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RAMANUJAN'S REMARKABLE SUMMATION FORMULA AND AN INTERESTING CONVOLUTION IDENTITY

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In this note we obtain a convolution identity for the coefficients $B_n(\alpha, \theta, q)$ defined by

$$\frac{\prod_{n=1}^{\infty}\left(1+2xq^n\cos\theta+x^2q^{2n}\right)}{\prod_{n=1}^{\infty}\left(1+2\alpha xq^n\cos\theta+\alpha^2x^2q^{2n}\right)}=\sum_{n=-\infty}^{\infty}B_n(\alpha, \theta, q)x^n$$

using Ramanujan's ${}_1\Psi_1$ summation. The identity contains as special cases convolution identities of Kung-Wei Yang and a few more interesting analogue.

1. INTRODUCTION

In this note we apply Ramanujan's ${}_1\Psi_1$ summation [17, p.196, Entry 17] and obtain a new convolution identity which contains as special cases the identities of Kung-Wei Yang [20] and a few more interesting analogous. Connections with the generalised Frobenius partition functions of some of our identities are also pointed out.

Ramanujan's ${}_1\Psi_1$ summation can be stated as

$$\frac{(qz;q)_\infty(1/z;q)_\infty(q;q)_\infty(\alpha\beta q;q)_\infty}{(\alpha qz;q)_\infty(\beta/z;q)_\infty(\alpha q;q)_\infty(\beta q;q)_\infty}=\sum_{n=-\infty}^{\infty}\frac{(1/\alpha;q)_n(\alpha qz)^n}{(\beta q;q)_n} \tag{1.1}$$

where

$$|\beta|<|z|<1/|\alpha q|,\ |q|<1,$$

$$(a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n)$$

and

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

(1.1) contains Jacobi's triple product identity [16]

$$(-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty=\sum_{n=-\infty}^{\infty}q^{n^2}z^n \tag{1.2}$$

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and the q-binomial theorem [**18**]

$$\frac{(qz;q)_\infty}{(\alpha qz;q)_\infty} = \sum_{n=0}^{\infty} \frac{(1/\alpha;q)_n(\alpha qz)^n}{(q;q)_n} \tag{1.3}$$

as special cases (In (1.1) put $\alpha = 0 = \beta$ and $z = -z/\sqrt{q}$ and then change q to q^2 to obtain (1.2); put $\beta = 1$ to obtain (1.3)). There are several proofs of (1.1) in the literature [**2, 3, 6, 7, 13, 14, 15**] including direct proofs (see for example [**1, 19**]) which do not presuppose Jacobi's triple product identity (1.2) or the q-binomial theorem (1.3). There are many interesting applications of (1.1), see for instance [**7, 8, 9, 10 and 11**].

In Section 2 below we obtain by a simple application of (1.1) a convolution identity for the coefficients $B_n(\alpha, \theta, q)$ defined by

$$\frac{\prod_{n=1}^{\infty}\left(1+2xq^n\cos\theta+x^2q^{2n}\right)}{\prod_{n=1}^{\infty}\left(1+2\alpha xq^n\cos\theta+\alpha^2x^2q^{2n}\right)} = \sum_{n=-\infty}^{\infty} B_n(\alpha, \theta, q)x^n. \tag{1.4}$$

Indeed, we show

$$\begin{aligned}\sum_{n=-\infty}^{\infty} & q^{-n}B_n(\beta, \theta, q)B_{m+n}(\alpha, \theta, q)\\ &= \frac{(-1)^m\left(e^{i\theta}\alpha q\right)^m(1/\alpha;q)_m(\beta q;q)_\infty^2(\alpha q;q)_\infty^2}{(q;q)_\infty^2(\alpha\beta q;q)_\infty^2(\beta q;q)_m}\\ &\quad\times {}_2\Psi_2\left[\begin{matrix} q^m/\alpha, & 1/\beta & ; & \\ & & & q;\alpha\beta qe^{2i\theta} \\ \alpha q, & \beta q^{m+1} & ; & \end{matrix}\right].\end{aligned} \tag{1.5}$$

Here as usual

$${}_2\Psi_2\left[\begin{matrix} a, & b; & \\ & & q;z \\ c, & d; & \end{matrix}\right] = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(d;q)_n}z^n. \tag{1.6}$$

In Section 3, we obtain special cases of (1.5) and in this context we shall need the following evaluation of ${}_2\Psi_2$ [**12**, p.305].

$$\begin{aligned}{}_2\Psi_2 & \left[\begin{matrix} b, & c & ; & \\ & & & q;-aq/bc \\ aq/b, & aq/c & ; & \end{matrix}\right]\\ &= \frac{(aq/bc;q)_\infty\left(aq^2/b^2;q^2\right)_\infty\left(aq^2/c^2;q^2\right)_\infty\left(q^2;q^2\right)_\infty}{(q/b;q)_\infty(q/c;q)_\infty(aq/b;q)_\infty}\\ &\quad\times\frac{\left(aq;q^2\right)_\infty\left(q/a;q^2\right)_\infty}{(aq/c;q)_\infty(-aq/bc;q)_\infty}.\end{aligned} \tag{1.7}$$

Indeed, in Section 3 we show that (1.5) yields as special cases, convolution identities of Yang [20] for the coefficients A_n defined by

$$\prod_{n=1}^{\infty}\left(1+xq^n+x^2q^{2n}\right)=\sum_{n=-\infty}^{\infty}A_nx^n \tag{1.8}$$

and a convolution identity for the coefficients C_n defined by

$$\prod_{n=1}^{\infty}(1+xq^n)=\sum_{n=-\infty}^{\infty}C_nx^n. \tag{1.9}$$

Also we deduce from (1.5) convolution identities analogous to those of Yang which seem new. In fact, we obtain convolution identities

$$\sum_{n=-\infty}^{\infty}q^{-n}D_nD_{2m+n}=\frac{q^{m(m+1)}\left(-q;q^2\right)_\infty^2}{(q;q)_\infty(q;q^2)_\infty}$$

and

$$\sum_{n=-\infty}^{\infty}q^{-n}D_nD_{2m+n-1}=\frac{2q^{m^2}\left(-q^2;q^2\right)_\infty^2}{(q;q)_\infty(q;q^2)_\infty}$$

where the coefficients D_n are defined by

$$\prod_{n=1}^{\infty}\left(1+2xq^n+x^2q^{2n}\right)=\sum_{n=-\infty}^{\infty}D_nx^n. \tag{1.10}$$

2. The main theorem

Theorem 2.1. *If $B_n(\alpha, \theta, q)$ is as defined by (1.4) then*

$$\begin{aligned}\sum_{n=-\infty}^{\infty}&q^{-n}B_n(\beta,\theta,q)B_{m+n}(\alpha,\theta,q)\\&=\frac{(-1)^m\left(e^{i\theta}\alpha q\right)^m(1/\alpha;q)_m(\beta q;q)_\infty^2(\alpha q;q)_\infty^2}{(q;q)_\infty^2(\alpha\beta q;q)_\infty^2(\beta q;q)_m}\\&\quad\times {}_2\psi_2\left[\begin{matrix}q^m/\alpha,\ 1/\beta;\\ \alpha q,\ \beta q^{m+1};\end{matrix}\ q;\alpha\beta q e^{2i\theta}\right].\end{aligned} \tag{2.1}$$

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PROOF: By (1.4)

$$\left[\sum_{n=-\infty}^{\infty} B_n(\alpha, \theta, q)x^n\right]\left[\sum_{n=-\infty}^{\infty} B_n(\beta, \theta, q)q^{-n}x^{-n}\right]$$
$$= \frac{\prod_{n=1}^{\infty}\left(1+2xq^n\cos\theta+x^2q^{2n}\right)\left(1+2x^{-1}q^{n-1}\cos\theta+x^{-2}q^{2(n-1)}\right)}{\prod_{n=1}^{\infty}\left(1+2\alpha xq^n\cos\theta+\alpha^2x^2q^{2n}\right)\left(1+2\beta x^{-1}q^{n-1}\cos\theta+\beta^2x^{-2}q^{2(n-1)}\right)}$$
$$= \frac{\left(-xe^{i\theta}q;q\right)_\infty\left(-e^{-i\theta}/x;q\right)_\infty\left(-xe^{-i\theta}q;q\right)_\infty\left(-e^{i\theta}/x;q\right)_\infty}{(-\alpha xe^{i\theta}q;q)_\infty(-\beta e^{-i\theta}/x;q)_\infty(-\alpha xe^{-i\theta}q;q)_\infty(-\beta e^{i\theta}/x;q)_\infty}$$
$$= \frac{(\beta q;q)^2_\infty(\alpha q;q)^2_\infty}{(q;q)^2_\infty(\alpha\beta q;q)^2_\infty}\left[\sum_{k=-\infty}^{\infty}\frac{(1/\alpha;q)_k\left(-\alpha xe^{i\theta}q\right)^k}{(\beta q;q)_k}\right]$$
$$\times\left[\sum_{j=-\infty}^{\infty}\frac{(1/\alpha;q)_j\left(-\alpha xe^{-i\theta}q\right)^j}{(\beta q;q)_j}\right], \qquad \text{on using (1.1).}$$

Now, comparing the coefficients of x^m we get

$$\sum_{n=-\infty}^{\infty} q^{-n}B_n(\beta, \theta, q)B_{m+n}(\alpha, \theta, q)$$
$$= \frac{(\beta q;q)^2_\infty(\alpha q;q)^2_\infty}{(q;q)^2_\infty(\alpha\beta q;q)^2_\infty}$$
$$\times\sum_{n=-\infty}^{\infty}\frac{(1/\alpha;q)_{-n}\left(-\alpha e^{-i\theta}q\right)^{-n}(1/\alpha;q)_{m+n}\left(-\alpha e^{i\theta}q\right)^{m+n}}{(\beta q;q)_{-n}(\beta q;q)_{m+n}}$$
$$= \frac{(-1)^m\left(e^{i\theta}\alpha q\right)^m(1/\alpha;q)_m(\beta q;q)^2_\infty(\alpha q;q)^2_\infty}{(\beta q;q)_m(q;q)^2_\infty(\alpha\beta q;q)^2_\infty}$$
$$\times\sum_{n=-\infty}^{\infty}\frac{(1/\alpha;q)_{-n}(q^m/\alpha;q)_n e^{2in\theta}}{(\beta q;q)_{-n}(\beta q^{m+1};q)_n}$$
$$= \frac{(-1)^m\left(e^{i\theta}\alpha q\right)^m(1/\alpha;q)_m(\beta q;q)^2_\infty(\alpha q;q)^2_\infty}{(\beta q;q)_m(q;q)^2_\infty(\alpha\beta q;q)^2_\infty}$$
$$\times\sum_{n=-\infty}^{\infty}\frac{(q^m/\alpha;q)_n(1/\beta;q)_n\left(\alpha\beta qe^{2i\theta}\right)^n}{(\alpha q;q)_n(\beta q^{m+1};q)_n}.$$

This completes the proof of the Theorem 2.1. □

3. Some special cases

Let $\overline{B}_n(\alpha, q)$ be defined by

(3.1) $$\frac{\prod_{n=1}^{\infty}\left(1+x^2q^{2n}\right)}{\prod_{n=1}^{\infty}\left(1+\alpha^2x^2q^{2n}\right)} = \sum_{n=-\infty}^{\infty}\overline{B}_n(\alpha, q)x^n$$

so that, by (1.4)

$$\overline{B}_n(\alpha, q) = B_n(\alpha, \pi/2, q). \tag{3.2}$$

Putting $\theta = \pi/2$ in (2.1) and then using (3.2) and (1.7) we have the following Theorem.

THEOREM 3.1. *If $\overline{B}_n(\alpha, q)$ is as defined by (3.1), then*

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{-n}\overline{B}_n(\beta, q)\overline{B}_{m+n}(\alpha, q) &= \frac{(-i)^m(\alpha q)^m(1/\alpha; q)_m\left(\alpha^2/q^{m-2}; q^2\right)_\infty(\alpha q; q)_\infty}{(q; q)_\infty^2(\alpha\beta q; q)_\infty} \\ &\times \frac{\left(\beta^2 q^{m+2}; q^2\right)_\infty\left(q^2; q^2\right)_\infty\left(q^{m+1}; q^2\right)_\infty\left(1/q^{m-1}; q^2\right)_\infty}{(\alpha/q^{m-1}; q)_\infty(-\alpha\beta q; q)_\infty}.\end{aligned} \tag{3.3}$$

Changing m to $2m$ in (2.1) and putting $\theta = \pi/3$, $\alpha = 0 = \beta$ and noting from (1.4) and (1.8) that $A_n = A_n(q) = B_n(0, \pi/3, q)$ we obtain

$$\sum_{n=-\infty}^{\infty} q^{-n}A_n A_{2m+n} = \frac{q^{m(m+1)}}{(q; q)_\infty^2}\sum_{n=-\infty}^{\infty} q^{(m+n)^2}\omega^{(m+n)} \tag{3.4}$$

where $\omega = e^{2\pi i/3}$. Employing (1.2) (with $z = \omega$) in the right side of (3.4) and then using the easily verified Euler's identity

$$\left(-q; q^2\right)_\infty = 1/\left(q; q^2\right)_\infty\left(-q^2; q^2\right)_\infty$$

we at once have the following Theorem of Yang [**20**].

THEOREM 3.2. (K.W. Yang)

$$\sum_{n=-\infty}^{\infty} q^{-n}A_n A_{2m+n} = \frac{q^{m(m+1)}\left(-q^3; q^6\right)_\infty\left(-q^2; q^2\right)_\infty}{(q; q)_\infty}. \tag{3.5}$$

Similarly on changing m to $2m-1$ in (2.1) and then proceeding as above, we obtain another result of Yang [**20**].

THEOREM 3.3. (K.W. Yang).

$$\sum_{n=-\infty}^{\infty} q^{-n}A_n A_{2m+n-1} = \frac{q^{m^2}\left(-q^6; q^6\right)_\infty\left(-q; q^2\right)_\infty}{(q; q)_\infty}. \tag{3.6}$$

Changing m to $2m$ in (2.1) and then putting $\theta = \pi/2$, $\alpha = 0 = \beta$ and noting from (1.4) and (1.9) that $B_{2n+1} = 0$ and $B_{2n}(0, \pi/2, \sqrt{q}) = C_n$, we get

$$\sum_{n=-\infty}^{\infty} q^{-2n} C_n(q^2) C_{m+n}(q^2) = \frac{q^{m(m+1)}}{(q;q)_\infty^2} \sum_{n=-\infty}^{\infty} (-1)^{m+n} q^{(m+n)^2}. \tag{3.7}$$

Employing (1.2) (with $z = -1$) in the right side of (3.7) and then changing q to $\sqrt{q}$ we have the following theorem equivalent to an identity of Cauchy [4, p.22].

Theorem 3.4.

$$\sum_{n=-\infty}^{\infty} q^{-n} C_n C_{m+n} = \frac{q^{m(m+1)/2}}{(q;q)_\infty}. \tag{3.8}$$

We may remark that (3.8) can of course be obtained from (3.3) by changing m to $2m$ and then putting $\alpha = 0 = \beta$. To see the equivalence of (3.8) with the aforementioned identity of Cauchy [4, p.22], first put $\alpha = 0$ and $z = -x$ in (1.3) to get

$$\prod_{n=1}^{\infty} (1 + xq^n) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} x^n}{(q;q)_n}$$

which is nothing but an identity of Euler [4]. Comparing this with (1.9), we have

$$C_n = \frac{q^{n(n+1)/2}}{(q;q)_n}$$

using which in (3.8) we have

$$\sum_{n=0}^{\infty} \frac{q^{n(n+m)}}{(q;q)_n (q;q)_{n+m}} = \frac{1}{(q;q)_\infty}, \tag{3.8$'$}$$

the required identity.

Changing m to $2m$ in (2.1) and then putting $\theta = 0$, $\alpha = 0 = \beta$ and noting from (1.4) and (1.10) that $D_n = D_n(q) = B_n(0, 0, q)$ we obtain

$$\sum_{n=-\infty}^{\infty} q^{-n} D_n D_{2m+n} = \frac{q^{m(m+1)}}{(q;q)_\infty^2} \sum_{n=-\infty}^{\infty} q^{(m+n)^2}. \tag{3.9}$$

Using (1.2) (with $z = 1$) in (3.9) we have the following Theorem.

Theorem 3.5.

$$\sum_{n=-\infty}^{\infty} q^{-n} D_n D_{2m+n} = \frac{q^{m(m+1)} (-q;q^2)_\infty^2}{(q;q)_\infty (q;q^2)_\infty}. \tag{3.10}$$

Similarly, on changing m to $2m-1$ in (2.1) and then proceeding as in the derivation of (3.10) we have the following Theorem:

Theorem 3.6.

$$\sum_{n=-\infty}^{\infty} q^{-n} D_n D_{2m+n-1} = \frac{2q^{m^2} (-q^2;q^2)_\infty^2}{(q;q)_\infty (q;q^2)_\infty}. \tag{3.11}$$

4. Some partition theoretic interpretations and open questions

In this section we bring about connections of some results of Section 3 with the generalised Frobenius partition functions and raise some questions.

1. An application of the q-binomial theorem to the left side of (3.1) at once yields the finite product representation of $\overline{B}_n(\alpha, q)$ namely

$$\overline{B}_{2n}(\alpha, q) = \frac{(-1)^n (\alpha^{-2}; q^2)_n \alpha^{2n} q^{2n}}{(q^2; q^2)_n}$$

and $B_{2n+1}(\alpha; q) = 0$. Consequently, Theorem 3.1 must be an instance of one of the ${}_2\Phi_1$ summations. Which one?

2. Considering the right side of (3.5) we have

$$\frac{q^{m(m+1)}(q^2;q^2)_\infty(-q^3;q^6)_\infty}{(q;q)_\infty} = \frac{q^{m(m+1)}(q^2;q^2)_\infty(-q^3;q^6)_\infty}{(q;q)^2_\infty(-q;q^2)_\infty}$$

$$= \frac{q^{m(m+1)}}{(q;q)_\infty(q^2;q^{12})_\infty(q^3;q^{12})_\infty(q^9;q^{12})_\infty(q^{10};q^{12})_\infty}$$

$$= q^{m(m+1)} \sum_{n=0}^{\infty} \phi_2(n) q^n$$

by Corollary 5.1 of [5], where $\phi_2(n)$ is the number of generalised Frobenius partitions of n that allow up to 2 repetitions of an integer in any row [5, p.6]. Thus we have proved that $\sum_{n=-\infty}^{\infty} q^{-n-m^2-m} A_n A_{2m+n}$ is the generating function for $\phi_2(n)$. Thus Theorem 3.2 takes on combinatorial significance. Is there a direct arithmetic proof of this fact?

3. Similarly, in Theorem 3.5, considering the right side of (3.10) and using Corollary 5.2 of [5] we can show that $\sum_{n=-\infty}^{\infty} q^{-n-m^2-m} D_n D_{2m+n}$ is the generating function for $c\phi_2(n)$, the number of 3-coloured generalised Frobenius partitions of n [5, p.7]. Is there a direct arithmetic proof of this fact also?

4. Are there any combinatorial facts in the analogous Theorems 3.3 and 3.6? Unlike in Theorems 3.2 and 3.5 the infinite products in the right sides of (3.6) and (3.11) do not seem to generate any partition functions.

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