The case of general $x \in \left(-\pi/2,\ 3\pi/2\right)$ can be treated similarly; in particular, when $x \in \left(-\pi/2,\ \pi/2\right)/\left(\pi/2,\ 3\pi/2\right)$ the sequence $S_n(x)$, $n \in \mathbb{N}$, increases/decreases monotonically.

As an alternative perspective of the above calculation, we mention that the iterative procedure $S_1(x) = x$, $S_{n+1}(x) = S_n(x) + \cos\left(S_n(x)\right)$, $n \in \mathbb{N}$, is just the fixed-point iteration for the map $x \to x + \cos x$. We conclude by noting that, due to the above result, the finite sum $S_n(\cdot)$ serves as a smooth approximation of a staircase function. Consequently, the derivative of $S_n(\cdot)$ can be viewed as a smooth approximation of a *Dirac comb*, i.e. of a periodic pulse wave consisting of Dirac delta functions.

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107.35 Two definite integrals that are (not surprisingly) equal

1. *Introduction*

In their recent note Ekhad, Zeilberger and Zudilin [1] gave a clever proof of the identity

$$\int_0^1 \frac{x^n (1-x)^n}{((x+a)(x+b))^{n+1}} dx = \int_0^1 \frac{x^n (1-x)^n}{((a-b)x + (a+1)b)^{n+1}} dx, \qquad (1)$$

for $n = 0, 1, 2, \ldots$ and $a > b > 0$, using the Almkvist–Zeilberger creative telescoping algorithm. If $L(n)$ and $R(n)$ denote the integrals on the left and right sides, respectively, for fixed a and b, then they show that $L(n)$ and $R(n)$ satisfy the same linear recursive formula of order two. Confirming that $L(0) = R(0)$ and $L(1) = R(1)$, the identity follows by mathematical induction. The authors mentioned that three other proofs of (1) exist. Bostan, Chamizo and Sundqvist [2] recognized in identity (1) a particular case of a known relation for Appell's bivariate hypergeometric function and gave three different proofs of (1).

The authors of [1, Remark 3] mention that the right-hand side $R(n)$ covers a famous sequence of rational approximations to $\log\left(1 + \dfrac{a-b}{(a+1)b}\right)$, and hence the left-hand side $L(n)$ does, too, and cite [3]. The estimate of the irrationality measure is based on considering certain integrals involving the nth Legendre type polynomial $L_n(x) = (n!)^{-1}\left(x^n(1-x)^n\right)^{(n)}$.

In the following we consider a more natural representation of (1). We rewrite identity (1) by replacing $a, b > 0$ with their reciprocals, in the form

$$\int_0^1 \frac{x^n (1-x)^n}{((1+ax)(1+bx))^{n+1}} dx = \int_0^1 \frac{x^n (1-x)^n}{(1 + a(1-x) + bx)^{n+1}} dx, \qquad (2)$$

for $n = 0, 1, 2, \dots$ and $b > a > 0$. Obviously, the identities (2) and (1) are equivalent.

The purpose of this Note is to generalise (2) in various directions. As was done in [2, Corollary 2, (7)], we consider different exponents for the terms $x^p(1 - x)^q$ and $(1 + ax)^{n+1}(1 + bx)^{m+1}$ appearing in the numerator and denominator, respectively, of the integral on the left-hand side. Moreover, we extend the identity to arbitrarily many parameters. Instead of the real unit interval of one variable of integration in (1) and (2) we will work with r integration variables in a particular region of $\mathbb{R}^r$, the so-called standard simplex S_r. This multi-dimensional simplex, which is more complicated than the unit cube, will be explained below. The outcome is an integral over the S_r which is analogous to the right-hand side of (2). We start with a short direct proof of the original identity in its equivalent form (2) which can be extended to more general results. No deeper knowledge of special functions or sophisticated tools are needed. We use only elementary calculus. Throughout the paper we suppose that $n \in \{0, 1, 2, \dots\}$.

2. *The original identity*

We present a direct proof of (1) in its equivalent form (2). Denote by $\mathcal{L}(n)$ and $\mathcal{R}(n)$ the integrals on the left and right sides of (2), respectively, for fixed a and b. First, let a, b be reals such that $0 < a \leqslant b < 1$. Observe that $a(1 - x) + bx \leqslant b < 1$, for all $x \in [0, 1]$. Using power-series expansion

$$\frac{1}{(1 + a(1 - x) + bx)^{n+1}} = \sum_{i,j=0}^{\infty} \frac{(n + i + j)!}{n!\,i!\,j!}(-a(1 - x))^i(-bx)^j$$

we obtain

$$\mathcal{R}(n) = \sum_{i,j=0}^{\infty} \frac{(n + i + j)!}{n!\,i!\,j!}(-1)^{i+j}a^ib^j\int_0^1 x^{n+j}(1 - x)^{n+i}\,dx,$$

where the integral is the Euler beta function $B(n + j + 1, n + i + 1)$. It is easy to check that

$$\frac{(n+i+j)!}{n!\,i!\,j!}B(n+j+1, n+i+1) = \binom{n+i}{i}\binom{n+j}{j}B(n+i+j+1, n+1).$$

Hence,

$$\mathcal{R}(n) = \int_0^1 \sum_{i=0}^{\infty}\binom{n + i}{i}(-ax)^i \sum_{j=0}^{\infty}\binom{n + j}{j}(-bx)^j x^n(1 - x)^n\,dx = \mathcal{L}(n).$$

Analytic continuation with respect to a and b implies that identity (2) is valid for all complex constants a, b not belonging to the interval $(-\infty, -1]$. If $b \geqslant a > 0$ the identity (2) is true also for real values of $n \geqslant 0$.

Returning to the original identity, we see that (1) is valid for all complex constants a, b not belonging to the interval $[-1, 0]$. If $a \geqslant b > 0$ the identity is true also for real values of $n \geqslant 0$.

3. *Generalisations*

Now we are going to generalise the equivalent form (2) of identity (1) in various directions. In particular, we prove, for $a, b > 0$ and reals $p, q, n, m > -1$ satisfying $p + q = n + m > -1$, the equation

$$\int_0^1 \frac{x^p (1 - x)^q}{(1 + ax)^{n+1}(1 + bx)^{m+1}} = \frac{p!\,q!}{n!\,m!} \int_0^1 \frac{x^n (1 - x)^m}{(1 + ax + b(1 - x))^{p+1}} dx. \quad (3)$$

Here the factorials are defined by the Euler gamma function, i.e., $z! = \Gamma(z + 1)$ for $z > -1$. Equation (2) is a direct consequence when $p = q = n = m$. Our main result is the following proposition. Its formulation needs the r-dimensional standard simplex S_r. It is the polytope whose vertices are the origin and the r standard unit vectors. In other words,

$$S_r = \left\{(x_1, \dots, x_r) \in \mathbb{R}^r \,|\, x_1 + \dots + x_r \leqslant 1 \text{ and } x_i \geqslant 0 \text{ for } i = 1, \dots, r\right\}$$

is the simplex, which is bounded by the coordinate hyperplanes $x_1 = 0, \dots, x_r = 0$, and the hyperplane $x_1 + \dots + x_r = 1$. In the simple case $r = 1$, the standard simplex reduces to the unit interval $S_1 = [0, 1]$. For $r = 2$, we have a triangle, and for $r = 3$, we have a tetrahedron.

Proposition 1: Suppose that $r \in \mathbb{N}$, $p, q > -1$, $\mathbf{a} = (a_1, \dots, a_r) \in (0, \infty)^r$, $a_{r+1} > 0$, and $(n_1, \dots, n_{r+1}) \in (-1, \infty)^{r+1}$. If $n_1 + \dots + n_{r+1} + r - 1 = p + q > -1$, then

$$\int_0^1 \frac{x^p (1 - x)^q}{\prod_{i=1}^{r+1} (1 + a_i x)^{1+n_i}} dx = \frac{p!\,q!}{\prod_{i=1}^{r+1} n!} \int_{S_r} \frac{x_1^{n_1} \dots x_r^{n_r} (1 - x_1 - \dots - x_r)^{n_{r+1}}}{(1 + \mathbf{a}\cdot\mathbf{x} + a_{r+1}(1 - x_1 - \dots - x_r))^{p+1}} dx_1 \dots dx_r,$$

where $\mathbf{a}\cdot\mathbf{x} = a_1 x_1 + \dots + a_r x_r$ and S_r denotes the standard simplex in $\mathbb{R}^r$.

Remark 2: Equation (3) is a special case of the proposition when $r = 1$ with $(n_1, n_2) = (n, m)$, $(a_1, a_2) = (a, b)$.

Remark 3: We leave it as an exercise to confirm Proposition 1 in the special case $r = 2$, $p = 1$, $q = 0$ and $n_1 = n_2 = n_3 = 0$ by direct computation of both integrals

$$\int_0^1 \frac{x}{(1 + a_1 x)(1 + a_2 x)(1 + a_3 x)} dx$$

$$= \int_0^1 \int_0^{1 - x_2} \frac{1}{(1 + a_1 x_1 + a_2 x_2 + a_3 (1 - x_1 - x_2))^2} dx_1 dx_2.$$

Remark 4: More generally, assuming that the positive numbers $a_i\,(i = 1, \dots, r + 1)$ are pairwise different, partial fraction decomposition

$$\frac{x^p}{\prod_{i=1}^{r+1}(1 + a_i x)} = (-1)^p \sum_{j=1}^{r+1} \frac{a_j^{r-p}}{(1 + a_j x)\prod_{\substack{i=1 \\ i \neq j}}^{r+1}(a_j - a_i)} \qquad (p = 0, 1, \dots, r)$$

leads to the formula

$$\int_0^1 \frac{x^p}{\prod_{i=1}^{r+1}(1 + a_i x)} = (-1)^p \sum_{j=1}^{r+1} \frac{a_j^{r-p-1}}{\prod_{\substack{i=1 \\ i \neq j}}^{r+1}(a_j - a_i)} \log(1 + a_j).$$

By Proposition 1, the integral

$$\int_{S_r} \frac{p!}{(1 + \mathbf{a}\cdot\mathbf{x} + a_{r+1}(1 - x_1 - \dots - x_r))^{p+1}} dx_1 \dots dx_r$$

has the same value. The special choice $p = r - 1 \geqslant 0$ yields the evaluation

$$\int_{S_r} \frac{1}{(1 + \mathbf{a}\cdot\mathbf{x} + a_{r+1}(1 - x_1 - \dots - x_r))^r} dx_1 \dots dx_r$$
$$= \frac{-1}{(r - 1)!} \sum_{j=1}^{r+1} \frac{\log(1 + a_j)}{\prod_{\substack{i=1 \\ i \neq j}}^{r+1}(a_i - a_j)}.$$

Remark 5: Partial differentiations with respect to a_1 for any $i \in \{1, \dots, r + 1\}$ can lead to closed-form expressions of some of the other integrals in the class appearing in Proposition 1.

Proof of Proposition 1: We use the following multi-index notation: For $\mathbf{v} = (v_1, \dots, v_{r+1}) \in \mathbb{N}_0^{r+1}$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, we set $|\mathbf{v}| = v_1 + \dots v_{r+1}$. For real m, the multinomial coefficient is defined as

$$\binom{m}{v_1, \dots, v_{r+1}} = \frac{\Gamma(m + 1)}{v_1! \dots v_{r+1}!\,\Gamma(m - |v| + 1)},$$

if $m > |\mathbf{v}| - 1$, where $k! = \Gamma(k + 1)$ is defined for real numbers $k > -1$.

By analytic continuation we can assume that $0 \leqslant a_i \leqslant a_{r+1} < 1$ $(i = 1, \dots, r)$. Then, we have

$$\int_0^1 \frac{x^p(1-x)^q}{\prod_{i=1}^{r+1}(1 + a_i x)^{1+n_i}} dx = \sum_{v_1, \dots, v_{r+1}=0}^{\infty} \left(\prod_{i=1}^{r+1}\binom{n_i + v_i}{v_i}(-a_i)^{v_i}\right)\int_0^1 x^{p+|v|}(1-x)^q dx,$$

where the last integral is equal to

$$B(p + |v| + 1, q + 1) = \frac{(p + |v|)!\, q!}{(p + |v| + q + 1)!}.$$

Noting that

$$\left(\prod_{i=1}^{r+1}\binom{n_i + v_i}{v_i}\right)\frac{(p + |v|)!\, q!}{(p + |v| + q + 1)!} = \frac{q!}{\prod_{i=1}^{r+1} n_i!} \cdot \frac{(p + |v|)!}{\prod_{i=1}^{r+1} v_i!} \cdot \frac{\prod_{i=1}^{r+1}(n_i + v_i)!}{(p + |v| + q + 1)!}$$

the assumption $p + q = |n| + r - 1$ implies that

$$\frac{\prod_{i=1}^{r+1}(n_i + v_i)!}{(p + |v| + q + 1)!} = \frac{\prod_{i=1}^{r+1}(n_i + v_i + 1)}{\Gamma(|n| + |v| + r + 1)}$$

$$= \int_{S_r}\left(\prod_{i=1}^{r} x_i^{n_i + v_i}\right)(1 - |x|)^{n_{r+1} + v_{r+1}}\, dx_1 \ldots dx_r,$$

where $|x| = x_1 + \ldots + x_r$. Therefore, we have

$$\int_0^1 \frac{x^p (1 - x)^q}{\prod_{i=1}^{r+1}(1 + a_i x)^{1 + n_i}} dx$$

$$= \frac{p!\, q!}{\prod_{i=1}^{r+1} n_i!} \sum_{v_1, \ldots, v_{r+1} = 0}^{\infty} \binom{p + |v|}{v_1, \ldots, v_{r+1}}\left(\prod_{i=1}^{r+1}(-a_i)^{v_i}\right)\int_{S_r}\left(\prod_{i=1}^{r} x_i^{n_i + v_i}\right)(1 - |x|)^{n_{r+1} + v_{r+1}} dx_1 \ldots dx_r$$

$$= \frac{p!\, q!}{\prod_{i=1}^{r+1} n_i!} \int_{S_r} \sum_{v_1, \ldots, v_{r+1} = 0} \binom{p + |v|}{v_1, \ldots, v_{r+1}}\left(\prod_{i=1}^{r} x_i^{n_i}(-a_i x_i)^{v_i}\right)(1 - |x|)^{n_{r+1} + v_{r+1}} dx_1 \ldots dx_r \cdot (-a_{r+1})^{v_{r+1}}$$

$$= \frac{p!\, q!}{\prod_{i=1}^{r+1} n_i!} \int_{S_r} \frac{\left(\prod_{i=1}^{r} x_i^{n_i}\right)(1 - |x|)^{n_{r+1}}}{\left(1 + \sum_{i=1}^{r} a_i x_i + a_{r+1}(1 - |x|)\right)^{p+1}} dx_1 \ldots dx_r$$

which is the desired formula. Here we used that

$$0 \leqslant \sum_{i=1}^{r} a_i x_i + a_{r+1}(1 - |x|) \leqslant \sum_{i=1}^{r}(a_i - a_{r+1})x_i + a_{r+1} \leqslant a_{r+1} < 1.$$

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107.36 Similarities and circle-preserving bijections of the plane

1. *Introduction*

A *similarity* of the complex plane $\mathbb{C}$ is a map of the form $z \to az + b$, or $z \to a\bar{z} + b$, where a and b are complex numbers with $a \neq 0$. Each similarity is a bijection of $\mathbb{C}$ onto itself, and maps a line onto a line, and a circle onto a circle. In addition, it is known that the converse is true: if f is a bijection of $\mathbb{C}$ onto itself that maps each line onto a line, and each circle onto a circle, then f is a similarity of $\mathbb{C}$. The sole purpose of this Note is to use this converse to provide an opportunity for students to experience and, more importantly, engage in, a substantial proof of a single result. So, instead of providing the details, we break the proof into a number of simpler (and, we hope, manageable) steps, and invite readers to formally justify these steps for themselves.

The *circle* $\{z : |z - a| = r\}$ is denoted by $C(a, r)$, and the (open) *disc* $\{z : |z - a| < r\}$ by $D(a, r)$, where (in each case) $a \in \mathbb{C}$ and $r > 0$. Although we are assuming that f maps each circle onto a circle, we are not assuming that f maps each disc onto a disc; in fact, we shall prove that this must be so. Note also that we are not assuming that f is continuous, and again we shall prove that this is so.

2. *The converse result*

We now give our sketch of the proof that if f is a bijection of $\mathbb{C}$ onto itself that maps each line onto a line, and each circle onto a circle, then f is a similarity of $\mathbb{C}$. It is important to recall that any three points in $\mathbb{C}$ are either *collinear* (they lie on a line), or *concyclic* (they lie on a circle), but not both. Also, as f is a bijection of $\mathbb{C}$ onto itself, f^{-1} exists and is also a bijection of $\mathbb{C}$ onto itself.