \cos\Big(\frac{2sn\pi}{u}\Big)\Big| > m.$$

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Now assume that $s = 0$. If $u \geq 15$, then $v \geq 5$ and

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{v}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| = \sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{v}\right) - 1\right| \geq 1 + \cos\left(\frac{\pi}{5}\right) > 1.8 > m.$$

Now assume that $s = 1$. With $\epsilon = \pm 1$,

$$\begin{aligned}\sup_{n\geq 1}&\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right)\right| \\ &= \sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{3v}\right) - \cos\left(\frac{2n\pi}{3v} + \frac{2n\epsilon\pi}{3}\right)\right| \\ &\geq \sup_{n\geq 1}\left|\cos\left(\frac{2(3n+1)\pi}{3v}\right) - \cos\left(\frac{2(3n+1)\pi}{3v} + \frac{2\epsilon\pi}{3}\right)\right| \\ &= \sqrt{3}\left|\sin\left(\frac{2n\pi}{v} + \frac{2\pi}{3v} + \frac{\epsilon\pi}{3}\right)\right|.\end{aligned}$$

There exists $p \geq 1$ and $q \in \mathbb{Z}$ such that $(\pi/2) - (\pi/v) \leq (2p\pi/v) + (2\pi/3v) + (\epsilon\pi/3) + 2q\pi \leq (\pi/2) + (\pi/v)$ and we obtain, for $u \geq 21$, $w = v \pm 1$,

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right)\right| \geq \sqrt{3}\cos\left(\frac{\pi}{v}\right) \geq \sqrt{3}\cos\left(\frac{\pi}{7}\right) \geq 1.56 > m.$$

We thus see that if $u \geq u_0$ is not divisible by three or if $u \geq \max(21, 3u_0)$ is divisible by three, for $2 \leq w \leq (u/2)$,

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2wn\pi}{u}\right)\right| \geq m,$$

so that $k(2\pi/u) > m$.

It follows from Corollary 3.6 that $k(a)$ depends only on the order u of a. Hence $k(a) > m$ if $u \geq \max(21, 3u_0)$, which shows that $\Omega(m)$ is finite. □

We now want to identify the real numbers a for which $k(a) \leq 1.5$.

If $a \in \pi\mathbb{Q}$ has order one, two or four, then $\sup_{n\geq 1}|\cos(an) - \cos(3an)| = 0$. We also know the following elementary facts.

Lemma 3.8. *Let $a \in \pi\mathbb{Q}$, and let $u \notin \{1, 2, 4\}$ be the order of a.*

(1) *If $u \notin \{3, 5, 6, 8, 9, 10, 11, 12, 15, 16, 18, 22, 24, 30\}$, then*

$$\sup_{n\geq 1}|\cos(an) - \cos(3an)| > 1.5.$$

(2) *If $u \in \{3, 6, 9, 12, 15, 18, 24, 30\}$, then*

$$\sup_{n\geq 1}|\cos(an) - \cos(3an)| = 1.5.$$

(3) *If $u \in \{5, 10\}$, then*

$$\sup_{n\geq 1}|\cos(an) - \cos(3an)| = \frac{\sqrt{5}}{2}.$$

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(4) *If* $u \in \{8, 16\}$, *then*

$$\sup_{n\geq 1} |\cos(an) - \cos(3an)| = \sqrt{2}.$$

(5) *If* $u \in \{11, 22\}$, *then*

$$\sup_{n\geq 1} |\cos(an) - \cos(3an)| = -\cos\Big(\frac{8\pi}{11}\Big) + \cos\Big(\frac{24\pi}{11}\Big) = \cos\Big(\frac{2\pi}{11}\Big) + \cos\Big(\frac{3\pi}{11}\Big) \approx 1.4961.$$

Proof. We know that $\{e^{ian}\}_{n\geq 1} = \{e^{2in\pi/u}\}_{1\leq n\leq u}$, and so

$$\sup_{n\geq 1} |\cos(an) - \cos(3an)| = \sup_{n\geq 1}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{6n\pi}{u}\Big)\Big| = \sup_{1\leq n\leq u}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{6n\pi}{u}\Big)\Big|$$

and the value of $\sup_{n\geq 1} |\cos(an) - \cos(3an)|$ depends only on the order u of a.

The function $x \to \cos(x) - \cos(3x)$ is increasing on $[0, \arccos(1/\sqrt{3})]$ and decreasing on $[\arccos(1/\sqrt{3}), -\arccos(1/\sqrt{3})]$, and $0.275\pi < \arccos(1/\sqrt{3}) < 0.333\pi$. Since $\cos(x) - \cos(3x) > 1.5$ if $x = 0.275\pi$ or if $x = 0.333\pi$, there exists a closed interval I of length 0.058π on which $\cos(x) - \cos(3x) > 1.5$. So, if $u \geq 35 > \frac{2}{0.058}$, there exists $n \geq 1$ such that $(2n\pi/u) \in I$, and

$$\sup_{n\geq 1} |\cos(an) - \cos(3an)| > 1.5 \quad \forall n \geq 35.$$

The other properties follow from computations of $\sup_{1\leq n\leq u} |\cos(2n\pi/u) - \cos(6n\pi/u)|$ for $3 \leq u \leq 34$ and are left to the reader. □

We now wish to obtain similar estimates for $\sup_{n\geq 1} |\cos(2\pi/n) - \cos(2s\pi/n)|$ for $s \in \{2, 4, 5, 6\}$. Set $f_s(x) = \cos(x) - \cos(sx), \theta_s = \sup_{x\geq 0} |f_s(x)|, \delta_s = \sup_{x\geq 0} |f_s''(x)|$. If s is even, $\theta_s = 2$, and a computer verification shows that $\theta_s > 1.8$ for $s = 5$. It follows from the Taylor–Lagrange inequality that if f_s attains it maximum at α_s, then

$$|f_s(x) - \theta_s| \leq \frac{\delta_s}{2}(x - \alpha_s)^2, \quad |f_s(x)| \geq \theta_s - \frac{\delta_s}{2}(x - \alpha_s)^2,$$

and so $|f_s(x)| > 1.5$ if $(x - \alpha_s)^2 \leq (2\theta_s - 3)/\delta_s$. So if $l_s < \sqrt{(2\theta_s - 3)/\delta_s}$, there exists a closed interval of length $2l_s$ on which $|f_s(x)| > 1.5$. Let $u_s \geq (\pi/l_s)$ be an integer.

$$\sup_{n\geq 1}\Big|\cos\Big(\frac{2n\pi}{u}\Big) - \cos\Big(\frac{2sn\pi}{u}\Big)\Big| > 1.5 \quad \forall u \geq u_s.$$

Values for u_s are given in Table 1.

We obtain the following lemma.

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TABLE 1. Values of u_s, $s = 2, 4, 5, 6$.

s	θ_s	δ_s	l_s	u_s
2	2	≤ 5	0.4472	8
4	2	≤ 17	0.2425	13
5	>1.8	≤ 26	0.1519	21
6	2	≤ 37	0.1644	20

LEMMA 3.9. *Let $u \geq 4$ be an integer and let $s \leq u/4$ be a nonnegative integer, with $s \neq 1$. If $s \neq 3$, then*

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2ns\pi}{u}\right)\right| > 1.5$$

PROOF. If $s = 0$, then

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2ns\pi}{u}\right)\right| = \sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - 1\right| > 1.8.$$

If $s \geq 7$, the result follows from Lemma 3.2(i). If $s \in \{2, 4, 6\}$, the result follows from Table 1 since $u \geq 4s$. If $s = 5$, the result also follows from the table for $u \geq 21$, and a direct computation shows that

$$\begin{aligned}\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{20}\right) - \cos\left(\frac{10n\pi}{20}\right)\right| &= \sup_{1\leq n\leq 20}\left|\cos\left(\frac{n\pi}{10}\right) - \cos\left(\frac{n\pi}{2}\right)\right| \\ &= 1 + \cos\left(\frac{\pi}{5}\right) > 1.8.\end{aligned}$$

□

Now set $g_s(x) = \cos(3x) - \cos(sx)$, $\theta_s = \sup_{x\geq 0}|g(s)|$, $\delta_s = \sup_{x\geq 0}|g_s''(x)|$. If s is even, $\theta_s = 2$, and a computer verification shows that $\theta_s > 1.85$ for $s = 5$, $\theta_s > 1.91$ for $s = 7$, $s = 11$, $\theta_s > 1.97$ for $s = 13$, $s = 17$, $\theta_s > 1.96$ for $s = 19$. We see, as above, that if $l_s < \sqrt{(2\theta_s - 3)/\delta_s}$ and if $u_s \geq \pi/l_s$ is an integer,

$$\sup_{n\geq 1}\left|\cos\left(\frac{2sn\pi}{u}\right) - \cos\left(\frac{6n\pi}{u}\right)\right| > 1.5 \quad \forall u \geq u_s.$$

Our results are shown in Table 2.

We will be interested here in the case where u is not divisible by three and where $(2s\pi/u) \leq (\pi/2)$, which means that $u \geq 4s$. So we are left with $s = 2$, $u = 8, 10$ or 11, and with $s = 5$, $u = 20$. We obtain, by direct computation,

$$\sup_{n\geq 1}\left|\cos\left(\frac{4n\pi}{8}\right) - \cos\left(\frac{6n\pi}{8}\right)\right| = \sup_{n\geq 1}\left|\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{3n\pi}{4}\right)\right| = 2.$$

$$\sup_{n\geq 1}\left|\cos\left(\frac{4n\pi}{10}\right) - \cos\left(\frac{6n\pi}{10}\right)\right| = \sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{5}\right) - \cos\left(\frac{3n\pi}{5}\right)\right| = 2.$$

$$\sup_{n\geq 1}\left|\cos\left(\frac{4n\pi}{11}\right) - \cos\left(\frac{6n\pi}{11}\right)\right| = \cos\left(\frac{20\pi}{11}\right) - \cos\left(\frac{30\pi}{11}\right) = \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) \approx 1.4961.$$

$$\sup_{n\geq 1}\left|\cos\left(\frac{10n\pi}{20}\right) - \cos\left(\frac{6n\pi}{20}\right)\right| = \sup_{n\geq 1}\left|\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{3n\pi}{10}\right)\right| > 1.80.$$

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TABLE 2. Values of u_s, $2 \le s \le 20$, s not divisible by three.

s	θ_s	δ_s	l_s	u_s
2	2	≤13	0.2774	12
4	2	≤23	0.2085	16
5	>1.85	≤34	0.1435	22
7	>1.91	≤58	0.1189	27
8	2	≤73	0.1170	27
10	2	≤109	0.0958	33
11	>1.91	≤130	0.0794	40
13	>1.97	≤178	0.0727	44
14	2	≤205	0.0698	45
16	2	≤275	0.0603	53
17	>1.97	≤298	0.0562	56
19	>1.96	≤390	0.0486	65
20	2	≤409	0.0494	64

We obtain the following lemma.

LEMMA 3.10. *Let u, s be positive integers satisfying $u \ge 4$ and $u/4 \le s \le 5u/12$, with $s \ge 2$, so that $u \ge 5$.*

$$\sup_{n\ge 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right|$$
$$\times \begin{cases} = \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) & \text{if } u = 5, s = 2 \text{ or if } u = 10, s = 3, \\ = \sqrt{2} & \text{if } u = 8, s = 3 \text{ or if } u = 16, s = 5, \\ = \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) & \text{if } u = 11, s = 3 \text{ or } s = 4 \text{ or if } u = 22, s = 7, \\ = 1.5 & \text{if } u = 9 \text{ or } u = 12, s = 3, \\ > 1.5 & \text{otherwise}. \end{cases}$$

PROOF. Set $r = |3s - u|$. Since $(2\pi/3) - (\pi/2) = (5\pi/6) - (2\pi/3) = (\pi/6)$, $0 \le (2\pi r/u) \le (\pi/2)$. If $r \ge 21$, it follows from the second assertion of Lemma 3.2(i) applied to $a = 2\pi/u$ and $b = 2s\pi/u$ that $\sup_{n\ge1} |\cos(2n\pi/u) - \cos(2sn\pi/u)| > 1.5$.

If u is not divisible by three, then r is not divisible by three either, and it follows from the discussion above that if $r \ne 1$, $r \ne 2$, $r \le 20$, or if $r = 2$, $u \ne 11$, then

$$\sup_{n\ge 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| \ge \sup_{n\ge 1}\left|\cos\left(\frac{6n\pi}{u}\right) - \cos\left(\frac{2rn\pi}{u}\right)\right| > 1.5.$$

If $r = 2$, $u = 11$, then $|s - \frac{11}{3}| = |s - (u/3)| = \frac{2}{3}$, and so $s = 3$ and

$$\sup_{n\ge 1}\left|\cos\left(\frac{2n\pi}{11}\right) - \cos\left(\frac{6n\pi}{11}\right)\right| = \sup_{1\le n\le 11}\left|\cos\left(\frac{2n\pi}{11}\right) - \cos\left(\frac{6n\pi}{11}\right)\right|$$
$$= \left|\cos\left(\frac{8\pi}{11}\right) - \cos\left(\frac{24\pi}{11}\right)\right| = \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) \approx 1.4961.$$

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The condition $r = 1$ gives $|s - (u/3)| = \frac{1}{3}$, and so $s = (u-1)/3$ if $u \equiv 1 \bmod 3$, and $s = (u+1)/3$ if $u \equiv 2 \bmod 3$. In this situation,

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| \geq \sup_{n\geq 1}\left|\cos\left(\frac{6n\pi}{u}\right) - \cos\left(\frac{6sn\pi}{u}\right)\right|$$
$$= \sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{6n\pi}{u}\right)\right|.$$

Since $|s - (u/3)| = \frac{1}{3}$, it follows from Lemma 3.8 that if $n \notin \{5, 8, 10, 11, 16, 22\}$, or if $u = 5$, $s \neq 2$, or if $u = 8$, $s \neq 3$, or if $u = 10$, $s \neq 3$, or if $u = 11$, $s \neq 4$, or if $u = 16$, $s \neq 5$, or if $u = 22$, $s \neq 7$, then

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| > 1.5.$$

A direct computation then shows that

$$\begin{aligned}
&\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| \\
&\quad = \sup_{1\leq n\leq u}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| \\
&\quad = \begin{cases} \cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) & \text{if } u = 5, s = 2 \text{ or if } u = 10, s = 3, \\ \sqrt{2} \quad \text{if } u = 8, s = 3 \text{ or if } u = 16, s = 5, \\ \cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) & \text{if } u = 11, s = 4 \text{ or if } u = 22, s = 7. \end{cases}
\end{aligned}$$

We now consider the case where $u = 3v$ is divisible by three. Then r is also divisible by three. If $r = 0$ and if $u \neq 9$, then

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| \geq \sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{v}\right) - 1\right| > 1.8.$$

If $u = 9$, then $s = 3$ and

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| = \sup_{1\leq n\leq 9}\left|\cos\left(\frac{2n\pi}{9}\right) - \cos\left(\frac{2n\pi}{3}\right)\right| = 1.5.$$

Now assume that $r = 3$, which means that $s = v + \epsilon$, with $\epsilon = \pm 1$.

$$\begin{aligned}
\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| &= \sup_{1\leq n\leq 3v}\left|\cos\left(\frac{2n\pi}{3v}\right) - \cos\left(\frac{2n\pi}{3} + \frac{2\epsilon n\pi}{3v}\right)\right| \\
&= 2\sup_{1\leq n\leq 3v}\left|\sin\left(\frac{n\pi}{3} + \frac{(1+\epsilon)n\pi}{3v}\right)\right|\left|\sin\left(-\frac{n\pi}{3} + \frac{(1-\epsilon)n\pi}{3v}\right)\right| \\
&= 2\sup_{1\leq n\leq 3v}\left|\sin\left(\frac{n\pi}{3}\right)\right|\left|\sin\left(\frac{n\pi}{3} + \frac{2n\pi}{3v}\right)\right| \\
&\geq \sqrt{3}\sup_{0\leq n\leq v}\left|\sin\left(\frac{(3n+1)\pi}{3} + \frac{2(3n+1)\pi}{3v}\right)\right| \\
&= \sqrt{3}\sup_{0\leq n\leq v}\left|\sin\left(\frac{2n\pi}{v} + \frac{(v+2)\pi}{3v}\right)\right|.
\end{aligned}$$

Since $\sin(x) > \sqrt{3}/2$ for $\pi/3 < x < 2\pi/3$, there exists $n \in \{1, \dots, v\}$ such that $\sin((2n\pi/v) + ((v+2)\pi/3v)) > \sqrt{3}/2$ if $v \geq 7$, so

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| > 1.5 \quad \text{if } u \geq 21.$$

We are left with the cases where $u = 6, v = 2, s = 1$ or 3, $u = 9, v = 3, s = 2$ or 4, $u = 12, v = 4, s = 3$ or 5, $u = 15, v = 5, s = 4$ or 6, $u = 18, v = 6, s = 5$ or 7. But $s = 1$ is not relevant, and the condition $u/4 \leq s \leq 5u/12$ is not satisfied for $u = 6, s = 3$ and for $u = 9, s = 2$ or 4.

Direct computations, which are left to the reader, show that

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| \begin{cases} >1.64 & \text{if } u = 15 \text{ and } s = 4, \\ >1.70 & \text{or if } u = 18 \text{ and } s = 5 \text{ or } s = 7, \\ >1.72 & \text{if } u = 15 \text{ and } s = 6, \\ >1.73 & \text{if } u = 12 \text{ and } s = 5. \end{cases}$$

So $\sup_{n\geq 1} |\cos(2n\pi/u) - \cos(2sn\pi/u)| > 1.5$ if $u/4 \leq s \leq 5u/12$ when u is divisible by three and when $s - (u/3) \in \{-1, 0, 1\}$, unless $u = 12$ and $s = 3$. If $u = 12$ and $s = 3$,

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| = \sup_{n\geq 1}\left|\cos\left(\frac{n\pi}{6}\right) - \cos\left(\frac{n\pi}{2}\right)\right| = 1.5.$$

Now assume that $u = 3v$ is divisible by three and that $2 \leq |s - v| \leq 6$. Set again $r = |3s - u|$ and set $p = r/3$, so that $2 \leq p \leq 6$. Notice also that $p \leq u/12$ since $r \leq u/4$, so that $u \geq 24$ and $v \geq 8$.

$$\begin{aligned}\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right) - \cos\left(\frac{2sn\pi}{u}\right)\right| &\geq \sup_{n\geq 1}\left|\cos\left(\frac{6n\pi}{u}\right) - \cos\left(\frac{2rn\pi}{u}\right)\right| \\ &= \sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{v}\right) - \cos\left(\frac{2pn\pi}{v}\right)\right|.\end{aligned}$$

It follows then from Lemma 3.9 that $\sup_{n\geq 1} |\cos(2n\pi/u) - \cos(2sn\pi/u)| > 1.5$ if $p \neq 3$.

If $p = 3$, then $u \geq 36$, and so $v \geq 12$. Since $s - v = \pm 3$, it follows from Lemma 3.8 that we only have to consider the cases when:

- $u = 36, s = 9$ or 15,
- $u = 45, s = 12$ or 18,
- $u = 54, s = 15$ or 21,
- $u = 72, s = 21$ or 27,
- $u = 90, s = 27$ or 33.

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Direct computations, which are left to the reader, show that

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right)-\cos\left(\frac{2sn\pi}{u}\right)\right|$$
$$\times\begin{cases} >1.93 & \text{if } u=36 \text{ and } s=9 \text{ or if } u=45 \text{ and } s=12 \text{ or } 18 \\ & \text{or if } u=72 \text{ and } s=27 \text{ or if } u=90 \text{ and } s=27 \text{ or } 33, \\ >1.91 & \text{or if } u=54 \text{ and } s=15, \\ >1.87 & \text{or if } u=72 \text{ and } s=24, \\ >1.85 & \text{or if } u=36 \text{ and } s=15, \\ >1.83 & \text{or if } u=54 \text{ and } s=21. \end{cases}$$

This concludes the proof of the lemma. □

Lemma 3.11. *Let u, s be positive integers satisfying $5u/12 \leq s \leq u/2$, with $s \geq 2$, so that $u \geq 4$.*

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right)-\cos\left(\frac{2sn\pi}{u}\right)\right| = \begin{cases} =1.5 & \text{if } u=6 \text{ and } s=3, \\ >1.5 & \text{otherwise.} \end{cases}$$

Proof. If $s \geq 4$, it follows from Lemma 3.2(ii) that

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right)-\cos\left(\frac{2sn\pi}{u}\right)\right| > 1.57.$$

So we only have to consider the cases $s=3$, $u=6$ or 7 and $s=2$, $u=4$.

A direct computation then shows that

$$\sup_{n\geq 1}\left|\cos\left(\frac{2n\pi}{u}\right)-\cos\left(\frac{2sn\pi}{u}\right)\right| \begin{cases} =2 & \text{if } u=4 \text{ and } s=2, \\ =1.5 & \text{if } u=6 \text{ and } s=3, \\ =\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{\pi}{7}\right)\approx 1.5245 & \text{if } u=7 \text{ and } s=3. \end{cases}$$ □

We consider again the numbers $\theta(u)$ and $\sigma(u)$ introduced in Definition 3.5.

It follows from Lemmas 3.8–3.11 that we have the following results.

Lemma 3.12. $\theta(5)=\theta(10)=\cos(\pi/5)+\cos(2\pi/5)$, $\theta(8)=\theta(16)=\sqrt{2}$, $\theta(11)=\theta(22)=\cos(2\pi/11)+\cos(3\pi/11)$, *and* $\theta(u)>1.5$ *for* $u\geq 4, u\neq 5, u\neq 8, u\neq 10, u\neq 11, u\neq 16, u\neq 22$.

Lemma 3.13. $\sigma(u)=1.5$ *if* $u\in\{1,2,3,4,5,6,8,10\}$ *and* $\sigma(u)>1.5$ *otherwise.*

Hence, if u is divisible by three, $\sigma(u/3)=1.5$ if $u\in\{3,6,9,12,15,18,24,30\}$ and $\sigma(u)>1.5$ otherwise. We then deduce from Corollary 3.6 a complete description of the set $\Omega(1.5)=\{a\in[0,\pi] \mid k(a)\leq 1.5\}$.

Theorem 3.14. *Let $a\in[0,\pi]$.*

- *If* $a\in\{\pi/5, 2\pi/5, 3\pi/5, 4\pi/5\}$, *then* $k(a)=\cos(\pi/5)+\cos(2\pi/5)\approx 1,1180$.
- *If* $a\in\{\pi/8, \pi/4, 3\pi/8, 5\pi/8, 5\pi/4, 7\pi/8\}$, *then* $k(a)=\sqrt{2}\approx 1,4142$.

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- *If* $a \in \{\pi/11, 2\pi/11, 3\pi/11, 4\pi/11, 5\pi/11, 6\pi/11, 7\pi/11, 8\pi/11, 9\pi/11, 10\pi/11\}$, *then* $k(a) = \cos(2\pi/11) + \cos(3\pi/11) \approx 1,4961$.
- *If* $a \in \{0, \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6\} \cup \{\pi/9, 2\pi/9, 4\pi/9, 5\pi/9, 7\pi/9, 8\pi/9\} \cup \{\pi/12, 5\pi/12, 7\pi/12\} \cup \{\pi/15, 2\pi/15, 4\pi/15, 7\pi/15, 8\pi/15, 11\pi/15, 13\pi/15, 14\pi/15\}$, *then* $k(a) = 1.5$.
- *For all other values of* a, $1.5 < k(a) \le 8/3\sqrt{3} \approx 1.5396$.

Corollary 3.15. *Let* G *be an abelian group and let* $(C(g))_{g\in G}$ *be a* G*-cosine family in a unital Banach algebra* A *such that* $\sup_{g\in G} \|C(g) - c(g)\| < \sqrt{5}/2$ *for some bounded scalar* G*-cosine family* $(c(g))_{g\in G}$*. Then* $C(g) = c(g)$ *for every* $g \in G$.

Proof. Let $g \in G$. Since the scalar cosine sequence $(c(ng))_{n\in\mathbb{Z}}$ is bounded, a standard argument shows that there exists $a(g) \in \mathbb{R}$ such that $c(ng) = \cos(na(g))1_A$ for $n \in \mathbb{Z}$. Since $k(a(g)) \ge \sqrt{5}/2$, it follows from Corollary 2.4 that $C(ng) = \cos(na(g))1_A = c(ng)$ for $n \in \mathbb{Z}$, and $C(g) = c(g)$. □

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