PARTIAL SOLUTION TO MACKEY'S PROBLEM ABOUT MODULAR PAIRS AND COMPLETENESS

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1. Introduction and statement of the theorem. Two elements A, B of a lattice are said to form a modular pair when $(X \vee A) \wedge B = X \vee (A \wedge B)$ holds for all $X \leqq B$, and are said to form a dual-modular pair when $(X \wedge A) \vee B = X \wedge (A \vee B)$ holds for all $X \geqq B$.

We are concerned here with a particular lattice, the lattice of closed subspaces of a normed linear space, and with a question posed by Mackey in 1945 (**6**, p. 206, problem 2), namely:

"Are there any incomplete normed linear spaces in whose lattices of closed subspaces modularity and d-modularity are equivalent?".

The principal result of this paper is the following.

THEOREM 1. *If, in the lattice of all norm-closed subspaces of a real or complex inner product space X, every modular pair is dual-modular, then X is complete.*

This provides the answer "no" to Mackey's question for spaces whose norm can be derived from an inner product.

In § 2 dual modularity is studied, in § 3 modularity is covered, and § 4 contains the proof of Theorem 1. Ideas of Amemiya and Araki (**1**) are used in the proof.

2. Dual-modular pairs of subspaces. A closure operation on a lattice $\mathscr{K}$ is a mapping $M \to \bar{M}$ of $\mathscr{K}$ into itself that satisfies the following three conditions: (1) $M \leqq \bar{M}$, (2) $\bar{M} = \bar{\bar{M}}$, and (3) $M \leqq N \Rightarrow \bar{M} \leqq \bar{N}$. We shall take for granted an acquaintance with the elementary properties of closure operations as is contained for example in Birkhoff's book (**3**; Chapter V, § 1), the principal fact we wish to recall being that the elements of $\mathscr{K}$ that satisfy $M = \bar{M}$ (elements we shall call closed) constitute a complete lattice $\mathscr{L}$ under the same order as that in $\mathscr{K}$, where now the meet operation $(\wedge)$ in $\mathscr{L}$ is the same as that in $\mathscr{K}$, but the join operation $(\vee)$ is in general different.

We are solely interested in the case when $\mathscr{K}$ is the lattice of all subspaces of a linear space X and when the closure operation satisfies these two additional criteria: (4) The zero subspace is closed, and (5) If M is a closed subspace and $x \in X$, then $M + \mathrm{sp}(x)$ is closed ($\mathrm{sp}(x)$ stands for the zero-dimensional or one-dimensional subspace spanned by the vector x). We shall call a mapping $M \to \bar{M}$ that satisfies these two conditions a *Mackey closure operation*.

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The following theorem, while somewhat more general, is still substantially Mackey's (**6**, Theorem III–6).

THEOREM 2 (Mackey). *Let X be a linear space over an arbitrary field, let there be given a Mackey closure operation on the lattice of all subspaces of X, and let $\mathscr{L}$ symbolize the lattice of closed subspaces. Then the closed subspaces M, N form a dual-modular pair in $\mathscr{L}$ when and only when the linear span $M + N$ is closed.*

Proof. If $M + N$ is closed, so that $M + N = M \vee N$, and if $y \in K \wedge (M \vee N)$, where K is a closed subspace, $K \geqq N$, then $y \in M \vee N = M + N$ hence $y = m + n$, $m \in M$, $n \in N$. Then $m = y - n \in K$, inasmuch as $y \in K$ and $n \in N \leqq K$. Thus $m \in M \wedge K$, hence

$$y = m + n \in (M \wedge K) + N \leqq (M \wedge K) \vee N.$$

As the reverse inequality is valid in any lattice, we have equality.

Conversely, suppose that M, N is a dual-modular pair. We wish to prove that $M + N = M \vee N$. If there exists an $x \in M \vee N$, $x \notin M + N$, then, setting $K = N + \mathrm{sp}(x)$, we would have $x \in K \wedge (M \vee N) = (K \wedge M) \vee N$ (using dual modularity). However, at the same time, we would have to have $K \wedge M = N \wedge M$ since

$$y \in K \wedge M \Rightarrow y = n + \lambda x \in M \Rightarrow \lambda x = y - n \in M + N$$

which implies that $\lambda = 0$, since $x \notin M + N$. Thus $y = n \in M \wedge N$ so that $K \wedge M = N \wedge M$ and thus $x \in (K \wedge M) \vee N = (N \wedge M) \vee N = N$, a contradiction. Thus no such x exists, from which we infer that $M + N = M \vee N$; in other words, $M + N$ is closed.

One immediate consequence of this theorem is that, in such a lattice $\mathscr{L}$, the relation "(M, N) form a dual-modular pair" is symmetric in M and N.

Topological closure in a Hausdorff topological vector space is a Mackey closure operation (property (5), the most difficult to verify, is proved, for example, in (**7**, Chapter I, § 3.3)). Consequently, a pair M, N of topologically closed subspaces is dual-modular precisely when their sum $M + N$ is topologically closed. This is in particular true for normed linear spaces, and, since this case has special importance for us, we state it explicitly.

COROLLARY 3. *In the lattice of all closed subspaces of a normed linear space, a pair M, N is dual-modular when and only when the linear span $M + N$ is closed.*

A linear system X_L consists of a linear space X together with a distinguished subspace L of the algebraic dual of X. We shall also always assume that the subspace L is total (that is, $l(x) = 0$ for all $l \in L$ implies $x = 0$) although this is not commonly part of the definition of a linear system. If X_L is a linear system, for a subset $M \subseteq X$, set $M' = (l \in L;\ l(M) = 0)$ and for a subset $N \subseteq L$, set $N' = (x \in X; l(x) = 0$ for all $l \in N)$. Then the mapping $M \rightarrow M''$

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is a Mackey closure operation on the subspaces of X (**6**, Theorem III–1). Consequently, we have the following result.

COROLLARY 4. *In the lattice of all closed subspaces of a linear system, the pair* (M, N) *is dual-modular if and only if* $M + N$ *is closed.*

There is a Mackey closure operation, namely bounded closure (**6**, Chapter IV, § 1 and Theorem IV–8), that is not closure with respect to any linear system (**6**, Chapter IV, last paragraph of § 2). Consequently, Theorem 2 cannot be derived from Corollary 4 (which is Mackey's original result). Of course, Corollary 3 can be derived from Corollary 4 by using the linear system X_L, where X is the topological linear space and L is its conjugate space consisting of all continuous linear functionals on X.

Since bounded closure is a Mackey closure operation, we see by Theorem 2 that the pair (M, N) is dual-modular in the lattice of all boundedly closed subspaces when and only when $M + N$ is boundedly closed. Is the sum of boundedly closed subspaces always boundedly closed? This is Mackey's problem (**6**, problem 4(a)) which, like many of his other fourteen problems, remains unsolved.

3. Modular pairs of subspaces in a normed linear space. The characterization of modularity is based on the concept contained in the following lemma, whose simple proof we leave to the reader.

LEMMA 5. *Let* X *be a normed linear space, and let* M, N *represent closed subspaces of* X. *The following conditions on* M *and* N *are equivalent:*

(1) *There exists* $\alpha > 0$ *such that* $||m|| + ||n|| \leqq \alpha||m + n||$ *for all* $m \in M$, $n \in N$;

(2) *There exists* $\alpha > 0$ *such that* $m \in M$, $||m|| = 1$, $n \in N$ *together imply* $||m - n|| \geqq \alpha$;

(3) $M \cap N = 0$ *and the (single-valued) linear operators* $S(m + n) = m$, $T(m + n) = n$ *are both bounded as maps on the normed linear space* $M + N$.

We shall call subspaces that satisfy the equivalent conditions of Lemma 5 *completely disjoint*; modularity is characterized in terms of complete disjointness.

THEOREM 6 (Mackey). *Let* X *be a normed linear space,* $\mathscr{L}$ *its lattice of closed subspaces, and let* M, N *represent closed subspaces of* X *satisfying* $M \cap N = 0$. *Then in order that* M, N *form a modular pair in* $\mathscr{L}$, *it is necessary and sufficient that they be completely disjoint.*

The proof of this characterization of modular pairs is quite difficult, using as it does deep topological facts in an apparently essential way, in surprising contrast to the simple characterization of dual-modularity (Theorem 2). Our proof of this theorem contains some technical innovations, the principal one being the use of unbounded operators, and thus provides a hopefully clearer but at least different derivation of the result.

Indeed, it is just this strong contrast between the concepts of modular and dual-modular that accounts for validity of the main theorem, the rationale being that the condition "modular $\Rightarrow$ dual-modular" derives its strength (a strength sufficient to imply metric completeness) from the fact that it forces the identification of two markedly different concepts.

The idea of our proof is to transfer the problem to the conjugate space, which is a Banach space so that the closed graph theorem can be used, and in which the original modular pair is transferred to a dual-modular pair which can then be characterized by Theorem 2. Since the conjugate space does not appear in the statement of the theorem, the use of the conjugate space can be regarded as an unsatisfactory aspect of the proof. A direct proof would be desirable.

We begin the proof of the theorem by recalling some facts about conjugate spaces and about adjoints of unbounded operators.

The conjugate space $L = X^*$, consisting of all continuous linear functionals on the normed linear space X, is a Banach space with the norm

$$||f|| = \text{lub}(|f(x)|;\ ||x|| \leqq 1).$$

A subspace M of X is norm-closed if and only if $M = M''$ in the linear system X_L; this is an easy consequence of the Hahn-Banach theorem. However, norm-closure in X^* is different from closure in the linear system $(X^*)_X$, those subspaces P of X^* that satisfy $P = P''$ (relative to the linear system $(X^*)_X$), which are what Banach called the regularly closed subspaces of X^*, being norm-closed in X^*, but not all norm-closed subspaces of X^* having this form (**2**, Chapter VIII, §§ 1, 2) (unless X is a reflexive Banach space.) Nonetheless, $P \to P''$ is a Mackey closure operation on the subspaces of X^*, and the maps $M \to M'$, $P \to P'$ are each one-to-one and onto as maps between the lattice $\mathscr{L}(X)$ of all norm-closed subspaces of X and the lattice $\mathscr{L}(X^*)$ of all regularly closed subspaces of X^*, are each order inverting, and are inverse to each other. Therefore a pair (M, N) of norm-closed subspaces is modular in $\mathscr{L}(X)$ if and only if (M', N') is dual-modular in $\mathscr{L}(X^*)$ which by Theorem 1 is equivalent to $M' + N'$ being regularly closed; i.e., $M' + N' = (M' + N')''$.

If S is a linear operator defined on the subspace $\text{dom}(S)$ of the normed linear space X, taking its values in X, we say that S is closed when $x_i \in \text{dom}(S)$, $i = 1, 2, \ldots, x_i \to x$, $Sx_i \to y$ together imply $x \in \text{dom}(S)$ and $Sx = y$ (**4**, Remark II.1.3). We shall use frequently the important *closed graph theorem*: A closed linear operator, defined everywhere on a *complete* normed linear space, is necessarily bounded (**4**, II.1.9). If S is a linear operator, not necessarily closed, defined on the dense subspace $\text{dom}(S)$ of the normed linear space X, the adjoint operator S^* (on X^*) is defined as follows: $\text{dom}(S^*)$ is the subspace comprising those, and only those, $f \in X^*$ for which $\hat{f}(\cdot) = f(S(\cdot))$ is a bounded functional on $\text{dom}(S)$. Since $\text{dom}(S)$ is dense, every such functional has a unique continuous extension to all of X and we define the value

$\hat{f} = S^*(f)$ to be this unique element of X^* (**4**, II.2.2). We note this fact: S^* is always a closed operator on the Banach space X^* (**4**, II.2.6) and is bounded and everywhere defined when S is bounded (**4**, II.2.8). Supposing now that $\mathrm{dom}(S^*)$ is dense in X^* (which need not be the case in general), we can form its adjoint, S^{**}, which is a linear map on X^{**}. We regard X as being a subspace of X^{**} in the usual way by identifying the element x with the functional on X^* given by $x(f) = f(x)$. The domain of S^{**} comprises those $x \in X^{**}$ for which $\hat{x}(\cdot) = x(S^*(\cdot))$ is a bounded functional on $\mathrm{dom}(S^*)$, and when $x \in \mathrm{dom}(S^{**})$, the value $S^{**}(x)$ is the unique continuous extension of this functional $\hat{x}$ to all of X^*. However, if $x \in \mathrm{dom}(S)$, and $f \in \mathrm{dom}(S^*)$, then $S^*(f) = \hat{f}$, where $\hat{f}$ is the functional $\hat{f}(\cdot) = f(S(\cdot))$ which is bounded on $\mathrm{dom}(S)$, so that

$$|\hat{x}(f)| = |x(S^*(f))| = |x(\hat{f})| = |\hat{f}(x)| = |f(S(x))| \leqq ||f||\,||Sx|| = \alpha_x||f||.$$

Thus $\hat{x}(\cdot)$ is a bounded linear functional on $\mathrm{dom}(S^*)$ and thus $x \in \mathrm{dom}(S^{**})$. Moreover, $[S^{**}(x)](f) = \hat{x}(f) = x(S^*(f)) = \hat{f}(x) = f(Sx) = [Sx](f)$ (considering the element Sx as a functional on X^*), from which we infer that $\mathrm{dom}(S) \subseteq \mathrm{dom}(S^{**})$ and, when $x \in \mathrm{dom}(S)$, that $Sx = S^{**}x$. Thus, as we commonly say, S^{**} *extends* S. In particular, we are entitled to draw this conclusion: If S^{**} is bounded, so is S.

We shall be applying the above theory to two special operators, obtained as in Lemma 5 (3). Suppose that M and N are two closed subspaces of X satisfying $M \cap N = 0$ and having $M + N$ dense in X. We define two (single-valued) linear operators S and T, each of which has domain $M + N$, by setting $S(m + n) = m$, $T(m + n) = n$. Both S and T are closed operators (**4**, proof of II.1.14). The adjoints of S and T are easily computed. Note first that (in general) $M' \cap N' = (M + N)'$, from which we conclude that $M' \cap N' = 0$, $M + N$ being dense. Since $M' \cap N' = 0$, the formulas $S^*(m' + n') = n'$, $T^*(m' + n') = m'$ define (single-valued) linear operators on $M' + N'$, which, as the notation anticipates, turn out to be the adjoints of S and T. To prove this we must verify that $\hat{f}(\cdot) = f(S(\cdot))$ is bounded on $M + N$ when and only when $f \in M' + N'$. However, $\hat{f}(m + n) = f(S(m + n)) = f(m)$ and this is bounded on $M + N$ precisely when there exists an $\alpha > 0$ such that $|\hat{f}(m + n)| \leqq \alpha||m + n||$ or that $|f(m)| \leqq \alpha||m + n||$ for all $m \in M$, $n \in N$. This latter condition is equivalent to $|f(n)| \leqq \beta||m + n||$ for some $\beta > 0$, since

$$\begin{aligned}|f(n)| = |f(m + n - m)| = |f(m + n) - f(m)| &\leqq |f(m + n)| + |f(m)| \\ &\leqq ||f||\,||m + n|| + \alpha||m + n|| = (||f|| + \alpha)||m + n||.\end{aligned}$$

And, putting these inequalities together, we obtain the following equivalence: $\hat{f}$ is bounded on $M + N$ if and only if there exists an $\alpha > 0$ such that $|f(m)| + |f(n)| \leqq \alpha||m + n||$ for all $m \in M$, $n \in N$. However, this latter condition is equivalent to $f \in M' + N'$, for if $|f(m)| + |f(n)| \leqq \alpha||m + n||$, then, upon setting $f_2(m + n) = f(m)$, $f_1(m + n) = f(n)$, we obtain two linear functionals f_1, f_2 on $M + N$, both of which are bounded and therefore uniquely extendable to X^*, and which satisfy $f_1(M) = f_2(N) = 0$, $f = f_1 + f_2$, hence we have f written as desired. Conversely, if $f = f_1 + f_2$ with $f_1 \in M', f_2 \in N'$, then $|f(m)| = |f_1(m) + f_2(m)| = |f_2(m)| = |f_2(m + n)| \leqq ||f_2|| \, ||m + n||$, and similarly $|f(n)| \leqq ||f_1|| \, ||m + n||$ so that

$$|f(m)| + |f(n)| \leqq (||f_1|| + ||f_2||)(||m + n||).$$

The value $S^*(f)$, when $f \in M' + N'$, is determined by the formula

$$[S^*(f)](x) = f(S(x)), \qquad x \in M + N.$$

This reduces to $g(m + n) = f(m)$ (where we have set $g = S^*(f)$), which in turn implies that $g \in N'$. However, since every $f \in M' + N'$ is uniquely a sum $f = m' + n'$, $m' \in M'$, $n' \in N'$, we have $g = n'$ and thus $S^*(m' + n') = n'$, as desired. The operator T is handled in exactly the same way.

Using these constructions, we now prove Theorem 6. We first note that we may assume that $M + N$ is dense in X simply by replacing X by the closure Y of $M + N$, and noting that the pair M, N is modular or completely disjoint in Y precisely when it is modular or completely disjoint, in X, respectively. We maintain our previous notation using S, T for the operators $S(m + n) = m$, $T(m + n) = n$ both defined on the (now) dense subspace $M + N$.

If (M, N) is a modular pair in $\mathscr{L}(X)$, then (M', N') is a dual-modular pair in $\mathscr{L}(X^*)$ so that $M' + N'$ is regularly closed (Theorem 2). This means that $M' + N' = (M' + N')''$ which, using $M' \cap N' = (M + N)'$, yields $M' + N' = (M \cap N)' = 0' = X^*$. Thus S^* and T^* are closed, everywhere-defined linear operators on the Banach space X^* and, by the closed graph theorem, are consequently bounded. Therefore S^{**} and T^{**} are also bounded, and with them, S and T. This means that the pair (M, N) is completely disjoint (Lemma 5).

If the pair (M, N) is completely disjoint, then the operators S and T are bounded and densely defined, and consequently S^* and T^* are bounded operators defined everywhere on X^* (**4**, II.2.8). Consequently, since

$$\operatorname{dom}(S^*) = \operatorname{dom}(T^*) = M' + N',$$

we have $M' + N' = X^*$. Therefore $M' + N'$ is regularly closed, (M', N') is a dual-modular pair, thus (M, N) is a modular pair. That completes the proof of Theorem 6.

We close this section with a few related results that we shall need.

LEMMA 7. *If the closed subspaces M, N of a Banach space satisfy $M \cap N = 0$, then M and N are completely disjoint when and only when $M + N$ is closed.*

Proof. If M and N are completely disjoint, then the closed operators $S(m + n) = m$, $T(m + n) = n$ are bounded. As is easily checked, any closed, bounded operator has a closed domain, hence $M + N = \operatorname{dom}(S)$ is closed. Conversely, if $M + N$ is closed, then it is a Banach space, and by the closed graph theorem, S and T are bounded, so that M and N are completely disjoint.

Using Theorem 6 and the above lemma, we could now easily deduce Mackey's result: *In the lattice of all norm-closed subspaces of a Banach space, every modular pair is dual-modular, and every dual-modular pair is modular.*

The next result is from (**5**, Lemma 5).

THEOREM 8. *Let M be a closed subspace of a Banach space X, and let D be a dense subspace of X. If there exists a closed subspace N with the following properties:*

(1) $N \subseteq D$,

(2) $M \cap N = 0$, $M + N = X$,

then $D \cap M$ is dense in M. In particular, if M has finite codimension, then $D \cap M$ is dense in M for any dense subspace D.

Proof. Let $m \in M$ be chosen. Since D is dense, there exist $y_i \in D$, $i = 1, 2, \ldots$, such that $y_i \to m$. We have $D = (D \cap M) + N$ (as is easily checked), hence $y_i = r_i + z_i$, $r_i \in D \cap M$, $z_i \in N$, $i = 1, 2, \ldots$. By Lemma 7, (M, N) is a completely disjoint pair, and applying Lemma 5(1) we deduce that $||r_i - r_j|| + ||z_i - z_j|| \leqq \alpha||y_i - y_j||$ for some $\alpha > 0$. Accordingly, both sequences (r_i) and (z_i) are Cauchy and thus converge, say $r_i \to r \in M$, $z_i \to z \in N$. Then $m = r + z$ and at the same time $m = m + 0$, so that $r = m$, $z = 0$, the condition $M \cap N = 0$ forcing uniqueness of the representation of m. Thus $r_i \to m$, and since $r_i \in D \cap M$, $i = 1, 2, \ldots$, we conclude that $D \cap M$ is dense in M.

When M has finite codimension, it is easy to see that we can choose a finite-dimensional complement of M spanned by elements of D. This will serve as N. This last statement of the theorem is known (**1**; **4**, p. 103).

4. Proof of Theorem 1. Let X be the inner product space in question, and let $\bar{X}$ represent its completion. We wish to prove that $X = \bar{X}$ (evidently $X \subseteq \bar{X}$).

Let $a \in \bar{X}$. While we could argue directly that then $a \in X$, it is more convenient to assume the contrary and then argue to a contradiction. Thus we assume that $a \notin X$; in particular, we have $a \neq 0$. Then, X being dense in $\bar{X}$, there is a $w \in X$ such that $(a, w) \neq 0$. Let $b = a - (||a||^2/(w, a))w$. A simple computation shows that $(b, a) = 0$, and $b \notin X$ follows from the assumption: $a \notin X$. In particular, $b \neq 0$.

Now we shall select two sequences $(a_i; i = 1, 2, \ldots)$ and $(b_i; i = 1, 2, \ldots)$ with the following properties:

(1) $a_i \in X$, $b_i \in X$, $i = 1, 2, \ldots$,

(2) $a_i \to a$, $b_i \to b$ (in $\bar{X}$), and

(3) $a_i \perp b_j$, $a_i \perp b$, and $a \perp b_j$ for all $i, j = 1, 2, \ldots$.

The selection of these sequences goes as follows. Since the closed subspace $\text{sp}(b)^\perp$ has codimension 1, and X is dense in $\bar{X}$, it follows from Theorem 8 that $\text{sp}(b)^\perp \cap X$ is dense in $\text{sp}(b)^\perp$. Since $a \in \text{sp}(b)^\perp$, we can select $a_1 \in \text{sp}(b)^\perp \cap X$ such that $||a_1 - a|| < 1$. We have $a, a_1 \in \text{sp}(b)^\perp$ so that $b \in \text{sp}(a)^\perp \cap \text{sp}(a_1)^\perp$.

By Theorem 8, $\mathrm{sp}(a)^{\perp} \cap \mathrm{sp}(a_1)^{\perp} \cap X$ is dense in $\mathrm{sp}(a)^{\perp} \cap \mathrm{sp}(a_1)^{\perp}$ thus we may select $b_1 \in \mathrm{sp}(a)^{\perp} \cap \mathrm{sp}(a_1)^{\perp} \cap X$ so that $||b_1 - b|| < 1$. Then

$$a \in \mathrm{sp}(b)^{\perp} \cap \mathrm{sp}(b_1)^{\perp}$$

hence we can find $a_2 \in \mathrm{sp}(b)^{\perp} \cap \mathrm{sp}(b_1)^{\perp} \cap X$ with $||a_2 - a|| < 1/2$. Next $b \in \mathrm{sp}(a)^{\perp} \cap \mathrm{sp}(a_1)^{\perp} \cap \mathrm{sp}(a_2)^{\perp}$, thus we can find

$$b_2 \in \mathrm{sp}(a)^{\perp} \cap \mathrm{sp}(a_1)^{\perp} \cap \mathrm{sp}(a_2)^{\perp} \cap X$$

with $||b_2 - b|| < 1/2$; and so forth. Clearly this method of selection produces sequences with the above-listed three properties; we leave to the reader the setting up of the formal induction.

Let M be the norm-closure in X of the subspace $\mathrm{sp}(a_1, a_2, \ldots)$ and let N be the norm-closure in X of the subspace $\mathrm{sp}(b_1, b_2, \ldots)$. Clearly $M \perp N$, and it follows easily from this that M and N are completely disjoint. (We can, for example, apply the criterion (2) of Lemma 5: $||m - n||^2 = (m - n, m - n) = ||m||^2 + ||n||^2 = 1 + ||n||^2 \geqq 1$.) Then, by Theorem 6, (M, N) is a modular pair in the lattice of all norm-closed subspaces of X, thus by the assumption of our theorem, (M, N) is also a dual modular pair. Applying Corollary 3, we conclude in turn that $M + N$ is norm-closed in X. Now each of the vectors $c_i = a_i - b_i$, $i = 1, 2, \ldots$, belongs to the subspace $M + N$, and we have $\lim c_i = \lim(a_i - b_i) = \lim a_i - \lim b_i = a - b = (||a||^2/(w, a))w$, which is a vector in X according to our original choice of w. Thus, since $M + N$ is closed, $(||a||^2/(w, a))w \in M + N$, and since $||a|| \neq 0$, we have $w \in M + N$, so that $w = m + n$, $m \in M$, $n \in N$.

Now let $\bar{M}$, $\bar{N}$ represent the closure of M, N, respectively, in $\bar{X}$. Evidently, $\bar{M} \perp \bar{N}$, and clearly $a \in \bar{M}$, $b \in \bar{N}$. Now $(w, a) \neq 0$, thus if we set $\lambda = (w, a)/||a||^2$, then $\lambda \neq 0$ and $w = \lambda a - \lambda b$ with $\lambda a \in \bar{M}$, $\lambda b \in \bar{N}$. But also $w = m + n$, $m \in M \subseteq \bar{M}$, $n \in N \subseteq \bar{N}$, hence by the uniqueness of representation (since $\bar{M} \perp \bar{N}$, in particular $\bar{M} \cap \bar{N} = 0$), we have $\lambda a = m$. Inasmuch as $\lambda \neq 0$, $a = (1/\lambda)m \in M \subseteq X$, which is the desired contradiction.

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