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Equivalence of Besov spaces on p.c.f. self-similar sets

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Abstract. On post-critically finite self-similar sets, whose walk dimensions of diffusions are in general larger than 2, we find a sharp region where two classes of Besov spaces, the heat Besov spaces $B^{p,q}_\sigma(K)$ and the Lipschitz–Besov spaces $\Lambda^{p,q}_\sigma(K)$, are identical. In particular, we provide concrete examples that $B^{p,q}_\sigma(K) = \Lambda^{p,q}_\sigma(K)$ with $\sigma > 1$. Our method is purely analytical, and does not involve heat kernel estimate.

1 Introduction

In this paper, we study the identity of two classes of Besov spaces on post-critically finite (p.c.f.) self-similar sets with regular harmonic structure. One class is the heat Besov spaces $B^{p,q}_\sigma(K)$, defined with the Neumann Laplacian Δ_N, which was introduced in the study of Brownian motions on self-similar sets (see [5–7, 16, 30, 31]), and was later extended to general p.c.f. self-similar sets in a purely analytical way by Kigami [24, 25]. The heat Besov spaces $B^{p,q}_\sigma(K)$ are defined as potential spaces following [23],

$$B^{p,q}_\sigma(K) = \left\{ f \in L^p(K) : \left(\int_0^\infty \left(t^{-\sigma/2} \left\| (t\Delta_N)^k P_t f \right\|_{L^p(K)} \right)^q dt/t \right)^{1/q} < \infty \right\},$$

where $\{P_t\}_{t\geq 0}$ is the heat semigroup associated with Δ_N. Here, we take the measure μ to be self-similar and d_H-regular with respect to the effective resistance metric $R(\cdot,\cdot)$ on K, where d_H is the Hausdorff dimension of K under R. The other class $\Lambda^{p,q}_\sigma(K)$, named Lipschitz–Besov spaces, is defined in terms of difference of functions

$$\Lambda^{p,q}_\sigma(K) = \left\{ f \in L^p(K) : \left(\int_0^\infty \left(\int_K t^{-d_H} \int_{B_t(x)} \frac{|f(x)-f(y)|^p}{t^{\sigma p d_W/2}} d\mu(y) d\mu(x) \right)^{q/p} \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

for $1 \leq q < \infty$, and

$$\Lambda^{p,\infty}_\sigma(K) = \left\{ f \in L^p(K) : \sup_{t>0} \left(\int_K t^{-d_H} \int_{B_t(x)} \frac{|f(x)-f(y)|^p}{t^{\sigma p d_W/2}} d\mu(y) d\mu(x) \right) < \infty \right\},$$

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where $B_t(x)$ is the ball of radius t centered at x under the metric R, and $d_W = 1 + d_H$ is the walk dimension of the associated heat kernel. Roughly speaking, d_H reflects the growth of the measure, and d_W reflects the speed of the diffusion process. There is another exponent d_S that will be involved in the study of $B^{p,q}_\sigma(K)$ and $\Lambda^{p,q}_\sigma(K)$, called the spectral dimension. It is known that $d_S = \frac{2d_H}{d_W}$ and it reflects the asymptotical law of the eigenvalue counting function associate with Δ_N (see [26, 28]). More explanations on general metric measure spaces can be found in [18].

The relationship between the two classes of Besov spaces $B^{p,q}_\sigma(K)$ and $\Lambda^{p,q}_\sigma(K)$ has been a long-term problem [32] on general metric measure spaces, and whether the identity

$$B^{p,q}_\sigma(K) = \Lambda^{p,q}_\sigma(K) \tag{1.1}$$

holds is of particular interest. For $p = q = 2$ and $0 < \sigma < 1$, when the Besov spaces coincide with the Sobolev spaces, under some weak assumption of heat kernel estimates, Hu and Zähle [23] showed that (1.1) holds, as well as Strichartz [35] obtained the same result on products of p.c.f. self-similar sets at the same time. Later, Grigor'yan and Liu proved that (1.1) holds for any $1 < p, q < \infty$ and any $0 < \sigma < \frac{2\Theta}{d_W} \wedge 1$, where Θ denotes the Hölder exponent of the heat kernel (see [19]). Note that on p.c.f. self-similar sets, due to the sub-Gaussian heat kernel estimates (see [21, 29]), the existence of small Hölder exponent Θ was shown in [18]. Until now, a larger region where (1.1) holds or not is still hard to reach.

Recently, Cao and Grigor'yan made much progress on this problem (see [10, 11]). They showed that (1.1) holds on a larger region, under the assumption of Gaussian heat kernel estimates. Their work utilizes some new techniques, but the results and ideas are restricted to the $d_W = 2$ case. For the general $d_W > 2$ case, the study is still on going.

In this paper, we will focus on p.c.f. self-similar sets with regular harmonic structures, which are a class of well-known fractals where sub-Gaussian heat kernel estimates hold. In particular, we will describe a *sharp region* where (1.1) holds. See the left panel of Figure 1.

More precisely, we introduce a critical curve $\mathscr{C}$, for $1 \le p \le \infty$,

$$\mathscr{C}(p) = \sup\left\{\sigma > 0 : \mathcal{H}_0 \subset \Lambda^{p,\infty}_\sigma(K)\right\},$$

where $\mathcal{H}_0$ is the space of harmonic functions. Our main result in this paper is the following theorem.

Theorem 1.1 *For $1 < p < \infty$, $1 \le q \le \infty$ and $0 < \sigma < \mathscr{C}(p)$, we have $B^{p,q}_\sigma(K) = \Lambda^{p,q}_\sigma(K)$, and their norms are equivalent. In addition,*

$$\mathscr{C}(p) = \sup\left\{\sigma > 0 : B^{p,q}_\sigma(K) = \Lambda^{p,q}_\sigma(K), \textit{ for some } 1 \le q \le \infty\right\}.$$

One important fact about the theorem is that the identical region of (1.1) is sharp. Indeed, it is not hard to see that

$$B^{p,q}_\sigma(K) \ne \Lambda^{p,q}_\sigma(K) \text{ if } \sigma > \mathscr{C}(p).$$

We will explain this in Proposition 3.2 at the beginning of Section 3.

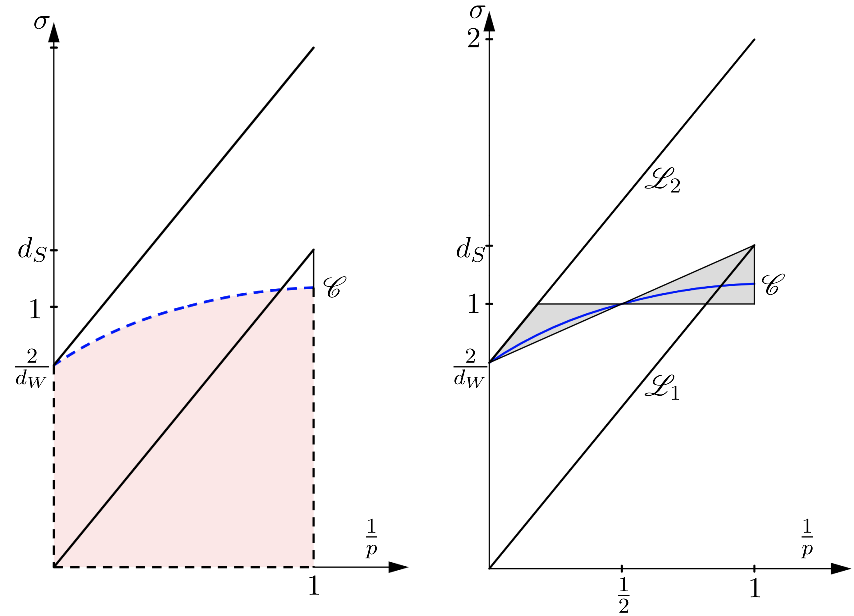

Figure 1: The sharp region for (1.1) and the possible area where $\mathscr{C}$ lies.

One may compare $\mathscr{C}(p)$ with another important critical exponent

$$\sigma_p^\# = \inf_{\sigma>0} \left\{\Lambda_\sigma^{p,\infty}(K) = constants\right\}.$$

Though, for $p = 2$, we always have $\sigma_2^\# = 1 = \mathscr{C}(2)$ [18], we have to say that $\sigma_p^\#$ is not in general equal to $\mathscr{C}(p)$. In fact, it has been shown that $\sigma_1^\# = d_S$ in [4] for nested fractals, while on the Sierpinski gasket, $\mathscr{C}(1) < d_S$ is indicated by Theorems 5.1 and 5.2 of [4] (also see Example 3 in Section 3 for a rough estimate by a simple calculation).

It is not hard to find a narrow region where $\mathscr{C}$ lives, see the right panel of Figure 1 for an illustration. Although much information of $\mathscr{C}$ can be derived with Proposition 5.6 in [2] and Theorem 3.11 in [4], to provide an intuitive understanding of Theorem 1.1, and to make our exposition self-contained, we will provide a short, elementary discussion on $\mathscr{C}$ in Section 3 (see Proposition 3.3). Write

$$\mathscr{L}_1(p) = \frac{d_S}{p}, \quad \mathscr{L}_2(p) = 2 - \frac{d_S}{p'}$$

with $p' = \frac{p}{p-1}$. $\mathscr{L}_1$ is naturally the critical line concerning the continuity of functions, and $\mathscr{L}_2$ is the critical line concerning the Hölder continuity of functions and thus the existence of normal derivatives at boundaries. In the authors' related works [12–14], there is a discussion on the role of these critical lines concerning the relationship between Sobolev spaces and (heat) Besov spaces on p.c.f. self-similar sets with different boundary conditions. We will see that the curve $\mathscr{C}$ is concave and increasing w.r.t. $\frac{1}{p}$, and in addition:

(1). for $1 \le p \le 2, 1 \le \mathscr{C}(p) \le \frac{2}{d_W} + \frac{2}{p} \cdot \frac{d_H - 1}{d_W}$,

(2). for $2 \le p \le \infty, \frac{2}{d_W} + \frac{2}{p} \cdot \frac{d_H-1}{d_W} \le \mathscr{C}(p) \le 1 \wedge \mathscr{L}_2(p)$.

See the right panel of Figure 1. In particular, it may happen that $\mathscr{C}(1) > 1$ (for example, it is true for the Vicsek set and the Sierpinski gasket in standard setting), so (1.1) even holds in some cases when $\sigma > 1$. This is a surprising result which was not mentioned in previous studies.

The exact characterization of the critical curve $\mathscr{C}$, and the problem of whether the identity (1.1) holds along $\mathscr{C}$, are still out of reach, and are left to the future study. It is of particular interest to see whether $\mathscr{C}(1) > 1$ always holds when $d_W > 2$. Despite of this, we are able to fully describe the curve $\mathscr{C}$ for the class of Vicsek sets (see Example 2 in Section 3).

At the end of this section, we mention that, due to the nested structure of p.c.f. self-similar sets, discrete characterizations of function spaces play natural and essential role throughout our study. This might also be a proper starting point for problems on general metric measure spaces by involving suitable partitions (see [27]).

Now, we briefly introduce the structure of this paper. Section 2 will serve as the background of this paper, where we introduce necessary knowledge and notations, including the p.c.f. self-similar sets, the Dirichlet forms and Laplacians on fractals, and the definitions of function spaces we consider here. In Section 3, we will discuss the critical curve $\mathscr{C}$ and provide several examples. This will help readers to understand the sharp region in the main theorem. In Section 4, we focus on the Lipschitz–Besov spaces $\Lambda^{p,q}_\sigma(K)$. We will provide two kinds of discrete type characterizations of $\Lambda^{p,q}_\sigma(K)$, which will serve as a main tool toward the main theorem. In Sections 5 and 6, we prove the main theorem, Theorem 1.1. In particular, we will show that $\Lambda^{p,q}_\sigma(K) \subset B^{p,q}_\sigma(K)$ for any $1 < p < \infty$, $1 \le q \le \infty$ and $0 < \sigma < 2$ in Section 5. In Section 6, we will prove the other direction, i.e., $B^{p,q}_\sigma(K) \subset \Lambda^{p,q}_\sigma(K)$ with $1 < p < \infty, 1 \le q \le \infty$ and $0 < \sigma < \mathscr{C}(p)$.

Throughout the paper, we will always write $f \lesssim g$ if there is a constant $C > 0$ such that $f \le Cg$ when we do not emphasize the constant C. In addition, we write $f \asymp g$ if both $f \lesssim g$ and $g \lesssim f$ hold.

2 Preliminary

The analysis on p.c.f. self-similar sets was originally developed by Kigami in [25, 26]. For convenience of readers, in this section, first, we will briefly recall the constructions of Dirichlet forms and Laplacians on p.c.f. fractals. We refer to books [26, 36] for details. Then we will provide the definitions of the two classes of Besov spaces, $B^{p,q}_\sigma(K)$ and $\Lambda^{p,q}_\sigma(K)$. There is a large literature on function spaces on fractals or on more general metric measure spaces (see [2–4, 12–15, 17, 22, 34] and the references therein).

Let $\{F_i\}_{i=1}^N$ be a finite collection of contractions on a complete metric space $(\mathcal{M}, d)$. The self-similar set associated with the *iterated function system (i.f.s.)* $\{F_i\}_{i=1}^N$ is the unique compact set $K \subset \mathcal{M}$ satisfying

$$K = \bigcup_{i=1}^{N} F_i K.$$

For $m \geq 1$, we define $W_m = \{1, \ldots, N\}^m$ the collection of *words* of length m, and for each $w \in W_m$, denote

$$F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}.$$

Set $W_0 = \emptyset$, and let $W_* = \bigcup_{m \geq 0} W_m$ be the collection of all finite words. For $w = w_1 w_2 \ldots w_m \in W_* \setminus W_0$, we write $w^* = w_1 w_2 \ldots w_{m-1}$ by deleting the last letter of w.

Define the shift space $\Sigma = \{1, 2, \ldots, N\}^{\mathbb{N}}$. There is a continuous surjection $\pi : \Sigma \to K$ defined by

$$\pi(\omega) = \bigcap_{m \geq 1} F_{[\omega]_m} K,$$

where for $\omega = \omega_1 \omega_2 \ldots$ in Σ we write $[\omega]_m = \omega_1 \omega_2 \ldots \omega_m \in W_m$ for each $m \geq 1$. Let

$$C_K = \bigcup_{i \neq j} F_i K \cap F_j K, \quad \mathcal{C} = \pi^{-1}(C_K), \quad \mathcal{P} = \bigcup_{n \geq 1} \sigma^n \mathcal{C},$$

where σ is the shift map define as $\sigma(\omega_1 \omega_2 \ldots) = \omega_2 \omega_3 \ldots$. $\mathcal{P}$ is called the *post-critical set*. Call K a *p.c.f. self-similar set* if $\#\mathcal{P} < \infty$. In what follows, we always assume that K is a connected p.c.f. self-similar set.

Let $V_0 = \pi(\mathcal{P})$ and call it the *boundary* of K. For $m \geq 1$, we always have $F_w K \cap F_{w'} K \subset F_w V_0 \cap F_{w'} V_0$ for any $w \neq w' \in W_m$. Denote $V_m = \bigcup_{w \in W_m} F_w V_0$, and let $l(V_m) = \{f : f \text{ maps } V_m \text{ into } \mathbb{C}\}$. Write $V_* = \bigcup_{m \geq 0} V_m$.

Let $H = (H_{pq})_{p,q \in V_0}$ be a symmetric linear operator(matrix). H is called a *(discrete) Laplacian* on V_0 if H is nonpositive definite; $Hu = 0$ if and only if u is constant on V_0; and $H_{pq} \geq 0$ for any $p \neq q \in V_0$. Given a Laplacian H on V_0 and a vector $\boldsymbol{r} = \{r_i\}_{i=1}^N$ with $r_i > 0, 1 \leq i \leq N$, define the *(discrete) energy form* on V_0 by

$$\mathcal{E}_0(f, g) = -(f, Hg), \quad \forall f, g \in l(V_0),$$

and inductively on V_m by

$$\mathcal{E}_m(f, g) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}_{m-1}(f \circ F_i, g \circ F_i), \quad \forall f, g \in l(V_m),$$

for $m \geq 1$. Write $\mathcal{E}_m(f, f) = \mathcal{E}_m(f)$ for short.

Say $(H, \boldsymbol{r})$ is a *harmonic structure* if for any $f \in l(V_0)$,

$$\mathcal{E}_0(f) = \min\{\mathcal{E}_1(g) : g \in l(V_1), g|_{V_0} = f\}.$$

In this paper, we will always assume that there exists a harmonic structure associated with K, and in addition, $0 < r_i < 1$ for all $1 \leq i \leq N$. Call $(H, \boldsymbol{r})$ a *regular harmonic structure* on K. It is known that the question of when a p.c.f. self-similar set admits a regular harmonic structure is nontrivial, for example, see [26, Section 3.1].

Now, for each $f \in C(K)$, the sequence $\{\mathcal{E}_m(f)\}_{m \geq 0}$ is nondecreasing. Let

$$\mathcal{E}(f, g) = \lim_{m \to \infty} \mathcal{E}_m(f, g) \text{ and } dom\mathcal{E} = \{f \in C(K) : \mathcal{E}(f) < \infty\},$$

where $f, g \in C(K)$ and we write $\mathcal{E}(f) := \mathcal{E}(f, f)$ for short. Call $\mathcal{E}(f)$ the *energy* of *f*. It is known that $(\mathcal{E}, dom\mathcal{E})$ turns out to be a local regular Dirichlet form on $L^2(K, \mu)$ for any Radon measure μ on K.

An important feature of the form $(\mathcal{E}, dom\mathcal{E})$ is the *self-similar identity*,

$$\mathcal{E}(f, g) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}(f \circ F_i, g \circ F_i), \quad \forall f, g \in dom\mathcal{E}. \tag{2.1}$$

Furthermore, denote $r_w = r_{w_1} r_{w_2} \dots r_{w_m}$ for each $w \in W_m$, $m \geq 0$. Then, for $m \geq 1$, we have

$$\mathcal{E}_m(f, g) = \sum_{w \in W_m} r_w^{-1} \mathcal{E}_0(f \circ F_w, g \circ F_w), \quad \mathcal{E}(f, g) = \sum_{w \in W_m} r_w^{-1} \mathcal{E}(f \circ F_w, g \circ F_w).$$

2.1 The Laplacian and harmonic functions

To study the Besov spaces on K, we need a suitable metric and a comparable measure. Instead of the original metric d, a natural choice of metric is the effective resistance metric $R(\cdot, \cdot)$ [26], which matches the form $(\mathcal{E}, dom\mathcal{E})$.

Definition 2.1 For $x, y \in K$, the *effective resistance metric* $R(x, y)$ between x and y is defined by

$$R(x, y)^{-1} = \min \{\mathcal{E}(f) : f \in dom\mathcal{E}, f(x) = 0, f(y) = 1\}.$$

It is known that R is indeed a metric on K which is topologically equivalent to the metric d, and for each $w \in W_*$, we always have $diam(F_w K) \asymp r_w$, where $diam(F_w K) = \max \{R(x, y) : x, y \in F_w K\}$. For convenience, we normalize $diam K$ to be 1 and so that we additionally have $diam(F_w K) \leq r_w$, $\forall w \in W_*$. For $x \in K$ and $t > 0$, we will use $B_t(x)$ to denote a ball centered at x with radius t in the sense of metric R.

We will always choose the following self-similar measure μ on K.

Definition 2.2 Let μ be the unique self-similar measure on K satisfying

$$\mu = \sum_{i=1}^{N} r_i^{d_H} \mu \circ F_i^{-1},$$

and $\mu(K) = 1$, where d_H is determined by the equation $\sum_{i=1}^{N} r_i^{d_H} = 1$.

In this paper, we also let $d_W = 1 + d_H$ and $d_S = \frac{2d_H}{d_W}$.

Clearly, d_H is the *Hausdorff dimension* of K with respect to the metric R. Write $\mu_i := r_i^{d_H}$, then we have $\mu(F_w K) = \mu_w := \mu_{w_1} \mu_{w_2} \dots \mu_{w_m}$ for any $m \geq 0, w \in W_m$. In addition, it is well known that

$$C^{-1} t^{d_H} \leq \mu(B_t(x)) \leq C t^{d_H}$$

with some constant C independent of x, t.

The exponent d_W is called the *walk dimension*, which appears as an important index in the heat kernel estimates (see [21]). In this paper, $d_W = 1 + d_H$ holds because we use the resistance metric R. In general, this relationship is not true, for example, on the Sierpinski gasket equipped the Euclidean metric, $d_W = \frac{\log 5}{\log 2}$, $d_H = \frac{\log 3}{\log 2}$, and so $d_W \neq 1 + d_H$. The exponent d_S is called the *spectral dimension* since it reflects the asymptotic order of the eigenvalue counting function associated with the Dirichlet form $(\mathcal{E}, dom\mathcal{E})$ (see [26, Theorem 4.2.1].

With the Dirichlet form $(\mathcal{E}, dom\mathcal{E})$ and the self-similar measure μ, we can define the associated Laplacian on K with the weak formula.

Definition 2.3 (a). Let $dom_0\mathcal{E} = \{\varphi \in dom\mathcal{E} : \varphi|_{V_0} = 0\}$. For $f \in dom\mathcal{E}$, say $\Delta f = u$ if

$$\mathcal{E}(f,\varphi) = -\int_K u\varphi d\mu, \quad \forall \varphi \in dom_0\mathcal{E}.$$

(b). In addition, say $\Delta_N f = u$ if

$$\mathcal{E}(f,\varphi) = -\int_K u\varphi d\mu, \quad \forall \varphi \in dom\mathcal{E}.$$

Although, we will focus on Besov spaces (and Sobolev spaces) with Neumann boundary condition in this paper, it is convenient to consider Δ instead of Δ_N in the proof, to enlarge the domain a little bit.

Definition 2.4 Define $\mathcal{H}_0 = \{h \in dom\mathcal{E} : \Delta h = 0\}$, and call $h \in \mathcal{H}_0$ a harmonic function.

In fact, $\mathcal{H}_0$ is a finite dimensional space, and each $h \in \mathcal{H}_0$ is uniquely determined by its boundary value on V_0. In particular, we can see that $\mathcal{H}_0$ is always in the L^p domain of Δ for any $1 < p < \infty$.

2.2 Besov spaces on *K*

In this paper, we consider the (heat) Besov spaces $B^{p,q}_\sigma(K)$ with the Neumann boundary condition. Recall that $P_t = e^{\Delta_N t}$, $t > 0$ is a heat operator associated with Δ_N, and the Bessel potential can be defined as $(1-\Delta_N)^{-\sigma/2} = \Gamma(\sigma/2)^{-1}\int_0^\infty t^{\sigma/2-1}e^{-t}P_t dt$. We define potential spaces on K as follows, following [23] and [34].

Definition 2.5 (a). For $1 < p < \infty$, $\sigma \geq 0$, define the *Sobolev space*

$$H^p_\sigma(K) = (1-\Delta_N)^{-\sigma/2}L^p(K)$$

with norm $\|f\|_{H^p_\sigma(K)} = \left\|(1-\Delta_N)^{\sigma/2}f\right\|_{L^p(K)}$.

(b). For $1 < p < \infty$, $1 \leq q \leq \infty$ and $\sigma > 0$, define the *heat Besov space*

$$B^{p,q}_\sigma(K) = \left\{ f \in L^p(K) : \left(\int_0^\infty \left(t^{-\sigma/2} \left\|(t\Delta_N)^k P_t f\right\|_{L^p(K)} \right)^q dt/t \right)^{1/q} < \infty \right\}$$

with $k \in \mathbb{N} \cap (\sigma/2, \infty)$, and norm $\|f\|_{B^{p,q}_\sigma(K)} = \|f\|_{L^p(K)} + \left(\int_0^\infty (t^{-\sigma/2}\|(t\Delta_N)^k P_t f\|_{L^p(K)})^q dt/t\right)^{1/q}$. We take the usual modification when $q = \infty$.

Note that the above definition is independent of k, since different choices of k will provide equivalent norms (see [19] for example). The heat Besov spaces are related with Sobolev spaces by real interpolation. See book [20] for a proof, noticing that Δ_N is a sectorial operator. See also books [9, 38] for the real interpolation methods.

Lemma 2.6 *Let $\sigma_1 > 0$, $1 < p < \infty$ and $1 \le q \le \infty$. For $0 < \theta < 1$ and $\sigma_\theta = \theta\sigma_1$, we have*

$$\big(L^p(K), H^p_{\sigma_1}(K)\big)_{\theta,q} = B^{p,q}_{\sigma_\theta}(K).$$

In application, we will set $\sigma_1 = 2$ in the above lemma, where $H^p_2(K) = dom_{L^p(K)}\Delta_N := \{f \in L^p(K) : \Delta_N f \in L^p(K)\}$ (see Sections 5 and 6 for details).

Another class of function spaces that will be studied is the Lipschitz–Besov spaces, whose definition does not rely on the Laplacian.

Definition 2.7 Let $1 \le p < \infty$, $t > 0$ and f be a measurable function on K, we define

$$I_p(f,t) = \Big(\int_K t^{-d_H} \int_{B_t(x)} |f(x) - f(y)|^p d\mu(y) d\mu(x)\Big)^{1/p}.$$

In addition, we define $I_\infty(f,t) = \sup\{|f(x) - f(y)| : x, y \in K, R(x,y) < t\}$.

The *Lipschitz–Besov spaces*, denote by $\Lambda^{p,q}_\sigma(K)$, are defined as follows.

Definition 2.8 For $\sigma > 0$ and $1 \le p, q \le \infty$, we define

$$\Lambda^{p,q}_\sigma(K) = \big\{f \in L^p(K) : t^{-\sigma d_w/2} I_p(f,t) \in L^q_*(0,1]\big\}$$

with norm

$$\|f\|_{\Lambda^{p,q}_\sigma(K)} := \|f\|_{L^p(K)} + \big\|t^{-\sigma d_w/2} I_p(f,t)\big\|_{L^q_*(0,1]},$$

where $\|f\|_{L^q_*(0,1]} = \Big(\int_0^1 |f(t)|^q \frac{dt}{t}\Big)^{1/q}$ and we take the usual modification when $q = \infty$.

Remark 1 Since K is bounded, we can replace the integral of t over $(0,1]$ with $(0,\infty)$ in the above definition.

Remark 2 Note that $I_p(f,st) \ge \theta^{-d_H/p} I_p(f,t), \forall s \in [1,\theta]$ for each $\theta > 1$ and $1 \le p \le \infty$. We have $\big\|t^{-\sigma d_w/2} I_p(f,t)\big\|_{L^q_*(0,1]} \asymp \big\|\theta^{m(\sigma d_w/2)} I_p(f,\theta^{-m})\big\|_{l^q}$, where $\|a_m\|_{l^q} = \big(\sum_{m=0}^\infty |a_m|^q\big)^{1/q}$ if $q < \infty$ and $\|a_m\|_{l^\infty} = \sup_{m\ge0}|a_m|$ for each sequence of real numbers a_m, $m \ge 0$. The constants of "$\asymp$" depend only on p, K and the harmonic structure $(H, \boldsymbol{r})$.

Using the above equivalent norm, we can see

$$\Lambda^{p,q_1}_\sigma(K) \subset \Lambda^{p,q_2}_\sigma(K) \text{ for any } \sigma > 0,\ 1 \le p \le \infty \text{ and } 1 \le q_1 \le q_2 \le \infty.$$

3 A critical curve

In this section, we introduce a *critical curve* $\mathscr{C}$ in the $(\frac{1}{p}, \sigma)$-parameter plane as follows.

Definition 3.1 For $1 \le p \le \infty$, we define $\mathscr{C}(p) = \sup\big\{\sigma > 0 : \mathcal{H}_0 \subset \Lambda^{p,\infty}_\sigma(K)\big\}$.

The following proposition implies that $B^{p,q}_\sigma(K) \ne \Lambda^{p,q}_\sigma(K)$ when $\sigma > \mathscr{C}(p)$.

Proposition 3.2 *For $1 < p < \infty$, $1 \le q \le \infty$, $\sigma > \mathscr{C}(p)$, $B^{p,q}_\sigma(K)\backslash\Lambda^{p,q}_\sigma(K) \neq \emptyset$.*

Proof Fix $\sigma > \mathscr{C}(p)$. First, by the definition of $\mathscr{C}(p)$ and by Remark 2 after Definition 2.8, there is $h \in \mathcal{H}_0$, such that $h \notin \Lambda^{p,q}_\sigma(K)$. Next, we choose $m \ge 1$, $w \in W_m$ such that $F_wK \cap V_0 = \emptyset$, and fix $k \in \mathbb{N}$ so that $2k > \sigma$. Then one can find $f \in H^p_{2k}(K) \subset B^{p,q}_\sigma(K)$ such that $f \circ F_w = h$. The last step can be done by gluing together functions in $\{g \circ F_w : g \in dom\mathcal{E}, \Delta^k g \in C(K)\}$ with proper boundary conditions on different m-cells. We can make f smooth enough, guaranteed by Theorem 4.3 of [33]. However, $f \notin \Lambda^{p,q}_\sigma(K)$. To show this, we note that

$$\begin{aligned}
I_p(f,t) &= \Big(\int_K t^{-d_H} \int_{B_t(x)} |f(x) - f(y)|^p d\mu(y)d\mu(x)\Big)^{1/p} \\
&\ge \Big(\int_{F_wK} t^{-d_H} \int_{B_t(x)\cap F_wK} |f(x) - f(y)|^p d\mu(y)d\mu(x)\Big)^{1/p} \\
&\ge \mu_w^{2/p}\Big(\int_K t^{-d_H} \int_{B_{c_1\cdot t/r_w}(x)} |f\circ F_w(x) - f\circ F_w(y)|^p d\mu(y)d\mu(x)\Big)^{1/p} \\
&= c_1^{d_H/p} r_w^{d_H/p} I_p(f\circ F_w, c_1 r_w^{-1}t) = c_1^{d_H/p} r_w^{d_H/p} I_p(h, c_1 r_w^{-1}t),
\end{aligned}$$

where we choose a finite positive constant c_1 such that $F_w\big(B_{c_1t/r_w}(x)\big) \subset B_t\left(F_w(x)\right)$ for each $x \in K$. Hence, we see that $\left\|t^{-\sigma d_W/2} I_p(f,t)\right\|_{L^q_*(0,1]} = \infty$ since $\left\|t^{-\sigma d_W/2} I_p(h,t)\right\|_{L^q_*(0,1]} = \infty$. ∎

3.1 Two regions

In this part, we provide some qualitative behavior of the critical curve $\mathscr{C}$. We begin with the following easy observation.

Proposition 3.3 *(a). The critical curve $\mathscr{C}$ is concave and increasing with respect to the parameter $\frac{1}{p}$. In addition, $\mathscr{C}(\infty) = \frac{2}{d_W}$ and $\mathscr{C}(2) = 1$.*

(b). For $1 \le p \le 2$, we have $1 \le \mathscr{C}(p) \le 1 + (\frac{2}{p} - 1)(d_S - 1)$.

(c). For $2 \le p \le \infty$, we have $1 + (\frac{2}{p} - 1)(d_S - 1) \le \mathscr{C}(p) \le 1 \wedge (\frac{2}{d_W} + \frac{d_S}{p})$.

See Figure 2 for an illustration.

Proof We remark that (b) and the lower bound in (c) can be derived by Proposition 5.6 in [2] and Theorem 3.11 in [4]. Since the proof is very short, we still provide an elementary proof here for completeness.

Recall Definition 2.2 that $d_W = 1 + d_H$, $d_S = \frac{2d_H}{d_W}$, and note that $d_S - 1 = 1 - \frac{2}{d_W}$.

(a). The observation that $\mathscr{C}(\infty) = \frac{2}{d_W}$ follows from the fact that $0 < \sup_{x\neq y} \frac{|h(x)-h(y)|}{R(x,y)} < \infty$ for any nonconstant harmonic function h (see [37]). For $p = 2$, it is well known that $\Lambda^{2,\infty}_1(K) = dom\mathcal{E}$ and $\Lambda^{2,\infty}_\sigma(K) = constants$ provided $\sigma > 1$ (see [18]), which gives $\mathscr{C}(2) = 1$.

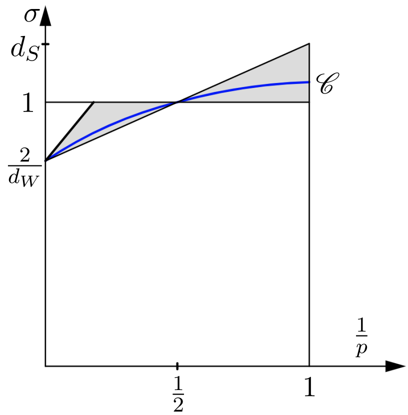

Figure 2: The critical curve $\mathscr{C}$ in the $(\frac{1}{p}, \sigma)$-parameter plane.

Next, let $1 \le p_1 < p_2 \le \infty$, $\sigma_1 < \mathscr{C}(p_1)$ and $\sigma_2 < \mathscr{C}(p_2)$. Also, let $s \in (0,1)$, and let $\frac{1}{p} = \frac{s}{p_1} + \frac{1-s}{p_2}$ and $\sigma = s\sigma_1 + (1-s)\sigma_2$. Then, for any $0 < t \le 1$ and $h \in \mathcal{H}_0$, it holds that

$$\begin{aligned}&\Big(\int_K t^{-d_H}\int_{B_t(x)} |h(x)-h(y)|^p d\mu(y)d\mu(x)\Big)^{1/p}\\ \le &\Big(\int_K t^{-d_H}\int_{B_t(x)} |h(x)-h(y)|^{s\cdot p_1/s} d\mu(y)d\mu(x)\Big)^{s/p_1}\\ &\cdot\Big(\int_K t^{-d_H}\int_{B_t(x)} |h(x)-h(y)|^{(1-s)\cdot p_2/(1-s)} d\mu(y)d\mu(x)\Big)^{(1-s)/p_2},\end{aligned}$$

and thus $t^{-\sigma d_W/2} I_p(h,t) \le t^{-\sigma d_W/2} I^s_{p_1}(h,t) I^{(1-s)}_{p_2}(h,t) \le \|h\|^s_{\Lambda^{p_1,\infty}_{\sigma_1}(K)} \|h\|^{1-s}_{\Lambda^{p_2,\infty}_{\sigma_2}(K)}$. This implies $\mathcal{H}_0 \subset \Lambda^{p,\infty}_\sigma(K)$. Thus, we conclude $\mathscr{C}(p) \ge s\mathscr{C}(p_1) + (1-s)\mathscr{C}(p_2)$. So $\mathscr{C}$ is concave.

Lastly, there is a constant $C > 0$ such that $\mu(B_t(x)) \le Ct^{d_H}$ for any $x \in K$ and $t \in (0,1]$. Thus, for $1 \le p_1 \le p_2 < \infty$ and $0 < t \le 1$, it is easy to see

$$I_{p_1}(h,t) \le C I_{p_2}(h,t)$$

by using the Hölder inequality. This implies that $\mathscr{C}$ is increasing with respect to $\frac{1}{p}$.

(b). Part (b) is a consequence of part (a) and the fact that $\mathscr{C}(\infty) = \frac{2}{d_W}$ and $\mathscr{C}(2) = 1$.

(c). Now, by part (a), we can conclude that $1 + (\frac{2}{p} - 1)(d_S - 1) \le \mathscr{C}(p) \le 1$. It remains to prove $\mathscr{C}(p) \le \frac{2}{d_W} + \frac{d_S}{p}$. We choose a nonconstant harmonic function h such that $h \circ F_1 = r_1 h$, whose existence is guaranteed by Theorem A.1.2 in [26]. For any $n \ge 0$, we see that

$$\begin{aligned} I_p(h,r_1^n) &= \left(\int_K \mu_1^{-n}\int_{B_{r_1^n}(x)} |h(x)-h(y)|^p d\mu(y)d\mu(x)\right)^{1/p} \\ &\geq \left(\int_{F_1^n K} \mu_1^{-n}\int_{B_{r_1^n}(x)} |h(x)-h(y)|^p d\mu(y)d\mu(x)\right)^{1/p} \\ &\geq r_1^n\mu_1^{n/p}\left(\int_K\int_K |h(x)-h(y)|^p d\mu(y)d\mu(x)\right)^{1/p}. \end{aligned}$$

This implies that $r_1^{-\mathscr{C}(p)d_W/2} r_1^{1+d_H/p} \leq 1$, and thus $\mathscr{C}(p) \leq \frac{2}{d_W}+\frac{d_S}{p}$. ■

Remark (a). When $d_W=2$, which happens when $d_H=1$ for the setting of the paper, we can see that $\mathscr{C}(p)\equiv 1$ for $p\in[1,\infty]$ by Proposition 3.3(b),(c) as $d_S=1$. This may happen when K is the unit interval, but to the best of the authors knowledge, it is unclear whether there are other interesting examples of p.c.f. self-similar sets.

(b). When $d_W>2$, which happens when $d_H>1$ for the setting of the paper, we can see $\mathscr{C}(\infty)<1$ by Proposition 3.3(c). So according to Proposition 3.2, $B^{p,q}_\sigma(K)=\Lambda^{p,q}_\sigma(K)$ does not hold for some large p and $\sigma<1$.

(c). For $1<p<2$, it is possible that $B^{p,q}_\sigma(K)=\Lambda^{p,q}_\sigma(K)$ for some $\sigma>1$. See the next subsection for examples with $\mathscr{C}(1)>1$.

There are two more critical lines $\mathscr{L}_1$, $\mathscr{L}_2$ in the $(\frac{1}{p},\sigma)$-parameter plane, that are of interest, with

$$\mathscr{L}_1(p)=\frac{d_S}{p} \text{ and } \mathscr{L}_2(p)=2-\frac{d_S}{p'},$$

where $p'=\frac{p}{p-1}$. See Figure 3 for an illustration for the positions of $\mathscr{C}$, $\mathscr{L}_1$, and $\mathscr{L}_2$. In particular, as illustrated in [18, 23, 34], the Sobolev spaces $H^p_\sigma(K)$ and the heat Besov spaces $B^{p,q}_\sigma(K)$ are embedded in $C(K)$ when the parameter point $(\frac{1}{p},\sigma)$ is above $\mathscr{L}_1$, and these function spaces with or without Neumann condition coincide if $(\frac{1}{p},\sigma)$ is below $\mathscr{L}_2$ (see [12–14]), which clearly covers the parameter region below $\mathscr{C}$ by Proposition 3.3.

In this paper, we are most interested in the region $\sigma<\mathscr{C}(p)$, and we can see that $\mathscr{C}$ and $\mathscr{L}_1$ intersect at some point with $1\leq p\leq d_S$ by Proposition 3.3. In particular, we divide the region below $\mathscr{C}$ into two parts (see Figure 4 for an illustration).

Region 1. $\mathscr{A}_1 := \left\{(\frac{1}{p},\sigma): 1<p<\infty \text{ and } \mathscr{L}_1(p)<\sigma<\mathscr{C}(p)\right\}$;

Region 2. $\mathscr{A}_2 := \left\{(\frac{1}{p},\sigma): 1<p<\infty \text{ and } 0<\sigma<\mathscr{L}_1(p)\wedge\mathscr{C}(p)\right\}$.

We will apply different methods when considering these two regions, for the proof of $B^{p,q}_\sigma(K)\subset\Lambda^{p,q}_\sigma(K)$. The border between the two regions can be dealt with by using real interpolation.

The reason that we need to divide the region $\sigma<\mathscr{C}(p)$ in this manner is due to the existence of the region $\mathscr{C}(p)<\sigma<\mathscr{L}_1(p)$ when $\mathscr{C}(1)<d_S$. For example, this happens for the Sierpinski gasket, see the next subsection.

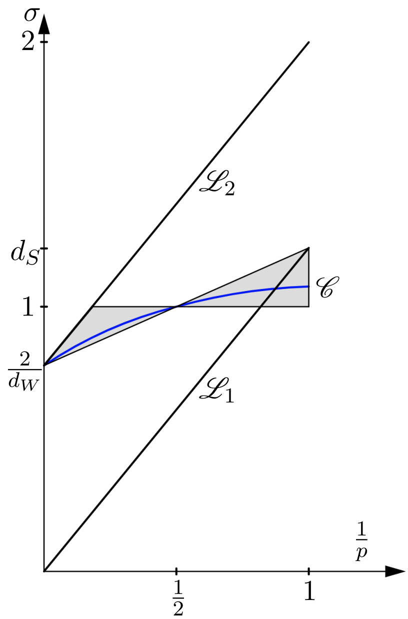

Figure 3: The critical curves $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{C}$.

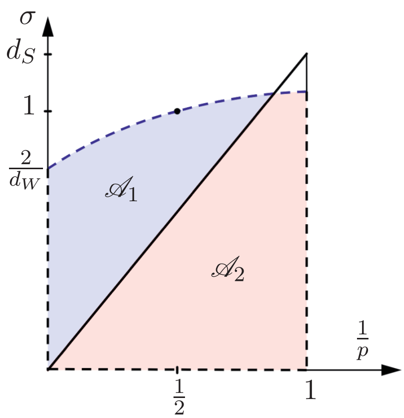

Figure 4: The regions $\mathscr{A}_1$ and $\mathscr{A}_2$.

3.2 Examples

In this subsection, we look at some typical p.c.f. self-similar sets, and describe their critical curves $\mathscr{C}$ or provide some rough estimates.

Example 1 The *unit interval* $I = [0,1]$, generated by $F_1(x) = \frac{x}{2}, F_2(x) = \frac{x}{2} + \frac{1}{2}$, is a simplest example of p.c.f. self-similar sets. We equip I with the standard Laplacian,

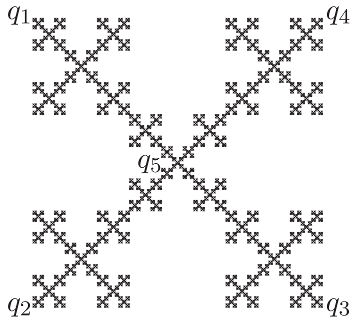

Figure 5: The Vicsek set $\mathcal{V}$.

then it has walk dimension $d_W = 2$ and spectral dimension $d_S = 1$. So the critical curve is simply a horizontal line segment, $\mathscr{C}(p) \equiv 1$, as pointed out in the remark after Proposition 3.3.

Example 2 A more interesting example is the *Vicsek set* $\mathcal{V}$. For this example, we show that

$$\mathscr{C}(p) = 1 + (\frac{2}{p} - 1)(d_S - 1) = \frac{2\log 3}{\log 15} + \frac{2}{p} \cdot \frac{\log 5 - \log 3}{\log 15}, \tag{3.1}$$

which corresponds to equality in two of the inequalities of Proposition 3.3(b),(c), specifically that $\mathscr{C}(p)$ is the line through $(p = \infty, \sigma = \frac{2}{d_W})$, $(p = 2, \sigma = 1)$, and $(p = 1, \sigma = d_S)$. See a similar consideration in [1]. In the following, we state the definition of $\mathcal{V}$ and show (3.1).

Let $\{q_i\}_{i=1}^4$ be the four vertices of a square in $\mathbb{R}^2$, and let q_5 be the center of the square. Define an i.f.s. $\{F_i\}_{i=1}^5$ by

$$F_i(x) = \frac{1}{3}(x - q_i) + q_i, \text{ for } 1 \le i \le 5.$$

The Vicsek set $\mathcal{V}$ is then the unique compact set in the square such that $\mathcal{V} = \bigcup_{i=1}^5 F_i\mathcal{V}$ (see Figure 5).

We equip $\mathcal{V}$ with the fully symmetric measure μ and energy form $(\mathcal{E}, dom\mathcal{E})$. In particular, μ is chosen to be the normalized Hausdorff measure on $\mathcal{V}$. As for $(\mathcal{E}, dom\mathcal{E})$, recall that it could be defined first on discrete graphs on V_m's then passing to the limit. Note that $V_m = \bigcup_{w\in W_m} F_w V_0$, where $V_0 = \{q_1, q_2, q_3, q_4\}$ is the boundary of $\mathcal{V}$. For convenience of the later calculation, we instead to use an equivalent definition of $(\mathcal{E}, dom\mathcal{E})$ by involving the point q_5 in the graph energy forms, i.e., letting $\tilde{V}_0 = \{q_i\}_{i=1}^5$ and $\tilde{V}_m = \bigcup_{w\in W_m} F_w \tilde{V}_0$, and defining the energy form on $\tilde{V}_0$ to be

$$\tilde{\mathcal{E}}_0(f, g) = \sum_{i=1}^4 (f(q_i) - f(q_5))(g(q_i) - g(q_5)),$$

and iteratively $\tilde{\mathcal{E}}_m(f, g) = 3\sum_{i=1}^5 \tilde{\mathcal{E}}_{m-1}(f \circ F_i, g \circ F_i)$ on $\tilde{V}_m$, which still approximate $(\mathcal{E}, dom\mathcal{E})$ on $\mathcal{V}$. In particular, we have $r = \frac{1}{3}$, and in addition,

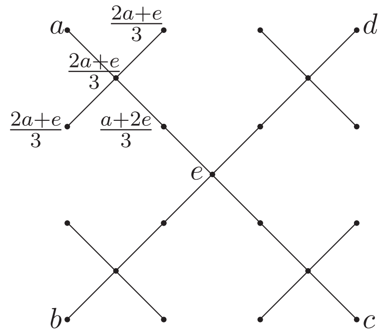

Figure 6: A harmonic function h on $\mathcal{V}$ with boundary value $h(q_1) = a$, $h(q_2) = b$, $h(q_3) = c$, $h(q_4) = d$, and $e = h(q_5) = (a + b + c + d)/4$.

$$d_H = \frac{\log 5}{\log 3}, \quad d_W = 1 + d_H = \frac{\log 15}{\log 3}, \quad d_S = \frac{2d_H}{d_W} = \frac{2\log 5}{\log 15}.$$

In particular, for $h \in \mathcal{H}_0$ and $t \in (0,1]$, we are interested in the estimate of $I_1(h,t)$. We denote by $\sum_{x\sim_m y} |h(x) - h(y)|$ the sum of absolute differences of h over edges of level m, where $x \sim_m y$ means that there exist a word $w \in W_m$ and an $1 \le i \le 4$ such that $x = F_w q_i$ and $y = F_w q_5$. Since $\mathcal{H}_0$ is of finite dimension, it is not hard to check that

$$\sum_{x\sim_m y} |h(x) - h(y)| \asymp 5^m I_1(h, 3^{-m}) = 3^{md_H} I_1(h, 3^{-m}).$$

On the other hand, due to the harmonic extension algorithm as shown in Figure 6, we immediately have

$$\sum_{x\sim_m y} |h(x) - h(y)| = \sum_{i=1}^{4} |f(q_i) - f(q_5)|, \quad \forall m \ge 0.$$

So $\sup_{m\ge 0} 3^{md_S d_W/2} I_1(h, 3^{-m}) \lesssim \|h\|_\infty$, which means $h \in \Lambda^{1,\infty}_{d_S}(\mathcal{V})$. Thus, $\mathscr{C}(1) = d_S$ by applying Proposition 3.3(b). This determines the formula of $\mathscr{C}(p)$ in (3.1), using Proposition 3.3(a).

The above description of $\mathscr{C}$ is also valid for a general $(2k+1)$-Vicsek set with $k \ge 1$, which is generated by an i.f.s. of $4k+1$ contractions, such that each of the two cross directions of the fractal consists of $2k+1$ sub-cells. We omit the details. Before ending, we refer to a recent study by Baudoin and Chen [8] on the equivalent Sobolev space characterization of the domain of p-energies on $\mathcal{V}$.

Example 3 The next example is the *Sierpinski gasket* $\mathcal{SG}$. Let $\{q_i\}_{i=0}^3$ be the three vertices of an equilateral triangle in $\mathbb{R}^2$, and define an i.f.s. $\{F_i\}_{i=1}^3$ by

$$F_i(x) = \frac{1}{2}(x - q_i) + q_i, \text{ for } 1 \le i \le 3.$$

The Sierpinski gasket $\mathcal{SG}$ is the unique compact set in $\mathbb{R}^2$ such that $\mathcal{SG} = \bigcup_{i=1}^3 F_i \mathcal{SG}$ (see Figure 7).

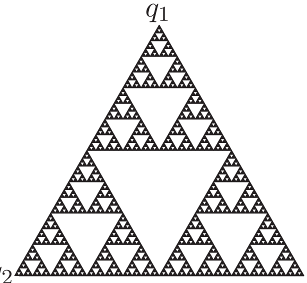

Figure 7: The Sierpinski gasket $\mathcal{SG}$.

On $\mathcal{SG}$, we take the normalized Hausdorff measure μ and the standard energy form $(\mathcal{E}, dom\mathcal{E})$ satisfying

$$\mathcal{E}(f,g) = \frac{5}{3}\sum_{i=1}^{3}\mathcal{E}(f\circ F_i, g\circ F_i), \quad \forall f,g \in dom\mathcal{E}.$$

In particular, we have $r = \frac{3}{5}$, and in addition,

$$d_H = \frac{\log 3}{\log 5 - \log 3}, \quad d_W = 1 + d_H = \frac{\log 5}{\log 5 - \log 3}, \quad d_S = \frac{2d_H}{d_W} = \frac{2\log 3}{\log 5} \approx 1.36521.$$

It seems hard to get the exact formula of $\mathscr{C}(p)$. However, by a simple calculation, we can see that $\mathscr{C}(p)$ is indeed a "curve" by observing that $1 < \mathscr{C}(1) < d_S$, and then using Proposition 3.3. In fact, this can be verified by estimating the maximal exponential growth ratio of $\sum_{x\sim_m y}|h(x)-h(y)|$ as $m\to\infty$, which should be $r^{\mathscr{C}(1)d_W/2-d_H}$ for harmonic functions h on $\mathcal{SG}$. Since any harmonic function h is a combination of h_1, h_2, h_3 with $h_i(q_j) = \delta_{i,j}$, by calculating $\sum_{x\sim_m y}|h_1(x)-h_1(y)|$ with $m = 3$, we see that

$$1.02 < \mathscr{C}(1) < 1.14.$$

Lastly, we remark that the fact $1 < \mathscr{C}(1) < d_S$ is also indicated by Theorem 5.2 of [4].

4 Discrete characterizations of $\Lambda_\sigma^{p,q}(K)$

In this section, we will provide some discrete characterizations of the Lipschitz–Besov spaces $\Lambda_\sigma^{p,q}(K)$. These characterizations will provide great convenience in proving Theorem 1.1. In particular, they heavily rely on the nested structure of K.

Definition 4.1 (a). For $m \geq 0$, define $\Lambda_m = \{w \in W_* : r_w \leq r^m < r_{w^*}\}$ with $r = \min_{1\leq i\leq N} r_i$. In particular, we denote $\Lambda_0 = \{\varnothing\}$.

(b). Define $V_{\Lambda_m} = \bigcup_{w\in\Lambda_m} F_w V_0$ for $m \geq 0$, and denote

$$\mathring{V}_{\Lambda_m} = \begin{cases} V_0, & \text{if } m = 0,\\ V_{\Lambda_m}\setminus V_{\Lambda_{m-1}}, & \text{if } m \geq 1.\end{cases}$$

In the rest of this section, we will consider two kinds of discrete characterizations of $\Lambda_\sigma^{p,q}(K)$, basing on the cell graphs approximation and vertex graphs approximation of K, respectively.

4.1 A Haar series expansion

We begin with a Haar series expansion of a function. We classify Haar functions on K into different levels based on the partition Λ_m.

Definition 4.2 (a). For each $f \in L^1(K)$, we define $E_w(f) = \frac{1}{\mu_w}\int_{F_wK} f d\mu$, and write

$$E[f|\Lambda_m] = \sum_{w\in\Lambda_m} E_w(f)1_{F_wK}, \quad m \geq 0,$$

which can be understood as the conditional expectation of f with respect to the sigma algebra generated by the collection $\{F_wK : w \in \Lambda_m\}$. In addition, we write

$$\tilde{E}[f|\Lambda_m] = \begin{cases} E[f|\Lambda_0], & \text{if } m = 0, \\ E[f|\Lambda_m] - E[f|\Lambda_{m-1}], & \text{if } m \geq 1. \end{cases}$$

(b). Define $\tilde{J}_m = \{\tilde{E}[f|\Lambda_m] : f \in L^1(K)\}$, and call $\tilde{J}_m$ the space of level-m Haar functions.

It is easy to see that, for $m \geq 1$, $\tilde{J}_m$ consists of functions u which are piecewise constant on $\{F_wK : w \in \Lambda_m\}$, and satisfy $E[u|\Lambda_{m-1}] = 0$. We have the following estimates.

Lemma 4.3 *Let $f \in L^p(K)$ with $1 < p < \infty$ and $u \in \tilde{J}_m$ with $m \geq 0$. Then:*
(a). $\|\tilde{E}[f|\Lambda_m]\|_{L^p(K)} \leq CI_p(f, r^{m-1})$ for any $m \geq 1$.
(b). $I_p(f,t) \leq C\|f\|_{L^p(K)}$ for any $0 < t \leq 1$.
(c). $I_p(u, r^n) \leq Cr^{(n-m)d_H/p}\|u\|_{L^p(K)}$ for any $n \geq m$.
The constant C can be chosen to be independent of f, u, and p.

Proof (a). For each point $x \in K\setminus V_{\Lambda_m}$, we define $Z_{\Lambda_m}(x) = F_wK$ with $w \in \Lambda_m$ such that $x \in F_wK$. Clearly, we have $Z_{\Lambda_m}(x) \subset B_{r^m}(x)$ since the diameter of each cell $F_wK, w \in \Lambda_m$ is at most r^m. For $m \geq 1$, we have

$$\begin{aligned}
\|\tilde{E}[f|\Lambda_m]\|_{L^p(K)} &= \Big(\int_K \big|E[f|\Lambda_m](x) - E[f|\Lambda_{m-1}](x)\big|^p d\mu(x)\Big)^{1/p} \\
&\leq \Big(\int_K \big|f(x) - E[f|\Lambda_{m-1}](x)\big|^p d\mu(x)\Big)^{1/p} \\
&\leq \Big(\int_K \Big(\mu^{-1}_{Z_{\Lambda_{m-1}(x)}} \int_{Z_{\Lambda_{m-1}(x)}} |f(x) - f(y)|^p d\mu(y)\Big) d\mu(x)\Big)^{1/p} \\
&\leq \Big(\int_K r^{-md_H} \int_{B_{r^{m-1}}(x)} |f(x) - f(y)|^p d\mu(y) d\mu(x)\Big)^{1/p} \leq CI_p(f, r^{m-1}),
\end{aligned}$$

where we ignore the finitely many points in V_{Λ_m} in the above estimate.
(b) is obvious, and C only depends on the estimate $\mu(B_t(x)) \lesssim t^{d_H}$.

(c). First, we have the estimate that

$$\begin{aligned}\|u\|_{L^p(K)} &= \Big(\sum_{w\in\Lambda_m}\mu_w|E_w(u)|^p\Big)^{1/p} \geq \Big(r^{(m+1)d_H}\sum_{w\in\Lambda_m}|E_w(u)|^p\Big)^{1/p} \\ &\gtrsim r^{md_H/p}\Big(\sum_{w\sim w' \text{ in } \Lambda_m}|E_w(u)-E_{w'}(u)|^p\Big)^{1/p},\end{aligned} \tag{4.1}$$

where we write $w \sim w'$ if $w \neq w'$ and $F_wK \cap F_{w'}K \neq \emptyset$, and use the fact that $\#\{w' \in \Lambda_m : w' \sim w\} \leq \#V_0\#\mathcal{C}$ for any $w \in \Lambda_m$.

Next, notice that there is $k > 0$ such that $R(x,y) > r^n$ for any $m \geq 0$, $n \geq m + k$ and $x \in F_wK, y \in F_{w'}K$ with $F_wK \cap F_{w'}K = \emptyset$, $w, w' \in \Lambda_m$. It suffices to consider $n \geq m + k$, since for $n < m + k$ we have (b). For $n \geq m + k$, we have the estimate

$$\begin{aligned}I_p(u,r^n) &= \Big(\int_K r^{-nd_H}\int_{B_{r^n}(x)}|u(x)-u(y)|^p d\mu(y)d\mu(x)\Big)^{1/p} \\ &\asymp \Big(\iint_{R(x,y)<r^n} r^{-nd_H}|u(x)-u(y)|^p d\mu(y)d\mu(x)\Big)^{1/p} \\ &= \Big(\sum_{w\sim w' \text{ in } \Lambda_m}\iint_{\{x\in F_wK,y\in F_{w'}K:R(x,y)<r^n\}} r^{-nd_H}|u(x)-u(y)|^p d\mu(y)d\mu(x)\Big)^{1/p} \\ &\lesssim \Big(r^{nd_H}\sum_{w\sim w' \text{ in } \Lambda_m}|E_w(u)-E_{w'}(u)|^p\Big)^{1/p}.\end{aligned}$$

Combining this with the estimate (4.1), we get (c). ■

Using Lemma 4.3, we can prove a Haar function decomposition of the spaces $\Lambda^{p,q}_\sigma(K)$ for $0 < \sigma < \mathscr{L}_1(p)$.

Proposition 4.4 *For $1 < p < \infty$, $1 \leq q \leq \infty$ and $0 < \sigma < \mathscr{L}_1(p) = \frac{d_S}{p}$, we have $f \in \Lambda^{p,q}_\sigma(K)$ if and only if $\big\|r^{-m\sigma d_W/2}\|\tilde{E}[f|\Lambda_m]\|_{L^p(K)}\big\|_{l^q} < \infty$. In addition, $\|f\|_{\Lambda^{p,q}_\sigma(K)} \asymp \big\|r^{-m\sigma d_W/2}\|\tilde{E}[f|\Lambda_m]\|_{L^p(K)}\big\|_{l^q}$.*

Proof We first observe that

$$\|f\|_{\Lambda^{p,q}_\sigma(K)} = \|f\|_{L^p(K)} + \big\|t^{-\sigma d_W/2}I_p(f,t)\big\|_{L^q_*(0,1)} \asymp \|f\|_{L^p(K)} + \|r^{-m\sigma d_W/2}I_p(f,r^m)\|_{l^q}. \tag{4.2}$$

By using Lemma 4.3(a), we then easily see that

$$\big\|r^{-m\sigma d_W/2}\|\tilde{E}[f|\Lambda_m]\|_{L^p(K)}\big\|_{l^q} \lesssim \|f\|_{\Lambda^{p,q}_\sigma(K)}.$$

For the other direction, we write $f_m = \tilde{E}[f|\Lambda_m]$ for $m \geq 0$, and assume that

$$\big\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\big\|_{l^q} < \infty.$$

First of all, by the Martingale convergence theorem, we see that $E[f|\Lambda_{m'}] = \sum_{m=0}^{m'} f_m \to f$ strongly in $L^p(K)$ as $m' \to \infty$, so

$$\|f\|_{L^p(K)} = \lim_{m'\to\infty} \|E[f|\Lambda_{m'}]\|_{L^p(K)} \le \sum_{m=0}^{\infty} \|f_m\|_{L^p(K)} \le \left\|r^{m\sigma d_W/2}\right\|_{l^{q'}} \cdot \left\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\right\|_{l^q} \tag{4.3}$$

with $q' = \frac{q}{q-1}$. Next, we notice that

$$\left\|r^{-m\sigma d_W/2} I_p(f, r^m)\right\|_{l^q} \le \left\|\sum_{n=0}^{m-1} r^{-m\sigma d_W/2} I_p(f_n, r^m)\right\|_{l^q} + \left\|\sum_{n=m}^{\infty} r^{-m\sigma d_W/2} I_p(f_n, r^m)\right\|_{l^q} \tag{4.4}$$

with

$$\begin{aligned}\left\|\sum_{n=0}^{m-1} r^{-m\sigma d_W/2} I_p(f_n, r^m)\right\|_{l^q} &\lesssim \left\|\sum_{n=0}^{m-1} r^{-m\sigma d_W/2+(m-n)d_H/p}\|f_n\|_{L^p(K)}\right\|_{l^q}\\ &= \left\|\sum_{n=1}^{m} r^{-m\sigma d_W/2+nd_H/p}\|f_{m-n}\|_{L^p(K)}\right\|_{l^q}\\ &= \left\|\sum_{n=1}^{m} r^{-(m-n)\sigma d_W/2+n(d_H/p-\sigma d_W/2)}\|f_{m-n}\|_{L^p(K)}\right\|_{l^q}\\ &\le \left(\sum_{n=1}^{\infty} r^{n(d_H/p-\sigma d_W/2)}\right) \cdot \left\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\right\|_{l^q}\end{aligned} \tag{4.5}$$

by using Lemma 4.3(c) and the Minkowski inequality, and

$$\begin{aligned}\left\|\sum_{n=m}^{\infty} r^{-m\sigma d_W/2} I_p(f_n, r^m)\right\|_{l^q} &\lesssim \left\|\sum_{n=m}^{\infty} r^{-m\sigma d_W/2}\|f_n\|_{L^p(K)}\right\|_{l^q}\\ &= \left\|\sum_{n=0}^{\infty} r^{-m\sigma d_W/2}\|f_{n+m}\|_{L^p(K)}\right\|_{l^q}\\ &\le \left(\sum_{n=0}^{\infty} r^{n\sigma d_W/2}\right) \cdot \left\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\right\|_{l^q}\end{aligned} \tag{4.6}$$

by using Lemma 4.3(b) and the Minkowski inequality again.

Combining equations 4.2–4.6, and noticing that $0 < \sigma < \frac{d_S}{p}$, we get

$$\|f\|_{\Lambda^{p,q}_\sigma(K)} \lesssim \left\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\right\|_{l^q}.$$

The proposition follows. ∎

4.2 Graph Laplacians and a tent function decomposition

Now, we turn to the case when $\sigma > \mathscr{L}_1(p)$. In this case, we have $B^{p,q}_\sigma(K) \subset C(K)$ as a well-known result[23], and we would expect this to happen for $\Lambda^{p,q}_\sigma(K)$. This can be easily seen from the following lemma.

Lemma 4.5 *Let $1 < p < \infty$ and $f \in L^p(K)$, we define $E_{B_t(x)}(f) = \frac{1}{\mu(B_t(x))}\int_{B_t(x)} f d\mu$ for any $x \in K$ and $0 < t \leq 1$. Then we have:*

(a). If $\{x_j\}_{j=1}^n \subset K$ is a finite set of points such that $\sum_{j=1}^n 1_{B_{rt}(x_j)} \leq \lambda$ for some finite number $\lambda < \infty$, then

$$\left(\sum_{j=1}^n |E_{B_t(x_j)}(f) - E_{B_{rt}(x_j)}(f)|^p\right)^{1/p} \lesssim \lambda^{1/p} t^{-d_H/p} I_p\left(f, (r+1)t\right).$$

(b). For $1 \leq q \leq \infty$, $\Lambda_\sigma^{p,q}(K) \subset C^{\sigma d_W/2 - d_H/p}(K)$ for $\sigma > \frac{d_S}{p}$, where $C^{\sigma d_W/2-d_H/p}(K)$ denotes the space of Hölder continuous functions on (K, R) with exponent $\sigma d_W/2 - d_H/p$.

Proof (a) follows from a direct estimate,

$$\begin{aligned}
&\left(\sum_{j=1}^n |E_{B_t(x_j)}(f) - E_{B_{rt}(x_j)}(f)|^p\right)^{1/p} \\
\lesssim& t^{-d_H/p}\left(\sum_{j=1}^n \int_{B_{rt}(x_j)} |f(y) - E_{B_t(x_j)}(f)|^p d\mu(y)\right)^{1/p} \\
\lesssim& t^{-d_H/p}\left(\sum_{j=1}^n \int_{B_{rt}(x_j)} ((r+1)t)^{-d_H} \int_{B_{(r+1)t}(y)} |f(y) - f(z)|^p d\mu(z) d\mu(y)\right)^{1/p} \\
\leq& \lambda^{1/p} t^{-d_H/p} I_p\left(f, (r+1)t\right).
\end{aligned}$$

(b). Let $f \in \Lambda_\sigma^{p,q}(K)$. By the Lebesgue differentiation theorem, we know that $f(x) = \lim_{m\to\infty} E_{B_{r^m}(x)}(f)$ for μ-a.e. $x \in K$. For such x and $m \geq 0$,

$$\begin{aligned}
\left|f(x) - E_{B_{r^m}(x)}(f)\right| &\leq \sum_{n=m}^\infty \left|E_{B_{r^n}(x)}(f) - E_{B_{r\cdot r^n}(x)}(f)\right| \\
&\leq C_1 \sum_{n=m}^\infty r^{-nd_H/p} I_p\left(f, ((r+1)r^n) \wedge 1\right) \leq C_2 r^{m(\sigma d_W/2 - d_H/p)} \|f\|_{\Lambda_\sigma^{p,q}(K)},
\end{aligned}$$

where we apply the special case of (a) with one point, one ball and $\lambda = 1$ in the second inequality, and we use the fact that $\frac{\sigma d_W}{2} > \frac{d_H}{p}$ in the last inequality. Next, we fix $x, y \in K$ such that $f(x) = \lim_{m\to\infty} E_{B_{r^m}(x)}(f)$, $f(y) = \lim_{m\to\infty} E_{B_{r^m}(y)}(f)$ and $r^{m+1} < R(x, y) \leq r^m$ for some $m \geq 0$. Choose $k > 0$ such that $r^k < 1/2$ and let $m' = (m-k) \vee 0$. It is not hard to see that

$$\left|E_{B_{r^m}(x)}(f) - E_{B_{r^m}(y)}(f)\right| \leq C_3 r^{-md_H/p} I_p(f, r^{m'}) \leq C_4 r^{m(\sigma d_W/2 - d_H/p)} \|f\|_{\Lambda_\sigma^{p,q}(K)}.$$

Combining the above two estimates, we can see that

$$\begin{aligned}
|f(x) - f(y)| &\leq (2C_2 + C_4) r^{m(\sigma d_W/2 - d_H/p)} \|f\|_{\Lambda_\sigma^{p,q}(K)} \\
&\leq C_5 \|f\|_{\Lambda_\sigma^{p,q}(K)} \cdot \left(R(x, y)\right)^{\sigma d_W/2 - d_H/p}.
\end{aligned}$$

Since the above estimate holds for μ-a.e. $x, y \in K$, there exists a Hölder continuous version of *f*, and the Hölder exponent is $\sigma d_W/2 - d_H/p$. ∎

In addition, we will have a characterization of $\Lambda_\sigma^{p,q}(K)$ based on the discrete Laplacian on Λ_m for $\mathscr{L}_1(p) < \sigma < \mathscr{C}(p)$.

Definition 4.6 (a). For $m \geq 0$, define the *graph energy form* on V_{Λ_m} by

$$\mathcal{E}_{\Lambda_m}(f,g) = \sum_{w\in\Lambda_m} r_w^{-1}\mathcal{E}_0(f\circ F_w, g\circ F_w), \quad \forall f,g\in l(V_{\Lambda_m}).$$

(b). Define $H_{\Lambda_m} : l(V_{\Lambda_m}) \to l(V_{\Lambda_m})$ the *graph Laplacian* associated with $\mathcal{E}_{\Lambda_m}$, i.e.,

$$\mathcal{E}_{\Lambda_m}(f,g) = -< H_{\Lambda_m} f, g >_{l^2(V_{\Lambda_m})} = -< f, H_{\Lambda_m} g >_{l^2(V_{\Lambda_m})}, \quad \forall f,g\in l(V_{\Lambda_m}),$$

where $l^2(V_{\Lambda_m})$ stands for the discrete l^2 inner product over V_{Λ_m} with the counting measure.

(c). For $m \geq 1$, define $J_m = \{f \in C(K) : f$ is harmonic in $F_wK, \forall w \in \Lambda_m$, and $f|_{V_{\Lambda_{m-1}}} = 0\}$, and call J_m the space of *level-m tent functions*. For convenience, set $J_0 = \mathcal{H}_0$.

(d). For $1 < p < \infty, 1 \leq q \leq \infty$ and $\sigma > \frac{d_S}{p}$, define

$$\Lambda^{p,q}_{\sigma,(1)}(K) = \left\{f\in C(K) : \{r^{m(-\sigma d_W/2+1+d_H/p)}\|H_{\Lambda_m} f\|_{l^p(V_{\Lambda_m})}\}\in l^q\right\}$$

with norm $\|f\|_{\Lambda^{p,q}_{\sigma,(1)}(K)} = \|f\|_{L^p(K)} + \left\|r^{m(-\sigma d_W/2+1+d_H/p)}\|H_{\Lambda_m} f\|_{l^p(V_{\Lambda_m})}\right\|_{l^q}$.

For each $f \in C(K)$, clearly f admits a unique expansion in terms of tent functions $f = \sum_{m=0}^\infty f_m$ with $f_m \in J_m, \forall m \geq 0$.

Before proceeding, let's first collect some easy observations. Recall the notation $\mathring{V}_{\Lambda_m}$ from Definition 4.1.

Lemma 4.7 *Let $1 < p < \infty$ and $u \in J_m$ with $m \geq 0$.*

(a). $H_{\Lambda_n} u|_{\mathring{V}_{\Lambda_n}} = 0$ for any $n > m$.

(b). $\|u\|_{L^p(K)} \asymp r^{m(1+d_H/p)}\|H_{\Lambda_m}u\|_{l^p(\mathring{V}_{\Lambda_m})} \asymp r^{m(1+d_H/p)}\|H_{\Lambda_m}u\|_{l^p(V_{\Lambda_m})}$ for $m \geq 1$.

(c). For any $\sigma < \mathscr{C}(p)$, we have $I_p(u, r^n) \leq Cr^{(n-m)\sigma d_W/2}\|u\|_{L^p(K)}$ for all $n \geq m$.

Proof (a) is trivial since u is harmonic on $\mathring{V}_{\Lambda_n}$ by definition.

(b). First, we see the following estimate between $L^p(K)$ and $l^p(\mathring{V}_{\Lambda_m})$ (or $l^p(V_{\Lambda_m})$) norm.

$$\begin{aligned}\|u\|_{L^p(K)} &= \left(\sum_{w\in\Lambda_m}\mu_w\|u\circ F_w\|^p_{L^p(K)}\right)^{1/p} \asymp r^{md_H/p}\left(\sum_{w\in\Lambda_m}\|u\circ F_w\|^p_{L^p(K)}\right)^{1/p} \\ &\asymp r^{md_H/p}\left(\sum_{x\in\mathring{V}_{\Lambda_m}}|u(x)|^p\right)^{1/p} = r^{md_H/p}\|u\|_{l^p(\mathring{V}_{\Lambda_m})} = r^{md_H/p}\|u\|_{l^p(V_{\Lambda_m})}.\end{aligned}$$

To proceed, on one hand, we notice that $\|H_{\Lambda_m} u\|_{l^p(V_{\Lambda_m})} \lesssim r^{-m}\|u\|_{l^p(V_{\Lambda_m})}$ since

$$\begin{aligned}\|H_{\Lambda_m} u\|_{l^p(V_{\Lambda_m})} &= \left(\sum_{x\in V_{\Lambda_m}} \left(\sum_{w\in\Lambda_m: F_w V_0 \ni x} r_w^{-1} \sum_{y\in F_w V_0} H_{(F_w^{-1}x)(F_w^{-1}y)}\,(u(x)-u(y))\right)^p\right)^{1/p}\\ &\lesssim \left(\sum_{x\in V_{\Lambda_m}} \left(\sum_{w\in V_{\Lambda_m}: F_w V_0\ni x} r_w^{-1}\sum_{y\in F_w V_0} u(x)\right)^p\right)^{1/p} \lesssim r^{-m}\|u\|_{l^p(V_{\Lambda_m})},\end{aligned}$$

where we recall that $H = (H_{pq})_{p,q\in V_0}$ is the Laplacian matrix associated with $\mathcal{E}_0$. On the other hand, noticing that there are essentially finitely many different types of $F_w K \cap \mathring{V}_{\Lambda_m}$, $w \in \Lambda_{m-1}$, we see that $\|H_{\Lambda_m} u\|_{l^p(\mathring{V}_{\Lambda_m}\cap F_w K)} \asymp r^{-m}\|u\|_{l^p(\mathring{V}_{\Lambda_m}\cap F_w K)}$ with "$\asymp$" independent of $w\in\Lambda_{m-1}$, $m \geq 1$, so by taking summation,

$$\|H_{\Lambda_m} u\|_{l^p(V_{\Lambda_m})} \geq \|H_{\Lambda_m} u\|_{l^p(\mathring{V}_{\Lambda_m})} \asymp r^{-m}\|u\|_{l^p(\mathring{V}_{\Lambda_m})}.$$

(b) follows by combining the above three estimates.

(c). For $m = 0$, the result trivially follows from the definition of $\mathscr{C}(p)$. It suffices to consider $m \geq 1$ case. Choose k such that $R(x,y) > r^{k+n}$ for any x, y not in adjacent cells in $\{F_\tau K : \tau \in \Lambda_n\}$ and for any $n \geq 0$. Now, for any fixed $w \in \Lambda_m$, we consider the integral

$$\left(\int_{F_w K} r^{-nd_H}\int_{B_{r^n}(x)} |u(x)-u(y)|^p d\mu(y)d\mu(x)\right)^{1/p} \leq I(1,w) + I(2,w)$$

with

$$\begin{aligned} I(1,w) &:= \left(\int_{F_w K} r^{-nd_H}\int_{B_{r^n}(x)\cap F_w K} |u(x)-u(y)|^p d\mu(y)d\mu(x)\right)^{1/p},\\ I(2,w) &:= \left(\int_{F_w K} r^{-nd_H}\int_{B_{r^n}(x)\setminus F_w K} |u(x)-u(y)|^p d\mu(y)d\mu(x)\right)^{1/p}.\end{aligned}$$

Notice that

$$I_p(u,r^n) \leq \left(\sum_{w\in\Lambda_m} (I(1,w)+I(2,w))^p\right)^{1/p} \leq \left(\sum_{w\in\Lambda_m} (I(1,w))^p\right)^{1/p} + \left(\sum_{w\in\Lambda_m} (I(2,w))^p\right)^{1/p}.$$

Let's estimate $I(1,w)$ and $I(2,w)$. For $I(1,w)$, we notice that

$$\begin{aligned} I(1,w) &= \left(\int_{F_w K} r^{-nd_H}\int_{B_{r^n}(x)\cap F_w K} |u(x)-u(y)|^p d\mu(y)d\mu(x)\right)^{1/p}\\ &\leq \left(r^{md_H}\int_K r^{(m-n)d_H}\int_{F_w^{-1}(B_{r^n}(F_w x))} |u\circ F_w(x) - u\circ F_w(y)|^p d\mu(y)d\mu(x)\right)^{1/p}\\ &\leq \left(r^{md_H}\int_K r^{(m-n)d_H}\int_{B_{c_2\cdot r^{n-m}}(x)} |u\circ F_w(x) - u\circ F_w(y)|^p d\mu(y)d\mu(x)\right)^{1/p}\\ &\lesssim r^{md_H/p} I_p(u\circ F_w, c_2 r^{n-m}),\end{aligned}$$

where c_2 is constant depending only on K and the harmonic structure, noticing that there is $c_1 \in (0,1)$ such that $R(F_w x, F_w y) \geq c_1 r_w R(x,y)$, $\forall w \in W_*, x, y \in K$.

In addition, since $u\circ F_w\in\mathcal{H}_0$ and $\sigma<\mathscr{C}(p)$, we know that

$$r^{(m-n)\sigma d_W/2}I_p(u\circ F_w,c_2r^{n-m})\lesssim\|u\circ F_w\|_{\Lambda^{p,\infty}_\sigma(K)}\lesssim\|u\circ F_w\|_{L^p(K)}\lesssim r^{-md_H/p}\|u\|_{L^p(F_wK)},$$

where the second inequality is due to the fact that all norms on the finite dimensional space $\mathcal{H}_0$ are equivalent. Combining the above two observations, we get

$$I(1,w)\lesssim r^{(n-m)\sigma d_W/2}\|u\|_{L^p(F_wK)}.$$

For $I(2,w)$, we can see that in fact $B_{r^n}(x)\backslash F_wK\neq\varnothing$ only if x stays in a cell $F_\tau K$ with $\tau\in\Lambda_{n-k}$ which contains a point $z\in F_wV_0$. Without loss of generality, we assume that $n-k\geq m$, and sum $I(2,w)$'s over Λ_m to get

$$\begin{aligned}\Big(\sum_{w\in\Lambda_m}I(2,w)^p\Big)^{1/p}&\leq\Big(\sum_{\tau,\tau'}\int_{x\in F_\tau K}\int_{y\in F_{\tau'}K}r^{-nd_H}|u(x)-u(y)|^pd\mu(y)d\mu(x)\Big)^{1/p}\\&\leq C\Big(\sum_{\tau}\sum_{z\in V_{\Lambda_m}\cap F_\tau K}\int_{x\in F_\tau K}|u(x)-u(z)|^pd\mu(x)\Big)^{1/p}\\&\leq C'\Big(r^{nd_H}\sum_{w\in\Lambda_m}r^{(n-m)p}\|u\circ F_w\|^p_{L^p(K)}\Big)^{1/p}\\&=C'r^{(n-m)(1+d_H/p)}\Big(\sum_{w\in\Lambda_m}\|u\|^p_{L^p(F_wK)}\Big)^{1/p},\end{aligned}$$

where $\tau,\tau'\in\{\tau''\in\Lambda_{n-k}:F_{\tau''}K\cap V_{\Lambda_m}\neq\varnothing\}$, and we require $\tau\neq\tau'$ with $F_\tau K\cap F_{\tau'}K\neq\varnothing$ in the first line.

Combining the estimates on $I(1,w)$'s and $I(2,w)$'s, and noticing that

$$1+\frac{d_H}{p}\geq\mathscr{C}(p)\frac{d_W}{2}>\frac{\sigma d_W}{2}$$

by Proposition 3.3, (c) follows. ∎

Now, we state the main result in this subsection.

Theorem 4.8 *Let $f\in C(K)$ with $f=\sum_{m=0}^\infty f_m$ and $f_m\in J_m,\forall m\geq0$. For $1<p<\infty$, $1\leq q\leq\infty$, for the claims:*

(1) $f\in\Lambda^{p,q}_\sigma(K)$; (2) $f\in\Lambda^{p,q}_{\sigma,(1)}(K)$; (3) $\{r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\}_{m\geq0}\in l^q$,

we can say:

(a). If $\sigma>\mathscr{L}_1(p)$, we have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) with

$$\|f\|_{\Lambda^{p,q}_\sigma(K)}\gtrsim\|f\|_{\Lambda^{p,q}_{\sigma,(1)}(K)}\gtrsim\big\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\big\|_{l^q}.$$

(b). If $\mathscr{L}_1(p)<\sigma<\mathscr{L}_2(p)$, we have (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) with

$$\|f\|_{\Lambda^{p,q}_\sigma(K)}\gtrsim\|f\|_{\Lambda^{p,q}_{\sigma,(1)}(K)}\asymp\big\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\big\|_{l^q}.$$

(c). If $(\frac{1}{p},\sigma)\in\mathscr{A}_1$, we have (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) with

$$\|f\|_{\Lambda^{p,q}_\sigma(K)}\asymp\|f\|_{\Lambda^{p,q}_{\sigma,(1)}(K)}\asymp\big\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\big\|_{l^q}.$$

Proof (a). We first prove (1) $\Longrightarrow$ (2). We follow the conventional notation to denote $x \sim_m y$ if $x, y \in F_w V_0$ for some $w \in \Lambda_m$. We fix $k > 0$ so that $B_{r^m}(y) \subset B_{r^{m-k}}(x)$ for any $x \sim_m y$ and $m \geq 0$. Then we can see

$$\left(\sum_{x \sim_m y} |E_{B_{r^m}(x)}(f) - E_{B_{r^m}(y)}(f)|^p \right)^{1/p} \lesssim r^{-md_H/p} I_p(f, r^{m-k}).$$

Since each vertex $x \in V_{\Lambda_m}$ is of bounded degree, by writing

$$f(x) = E_{B_{r^m}(x)}(f) + \sum_{n=m}^{\infty} \left(E_{B_{r^{n+1}}(x)}(f) - E_{B_{r^n}(x)}(f) \right),$$

we apply Lemma 4.5(a) to see that

$$\left(\sum_{x \sim_m y} |f(x) - f(y)|^p \right)^{1/p} \lesssim r^{-md_H/p} I_p(f, r^{m-k}) + \sum_{n=m}^{\infty} r^{-nd_H/p} I_p(f, r^n).$$

Write $\tilde{H}_{\Lambda_m} f = r^{m(1+d_H/p)} H_{\Lambda_m} f$ for convenience. We then have

$$\begin{aligned} \|\tilde{H}_{\Lambda_m} f\|_{l^p(V_{\Lambda_m})} &\lesssim I_p(f, r^{m-k}) + \sum_{n=m}^{\infty} r^{(m-n)d_H/p} I_p(f, r^n) \\ &= I_p(f, r^{m-k}) + \sum_{n=0}^{\infty} r^{-nd_H/p} I_p(f, r^{m+n}). \end{aligned}$$

Noticing that

$$\begin{aligned} \left\| r^{-m\sigma d_W/2} \sum_{n=0}^{\infty} r^{-nd_H/p} I_p(f, r^{m+n}) \right\|_{l^q} &= \left\| \sum_{n=0}^{\infty} r^{n(\sigma d_W/2 - d_H/p)} r^{-(m+n)\sigma d_W/2} I_p(f, r^{m+n}) \right\|_{l^q} \\ &\leq \sum_{n=0}^{\infty} r^{n(\sigma d_W/2 - d_H/p)} \cdot \left\| r^{-m\sigma d_W/2} I_p(f, r^m) \right\|_{l^q} \\ &\lesssim \left\| r^{-m\sigma d_W/2} I_p(f, r^m) \right\|_{l^q}, \end{aligned}$$

since we assume $\sigma > \mathscr{L}_1(p) = \frac{d_S}{p}$, the claim follows.

(2) $\Longrightarrow$ (3) is easy. We can see that $H_{\Lambda_m}\left(\sum_{n=0}^{m-1} f_m\right)|_{\mathring{V}_{\Lambda_m}} \equiv 0$ by Lemma 4.7(a), and $(\sum_{n=0}^{m} f_m)|_{V_{\Lambda_m}} = f|_{V_{\Lambda_m}}$. Thus, $H_{\Lambda_m} f|_{\mathring{V}_{\Lambda_m}} = H_{\Lambda_m} f_m|_{\mathring{V}_{\Lambda_m}}$ and then

$$\|f_m\|_{L^p(K)} \asymp \|\tilde{H}_{\Lambda_m} f\|_{l^p(\mathring{V}_{\Lambda_m})}$$

using Lemma 4.7(b). The claim follows immediately.

(b). It remains to show (3) $\Longrightarrow$ (2). By the definition of $\mathcal{E}_{\Lambda_m}$ and the fact that f_m is harmonic in $F_w K$ for each $w \in \Lambda_m$, we can see that

$$\|H_{\Lambda_n} f_m\|_{l^p(V_{\Lambda_n})} = \|H_{\Lambda_m} f_m\|_{l^p(V_{\Lambda_m})} \asymp r^{-m(1+d_H/p)} \|f_m\|_{L^p(K)}, \forall n \geq m.$$

Also, $\|H_{\Lambda_n} f_m\|_{l^p(V_{\Lambda_n})} = 0$ for $n < m$. Thus,

$$\begin{aligned}\left\| r^{m(-\sigma d_W/2+1+d_H/p)} \|H_{\Lambda_m} f\|_{l^p(V_{\Lambda_m})} \right\|_{l^q} &\lesssim \left\| \sum_{n=0}^{m} r^{(m-n)(1+d_H/p)-m\sigma d_W/2} \|f_n\|_{L^p(K)} \right\|_{l^q} \\ &= \left\| \sum_{n=0}^{m} r^{n(1+d_H/p)-m\sigma d_W/2} \|f_{m-n}\|_{L^p(K)} \right\|_{l^q} \\ &\lesssim \sum_{n=0}^{\infty} r^{n(1+d_H/p-\sigma d_W/2)} \cdot \left\| r^{-m\sigma d_W/2} \|f_m\|_{L^p(K)} \right\|_{l^q}.\end{aligned}$$

The claim follows since $1 + \frac{d_H}{p} - \frac{\sigma d_W}{2} > 0$ by $\sigma < \mathscr{L}_2(p)$.

(c). We only need to prove (3) $\Longrightarrow$ (1). Since $(\frac{1}{p}, \sigma) \in \mathscr{A}_1$, which means that we have $\sigma < \mathscr{C}(p)$ in addition to $\mathscr{L}_1(p) < \sigma < \mathscr{L}_2(p)$, we can find $\eta \in (\sigma, \mathscr{C}(p))$. The rest proof goes similar to the second part of Proposition 4.4. By using Lemma 4.7(c), we can see

$$I_p(f_n, r^m) \lesssim r^{(m-n)\eta d_W/2} \|f_n\|_{L^p(K)}, \forall m \geq n \geq 0.$$

Then, by the proof of (4.5), we get

$$\| \sum_{n=0}^{m-1} r^{-m\sigma d_W/2} I_p(f_n, r^m) \|_{l^q} \lesssim \left(\sum_{n=1}^{\infty} r^{n(\eta-\sigma)d_W/2} \right) \cdot \left\| r^{-m\sigma d_W/2} \|f_m\|_{L^p(K)} \right\|_{l^q}.$$

In addition, by Lemma 4.3(c) and the same proof of (4.6), we have

$$\| \sum_{n=m}^{\infty} r^{-m\sigma d_W/2} I_p(f_n, r^m) \|_{l^q} \lesssim \left(\sum_{n=0}^{\infty} r^{n\sigma d_W/2} \right) \cdot \left\| r^{-m\sigma d_W/2} \|f_m\|_{L^p(K)} \right\|_{l^q}.$$

The claim then follows by combining the above two estimates, and noticing that

$$\|r^{-m\sigma d_W/2} I_p(f, r^m)\|_{l^q} \leq \| \sum_{n=0}^{m-1} r^{-m\sigma d_W/2} I_p(f_n, r^m) \|_{l^q} + \| \sum_{n=m}^{\infty} r^{-m\sigma d_W/2} I_p(f_n, r^m) \|_{l^q}.$$

■

We end this section with the following theorem, whose proof will be completed in Section 5 and Section 6.

Theorem 4.9 *For $1 < p < \infty$, $1 \leq q \leq \infty$ and $\mathscr{L}_1(p) < \sigma < 2$, we have $B^{p,q}_\sigma(K) = \Lambda^{p,q}_{\sigma,(1)}(K)$ with $\|\cdot\|_{\Lambda^{p,q}_{\sigma,(1)}(K)} \asymp \|\cdot\|_{B^{p,q}_\sigma(K)}$.*

5 Embedding $\Lambda^{p,q}_\sigma(K)$ into $B^{p,q}_\sigma(K)$

In this section, we will use the J-method of real interpolation to prove that $\Lambda^{p,q}_\sigma(K) \subset B^{p,q}_\sigma(K)$ for $0 < \sigma < 2$. This will not involve the critical curve $\mathscr{C}$.

We will use the following fact about the real interpolation, which is obvious from the J-method. Readers may find details of the J-method in the book [9].

Lemma 5.1 *Let $\bar{X} := (X_0, X_1)$ be an interpolation couple of Banach spaces, $1 \leq q \leq \infty$, $0 < \theta < 1$, and $\bar{X}_{\theta,q} := (X_0, X_1)_{\theta,q}$ be the real interpolation space. For each $x \in X_0 \cap X_1$*

and $\lambda > 0$, *write*

$$J(\lambda, x) = \max\left\{\|x\|_{X_0}, \lambda\|x\|_{X_1}\right\}.$$

If $\{g_m\}_{m\geq 0} \subset X_0 \cap X_1$ *is a sequence with* $\left\{\lambda^{-m\theta}J(\lambda^m, g_m)\right\}_{m\geq 0} \in l^q$ *and* $\lambda > 0$, *then*

$$g = \sum_{m=0}^{\infty} g_m \in \bar{X}_{\theta,q}$$

with $\|g\|_{\bar{X}_{\theta,q}} \lesssim \left\|\lambda^{-m\theta}J(\lambda^m, g_m)\right\|_{l^q}$.

We start with the following lemma.

Lemma 5.2 *For* $1 < p < \infty$, $f \in L^p(K)$ *and* $m \geq 0$, *there is a function* u_m *in* $H_2^p(K)$ *such that*

$$E[u_m|\Lambda_m] = E[f|\Lambda_m],$$

and in addition,

$$\begin{cases} \|u_m - E[f|\Lambda_m]\|_{L^p(K)} \leq CI_p(f, r^{m-k}), \\ \|\Delta u_m\|_{L^p(K)} \leq Cr^{-md_W} I_p(f, r^{m-k}), \end{cases}$$

where $k \in \mathbb{N}$ *and* $C > 0$ *are constants independent of f and m.*

Proof For convenience, we write $E[f|\Lambda_m] = \sum_{w\in\Lambda_m} c_w 1_{F_w K}$ with $c_w \in \mathbb{C}$.

For each $x \in V_{\Lambda_m}$ and $n \geq m$, we write $U_{x,n} = \bigcup\{F_{w'}K : x \in F_{w'}K, w' \in \Lambda_n\}$, and take $m' \geq m$ to be the smallest one, such that

$$\#V_{\Lambda_m} \cap U_{x,m'} \leq 1, \quad U_{x,m'} \cap U_{y,m'} = \emptyset, \quad \forall x, y \in V_{\Lambda_m}.$$

Clearly, the difference $m' - m$ is bounded for all m.

Let $U_{m'} = \bigcup_{x\in V_{\Lambda_m}} U_{x,m'}$. We define u_m as follows:

(1). For $x \in K \backslash U_{m'}$, we define $u_m(x) = E[f|\Lambda_m](x)$.

(2). For $x \in V_{\Lambda_m}$, we let $M_x = \#\{w \in \Lambda_m : x \in F_w K\}$ and define

$$u_m(x) = \frac{1}{\#M_x}\sum_{w\in\Lambda_m, F_w K\ni x} c_w.$$

(3). It remains to construct u_m on each $U_{x,m'}$ with $x \in V_{\Lambda_m}$. In this case, for each $F_{w'}K \subset U_{x,m'}$ with $w' \in \Lambda_{m'}$, we have already defined its boundary values. For u_m in $F_{w'}K$, additionally, we require that u_m satisfies the Neumann boundary condition on $F_{w'}V_0$, and $E_{w'}(u_m) = c_w$ for w to be the word in Λ_m such that $F_{w'}K \subset F_w K$. It is easy to see the existence of such a function locally on $F_{w'}K$, and the following estimate can be achieved by scaling:

$$\begin{cases} \|u_m - c_w\|_{L^p(F_{w'}K)} \lesssim r^{md_H/p}\left|u_m(x) - c_w\right|, \\ \|\Delta u_m\|_{L^p(F_{w'}K)} \lesssim r^{m(d_H/p - d_W)}\left|u_m(x) - c_w\right|. \end{cases}$$

With (1)–(3), we obtain a function $u_m \in H_2^p(K)$ such that $E[u_m|\Lambda_m] = E[f|\Lambda_m]$. It remains to show the desired estimates for u_m. First, we have

$$\begin{aligned}\|u_m - E[f|\Lambda_m]\|_{L^p(K)} &= \Big(\sum_{w'} \|u_m - c_w\|^p_{L^p(F_{w'}K)}\Big)^{1/p} \\ &\lesssim \Big(r^{md_H} \sum_{w'} |u_m(x_{w'}) - c_w|^p\Big)^{1/p} \lesssim \Big(r^{md_H} \sum_{w\sim_m v} |c_w - c_v|^p\Big)^{1/p},\end{aligned}$$

where the summation $\sum_{w'}$ is over all $w' \in \Lambda_{m'}$ such that $F_{w'}K \cap V_{\Lambda_m} \neq \emptyset$ and $x_{w'}$ is the single vertex in $F_{w'}K \cap V_{\Lambda_m}$, w stands for the word in Λ_m such that $F_{w'}K \subset F_wK$, and $\sum_{w\sim_m v}$ is over all the pairs $w, v \in \Lambda_m$ with $F_wK \cap F_vK \neq \emptyset$. By choosing $k \in \mathbb{N}$ such that $r^{m-k} > 2\mathrm{diam}F_wK$ for any $w \in \Lambda_m$ (clearly, this k can be chosen to work for all m), we then have

$$\Big(r^{md_H} \sum_{w\sim v} |c_w - c_v|^p\Big)^{1/p} \lesssim I_p(f, r^{m-k}),$$

thus, we get the first desired estimate. The estimate for $\|\Delta u_m\|_{L^p(K)}$ is essentially the same. ∎

Now, we prove the main result of this section.

Proposition 5.3 *For $1 < p < \infty$, $1 \le q \le \infty$ and $0 < \sigma < 2$, we have $\Lambda^{p,q}_\sigma(K) \subset B^{p,q}_\sigma(K)$ with $\|\cdot\|_{B^{p,q}_\sigma(K)} \lesssim \|\cdot\|_{\Lambda^{p,q}_\sigma(K)}$.*

Proof Let $f \in \Lambda^{p,q}_\sigma(K)$. We define a sequence of functions u_m in $H_2^p(K)$ by Lemma 5.2, and we take

$$g_m = \begin{cases} u_0, & \text{if } m = 0,\\ u_m - u_{m-1}, & \text{if } m > 0.\end{cases}$$

For $m \ge 0$, by Lemma 5.2, we have the estimate

$$\begin{cases} \|g_m\|_{L^p(K)} \lesssim \big\|\tilde{E}[f|\Lambda_m]\big\|_{L^p(K)} + I_p(f, r^{m-k}) + I_p(f, r^{m-k-1}) \lesssim I_p(f, r^{m-k-1}),\\ \|\Delta g_m\|_{L^p(K)} \lesssim r^{-md_W}\big(I_p(f, r^{m-k}) + I_p(f, r^{m-k-1})\big) \lesssim r^{-md_W} I_p(f, r^{m-k-1}),\end{cases}$$

where k is the same as Lemma 5.2, and we use Lemma 4.3(a) in the second estimate of the first formula. Taking $\lambda = r^{d_W}$, $X_0 = L^p(K)$ and $X_1 = H_2^p(K)$ in Lemma 5.1, it then follows that

$$\begin{aligned}\Big\|r^{-m\sigma d_W/2} J(r^{md_W}, g_m)\Big\|_{l^q} &\le \Big\|r^{-m\sigma d_W/2}\|g_m\|_{L^p(K)}\Big\|_{l^q} + \Big\|r^{m(1-\sigma/2)d_W}\|g_m\|_{H_2^p(K)}\Big\|_{l^q} \\ &\lesssim \Big\|r^{-m\sigma d_W/2} I_p(f, r^{m-k-1})\Big\|_{l^q} \lesssim \|f\|_{\Lambda^{p,q}_\sigma(K)}.\end{aligned}$$

It is easy to see that $f = \sum_{m=0}^\infty g_m$, so combining with Lemma 2.6, we have $f \in B^{p,q}_\sigma(K)$ with $\|f\|_{B^{p,q}_\sigma(K)} \lesssim \|f\|_{\Lambda^{p,q}_\sigma(K)}$. ∎

Before ending this section, we mention that the same method can be applied to show that $\Lambda^{p,q}_{\sigma,(1)}(K) \subset B^{p,q}_\sigma(K)$ for $\mathscr{L}_1(p) < \sigma < 2$. In this case, for each $m \ge 0$, we choose a piecewise harmonic function of level m that coincides with f at V_{Λ_m}, then modify it in a neighborhood of V_{Λ_m} to get a function u_m in $H_2^p(K)$ analogous to that in Lemma 5.2. Since the idea is essentially the same, we omit the proof, and state the result as follows.

Proposition 5.4 *For $1 < p < \infty$, $1 \le q \le \infty$ and $\mathscr{L}_1(p) < \sigma < 2$, we have $\Lambda^{p,q}_{\sigma,(1)}(K) \subset B^{p,q}_\sigma(K)$ with $\|\cdot\|_{B^{p,q}_\sigma(K)} \lesssim \|\cdot\|_{\Lambda^{p,q}_{\sigma,(1)}(K)}$.*

6 Embedding $B^{p,q}_\sigma(K)$ into $\Lambda^{p,q}_\sigma(K)$

We will prove $B^{p,q}_\sigma(K) \subset \Lambda^{p,q}_\sigma(K)$ for $0 < \sigma < \mathscr{C}(p)$ in this section. Also, we will show that $B^{p,q}_\sigma(K) \subset \Lambda^{p,q}_{\sigma,(1)}(K)$ for $\mathscr{L}_1(p) < \sigma < 2$.

First, let's look at two easy lemmas.

Lemma 6.1 *Let $\{X_m, \|\cdot\|_{X_m}\}_{m\ge0}$ be a sequence of Banach spaces. For $1 \le q \le \infty$ and $\alpha > 0$, let*

$$l^q_\alpha(X_\cdot) = \{\boldsymbol{s} = \{s_m\}_{m\ge0} : s_m \in X_m, \forall m \ge 0, \text{ and } \{\alpha^{-m}\|s_m\|_{X_m}\} \in l^q\},$$

be the space with norm $\|\boldsymbol{s}\|_{l^q_\alpha(X_\cdot)} = \big\|\alpha^{-m}\|s_m\|_{X_m}\big\|_{l^q}$. Then, for $\alpha_0 \ne \alpha_1$, $0 < \theta < 1$ and $1 \le q_0, q_1, q \le \infty$, we have

$$\big(l^{q_0}_{\alpha_0}(X_\cdot), l^{q_1}_{\alpha_0}(X_\cdot)\big)_{\theta,q} = l^q_{\alpha_\theta}(X_\cdot), \text{ with } \alpha_\theta = \alpha_0^{(1-\theta)}\alpha_1^\theta.$$

This lemma is revised from Theorem 5.6.1 in book [9] with a same argument. The difference is that we allow each coordinate taking values in different spaces, which does not bring any difficult to the proof. As an immediate consequence, we can see the following interpolation lemma for $\Lambda^{p,q}_\sigma(K)$ and $\Lambda^{p,q}_{\sigma,(1)}(K)$.

Lemma 6.2 *Let $1 \le p, q, q_0, q_1 \le \infty$, $0 < \sigma_0, \sigma_1 < \infty$ and $0 < \theta < 1$. We have:*
(a). $\big(\Lambda^{p,q_0}_{\sigma_0}(K), \Lambda^{p,q_1}_{\sigma_1}(K)\big)_{\theta,q} \subset \Lambda^{p,q}_{\sigma_\theta}(K)$;
(b). $\Big(\Lambda^{p,q_0}_{\sigma_0,(1)}(K), \Lambda^{p,q_1}_{\sigma_1,(1)}(K)\Big)_{\theta,q} \subset \Lambda^{p,q}_{\sigma_\theta,(1)}(K)$,
with $\sigma_\theta = (1-\theta)\sigma_0 + \theta\sigma_1$.

From now on, we will separate our consideration into two cases, according to $(\frac{1}{p}, \sigma)$ belongs to $\mathscr{A}_1$ or $\mathscr{A}_2$. We will deal with the border between $\mathscr{A}_1$ and $\mathscr{A}_2$ by using Lemma 6.2. For the region $\mathscr{A}_1$, in fact, we mainly consider a larger region $\mathscr{B} := \Big\{(\frac{1}{p}, \sigma) : \mathscr{L}_1(p) < \sigma < \mathscr{L}_2(p)\Big\}$ instead.

6.1 On regions $\mathscr{A}_1$ and $\mathscr{B}$

To reach the goal that $B^{p,q}_\sigma(K) \subset \Lambda^{p,q}_\sigma(K)$ for $(\frac{1}{p}, \sigma) \in \mathscr{A}_1$, by Theorem 4.8(c), it suffices to prove $B^{p,q}_\sigma(K) \subset \Lambda^{p,q}_{\sigma,(1)}(K)$. We will fulfill this for the parameter region $\mathscr{B} = \Big\{(\frac{1}{p}, \sigma) : \mathscr{L}_1(p) < \sigma < \mathscr{L}_2(p)\Big\}$, which is the region between the two critical lines $\mathscr{L}_1$ and $\mathscr{L}_2$, and of cause contains $\mathscr{A}_1$.

Note that we can write each $f \in C(K)$ as a unique series,

$$f = \sum_{m=0}^{\infty} f_m, \text{ with } f_m \in J_m, \quad \forall m \geq 0,$$

and in addition, by Theorem 4.8(b), it always holds

$$\|f\|_{\Lambda^{p,q}_{\sigma,(1)}(K)} \asymp \left\| r^{-m\sigma d_W/2} \|f_m\|_{L^p(K)} \right\|_{l^q}.$$

Let's start with some observations.

Lemma 6.3 *Let $1 < p, q < \infty$ and $0 < \sigma < 2$. Write $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. Then there is a continuous sesquilinear form $\tilde{\mathcal{E}}(\cdot,\cdot)$ on $B^{p,q}_\sigma(K) \times B^{p',q'}_{2-\sigma}(K)$ such that*

$$\tilde{\mathcal{E}}(f,g) = \mathcal{E}(f,g), \quad \forall f \in B^{p,q}_\sigma(K) \cap dom\mathcal{E}, \quad g \in H^{p'}_2(K).$$

Proof First, by the definition of Δ_N, we can see that

$$\big|\mathcal{E}(f,g)\big| = \Big|\int_K f\Delta_N g d\mu\Big| \leq \|f\|_{L^p(K)} \|g\|_{H^{p'}_2(K)}, \tag{6.1}$$

for any $f \in L^p(K) \cap dom\mathcal{E}$ and $g \in H^{p'}_2(K)$. So there is a continuous sesquilinear form $\tilde{\mathcal{E}} : L^p(K) \times H^{p'}_2(K) \to \mathbb{C}$, such that $\tilde{\mathcal{E}}(f,g) = \mathcal{E}(f,g)$ for any $f \in L^p(K) \cap dom\mathcal{E}$ and $g \in H^{p'}_2(K)$. In addition, we can see that $|\mathcal{E}(f,g)| \leq \|f\|_{H^p_2(K)} \cdot \|g\|_{L^{p'}(K)}$ for any $f \in H^p_2(K)$ and $g \in H^{p'}_2(K)$.

As a consequence, the mapping $f \to \tilde{\mathcal{E}}(f,\cdot)$ is continuous from $L^p(K)$ to $\left(H^{p'}_2(K)\right)^*$, and is continuous from $H^p_2(K)$ to $\left(L^{p'}(K)\right)^*$ since $H^{p'}_2(K)$ is dense in $L^{p'}(K)$, where we use $*$ to denote the dual space. Using the theorem of real interpolation (see [9, Theorem 3.7.1] Theorem 3.7.1), we have $f \to \tilde{\mathcal{E}}(f,\cdot)$ is continuous from $B^{p,q}_\sigma(K)$ to $\left(B^{p',q'}_{2-\sigma}(K)\right)^*$. So $\tilde{\mathcal{E}}$ extends to a continuous sesquilinear form on $B^{p,q}_\sigma(K) \times B^{p',q'}_{2-\sigma}(K)$. ∎

Lemma 6.4 *For $1 \leq p, q \leq \infty$, we have $H^p_2(K) \subset \Lambda^{p,\infty}_{2,(1)}(K)$. In addition, this implies $H^p_2(K) \subset \Lambda^{p,q}_{\sigma,(1)}(K)$ for each $0 < \sigma < 2$.*

Proof In fact, for each $f \in H^p_2(K)$ and $x \in V_{\Lambda_m}$, we have

$$H_{\Lambda_m} f(x) = \int_{U_{x,m}} \psi_{x,m}(\Delta f) d\mu,$$

where $U_{x,m}$ is the same we defined in the proof of Lemma 5.2, and $\psi_{x,m}$ is a piecewise harmonic function supported on $U_{x,m}$, with $\psi_{x,m}(x) = 1$ and $\psi_{x,m}|_{V_{\Lambda_m}\setminus\{x\}} \equiv 0$, and is harmonic in each $F_w K$, $w \in \Lambda_m$. As a consequence, we get

$$r^{-md_H/p'} \|H_{\Lambda_m} f\|_{l^p(V_{\Lambda_m})} \lesssim \|\Delta f\|_{L^p(K)},$$

which yields that $H^p_2(K) \subset \Lambda^{p,\infty}_{2,(1)}(K)$, noticing that $-2d_W/2 + 1 + d_H/p = -d_H/p'$ since $d_W = 1 + d_H$. Finally, $H^p_2(K) \subset \Lambda^{p,\infty}_{2,(1)}(K) \subset \Lambda^{p,q}_{\sigma,(1)}(K)$ since $r^{m(-\sigma d_W/2+1+d_H/p)} \|H_{\Lambda_m} f\|_{l^p(V_{\Lambda_m})} \leq r^{m(2-\sigma)d_W/2} \|f\|_{\Lambda^{p,\infty}_{2,(1)}(K)}$. ∎

Proposition 6.5 *For $1<p<\infty$, $1\le q\le\infty$ and $\mathscr{L}_1(p)<\sigma<\mathscr{L}_2(p)$, we have $B^{p,q}_\sigma(K)=\Lambda^{p,q}_{\sigma,(1)}(K)$ with $\|\cdot\|_{\Lambda^{p,q}_{\sigma,(1)}(K)}\asymp\|\cdot\|_{B^{p,q}_\sigma(K)}$. In particular, if $(\frac{1}{p},\sigma)\in\mathscr{A}_1$, we have $B^{p,q}_\sigma(K)=\Lambda^{p,q}_\sigma(K)$ with $\|\cdot\|_{\Lambda^{p,q}_\sigma(K)}\asymp\|\cdot\|_{B^{p,q}_\sigma(K)}$.*

Proof By Theorem 4.8(c), it suffices to prove the first result. Also, by Lemma 6.2(b) and Proposition 5.4, it suffices to consider the $1<q<\infty$ case.

By Lemma 6.3, there exists $\tilde{\mathcal{E}}$ on $B^{p,q}_\sigma(K)\times B^{p',q'}_{2-\sigma}(K)$ with $p'=\frac{p}{p-1}$, $q'=\frac{q}{q-1}$ satisfying (6.1). In the following claims, we provide an exact formula of $\tilde{\mathcal{E}}$ on $\Lambda^{p,q}_{\sigma,(1)}(K)\times\Lambda^{p',q'}_{2-\sigma,(1)}(K)$.

Claim 1 *Let $M\ge0$, $f=\sum_{m=0}^M f_m$ and $g=\sum_{m=0}^M g_m$ with $f_m,g_m\in J_m$, $0\le m\le M$. We have*

$$\tilde{\mathcal{E}}(f,g)=\sum_{m=0}^M\mathcal{E}(f_m,g_m)=-\sum_{m=0}^M<H_{\Lambda_m}f_m,g_m>_{l^2(V_{\Lambda_m})}. \qquad\blacksquare$$

Proof Clearly, we have $f\in dom\mathcal{E}\cap B^{p,q}_\sigma(K)$, and $g\in B^{p',q'}_{2-\sigma}(K)$ by Proposition 5.4, thus, there is a sequence of functions $g^{(n)}$ in $H^{p'}_2(K)$ converging to g in $B^{p',q'}_{2-\sigma}(K)$. For each n, by Lemma 6.3, we have

$$\tilde{\mathcal{E}}(f,g^{(n)})=\mathcal{E}(f,g^{(n)})=\sum_{m=1}^M\mathcal{E}(f_m,g^{(n)})=-\sum_{m=1}^M<H_{\Lambda_m}f_m,g^{(n)}>_{l^2(V_{\Lambda_m})}.$$

Letting $n\to\infty$, we have the claim proved, since $g^{(n)}$ converges to g in $B^{p',q'}_{2-\sigma}(K)$ and thus converges uniformly as $2-\sigma>\mathscr{L}_1(p')$. $\blacksquare$

Claim 2 *Let $f=\sum_{m=0}^\infty f_m$ and $g=\sum_{m=0}^\infty g_m$ with $f_m,g_m\in J_m$, $\forall m\ge0$, and*

$$\left\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\right\|_{l^q}<\infty,\qquad\left\|r^{m(\sigma-2)d_W/2}\|g_m\|_{L^{p'}(K)}\right\|_{l^{q'}}<\infty.$$

We have

$$\tilde{\mathcal{E}}(f,g)=-\sum_{m=0}^\infty<H_{\Lambda_m}f_m,g_m>_{l^2(V_{\Lambda_m})}.$$

Proof By using Claim 1, we have $\tilde{\mathcal{E}}\left(\sum_{m=0}^M f_m,\sum_{m=0}^M g_m\right)=-\sum_{m=0}^M<H_{\Lambda_m}f_m,g_m>_{l^2(V_{\Lambda_m})}$ for any $M\ge0$. Letting $M\to\infty$, then the claim follows, since the left side converges to $\tilde{\mathcal{E}}(f,g)$ as $\sum_{m=0}^M f_m$ converges to f in $B^{p,q}_\sigma(K)$ and $\sum_{m=0}^M g_m$ converges to g in $B^{p',q'}_{2-\sigma}(K)$ by Proposition 5.4. $\blacksquare$

Claim 3 *Let $f=\sum_{m=0}^\infty f_m$ with $f_m\in J_m$, $\forall m\ge0$, and $\left\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\right\|_{l^q}<\infty$. We have $\left\|r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\right\|_{l^q}\lesssim\|f\|_{B^{p,q}_\sigma(K)}$.*

Proof The space $l^{q'}\left(l^{p'}(V_{\Lambda_\cdot})\right)$ can be identified with the dual space of $l^q\left(l^p(V_{\Lambda_\cdot})\right)$ in a natural way, and thus, we can find $g=\sum_{m=0}^\infty g_m$, with $g_m\in J_m$ and $0<\left\|r^{m(\sigma-2)d_W/2}\|g_m\|_{L^{p'}(K)}\right\|_{l^{q'}}<\infty$, such that

$$
\begin{aligned}
|\tilde{\mathcal{E}}(f,g)| &\geq \frac{1}{2}\left\| r^{-m\sigma d_W/2+m+md_H/p}\|H_{\Lambda_m} f_m\|_{l^p(V_{\Lambda_m})}\right\|_{l^q} \cdot \left\| r^{m(\sigma-2)d_W/2+md_H/p'}\|g_m\|_{l^{p'}(V_{\Lambda_m})}\right\|_{l^{q'}} \\
&\gtrsim \left\| r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\right\|_{l^q} \cdot \left\| r^{m(\sigma-2)d_W/2}\|g_m\|_{L^{p'}(K)}\right\|_{l^{q'}} \\
&\gtrsim \left\| r^{-m\sigma d_W/2}\|f_m\|_{L^p(K)}\right\|_{l^q} \cdot \|g\|_{B^{p',q'}_{2-\sigma}(K)}.
\end{aligned}
$$

On the other hand, we have $|\tilde{\mathcal{E}}(f,g)| \lesssim \|f\|_{B^{p,q}_\sigma(K)} \cdot \|g\|_{B^{p',q'}_{2-\sigma}(K)}$. The estimate follows. ■

Now, combining Claim 3 and Proposition 5.4, we can see that $\Lambda^{p,q}_{\sigma,(1)}(K)$ is a closed subset of $B^{p,q}_\sigma(K)$. On the other hand, we have $H^p_2(K) \subset \Lambda^{p,q}_{\sigma,(1)}(K)$ by Lemma 6.4. So the desired result follows since $H^p_2(K)$ is dense in $B^{p,q}_\sigma(K)$.

Finally, for $1 < p < \infty$, $q = 1, \infty$ and $\mathscr{L}_1(p) < \sigma < \mathscr{L}_2(p)$, we pick $\mathscr{L}_1(p) < \sigma_1 < \sigma < \sigma_2 < \mathscr{L}_2(p)$ and $\theta \in (0,1)$ such that $\sigma = (1-\theta)\sigma_1 + \theta\sigma_2$, then

$$
\Lambda^{p,q}_\sigma(K) \supset \left(\Lambda^{p,2}_{\sigma_1}(K), \Lambda^{p,2}_{\sigma_2}(K)\right)_{\theta,q} = \left(B^{p,2}_{\sigma_1}(K), B^{p,2}_{\sigma_2}(K)\right)_{\theta,q} = B^{p,q}_\sigma(K)
$$

by Lemma 6.2(b), Lemma 2.6, and the reiteration theorem of real interpolation. Hence, $\Lambda^{p,q}_\sigma(K) = B^{p,q}_\sigma(K)$ by Proposition 5.4.

By applying Propositions 5.4 and 6.5 and Lemma 6.2(b), we can finish the proof of Theorem 4.9.

Proof of Theorem 4.9 Let's fix $1 < p < \infty$, $1 \leq q \leq \infty$ and $\mathscr{L}_1(p) < \sigma < 2$, and we choose $\mathscr{L}_1(p) < \sigma_1 < \sigma \wedge \mathscr{L}_2(p)$ and $\theta \in (0,1)$ such that $\sigma = (1-\theta)\sigma_1 + 2\theta$. Then, by Proposition 6.5, we know that $B^{p,q}_{\sigma_1}(K) = \Lambda^{p,q}_{\sigma_1,(1)}(K)$; by Lemma 6.4, we know that $H^p_2(K) \subset \Lambda^{p,\infty}_{2,(1)}(K)$. Hence,

$$
B^{p,q}_\sigma(K) = \left(B^{p,q}_{\sigma_1}(K), H^p_2(K)\right)_{\theta,q} \subset \left(\Lambda^{p,q}_{\sigma_1,(1)}(K), \Lambda^{p,\infty}_{2,(1)}(K)\right)_{\theta,q} \subset \Lambda^{p,q}_{\sigma,(1)}(K)
$$

by Lemma 2.6, the reiteration theorem of real interpolation and Lemma 6.2(b). Combining this with Proposition 5.4, the theorem follows. ■

6.2 On region $\mathscr{A}_2$

It remains to show $B^{p,q}_\sigma(K) \subset \Lambda^{p,q}_\sigma(K)$ on $\mathscr{A}_2$. In fact, by Proposition 6.5 and Lemma 6.2(a), noticing that $L^p(K)$ is contained in "$\Lambda^{p,\infty}_0(K)$," we can simply cover a large portion of $\mathscr{A}_2$, see an illustration in Figure 8. However, it remains unclear for the strip region near $p = 1$, if $\mathscr{C}$ and $\mathscr{L}_1$ intersect at some point with $p > 1$. We will apply another idea to overcome this. Also, we mention here that a similar method can solve the $\mathscr{A}_1$ region as well with necessary modifications.

We will rely on Proposition 4.4 in this part, which says, for $0 < \sigma < \mathscr{L}_1(p)$, it holds that

$$
\|f\|_{\Lambda^{p,q}_\sigma(K)} \asymp \left\| r^{-m\sigma d_W/2}\|\tilde{E}[f|\Lambda_m]\|_{L^p(K)}\right\|_{l^q}.
$$

To get a reasonable estimate for $\left\|\tilde{E}[f|\Lambda_m]\right\|_{L^p(K)}$, we start with a new decomposition.

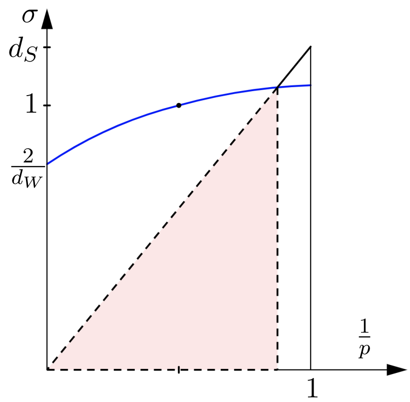

Figure 8: A portion of $\mathscr{A}_2$.

Definition 6.6 (a). For $m \geq 0$, we define $T_m = \left\{\sum_{w\in\Lambda_m} h_w \circ F_w^{-1} : h_w \in \mathcal{H}_0, \forall w \in \Lambda_m\right\}$.

(b). Write P_{T_m} for the orthogonal projection $L^2(K) \to T_m$ for $m \geq 0$, and

$$P_{\tilde{T}_m} = \begin{cases} P_{T_0}, & \text{if } m = 0, \\ P_{T_m} - P_{T_{m-1}}, & \text{if } m \geq 1. \end{cases}$$

Since P_{T_m} is realized by integration against an L^2 orthogonal basis of harmonic functions, which are bounded, it extends to a bounded linear map $L^p(K) \to T_m$ for any $1 \leq p \leq \infty$.

(c). Write $\tilde{T}_m = \left\{P_{\tilde{T}_m} f : f \in L^1(K)\right\}$ for $m \geq 1$, and write $\tilde{T}_0 = T_0$.

Remark The spaces T_m are collections of piecewise harmonic functions, but may not be continuous at $V_{\Lambda_m} \setminus V_0$.

We collect some useful results in the following lemma.

Lemma 6.7 *Let $1 < p < \infty$, $m \geq 0$ and $u \in \tilde{T}_m$.*

(a). We have $\tilde{E}[u|\Lambda_n] = 0$ if $n < m$.

(b). For any $0 < \sigma < \mathscr{C}(p)$, we have $\left\|\tilde{E}[u|\Lambda_n]\right\|_{L^p(K)} \lesssim r^{(n-m)\sigma d_W/2}\|u\|_{L^p(K)}$ for $n \geq m$.

Proof (a). By definition, for each $u \in \tilde{T}_m$, we have $P_{T_{m-1}}u = 0$. On the other hand, we can see that $\oplus_{l=0}^n \tilde{J}_l \subset T_{m-1}$ since clearly $\tilde{J}_l$ consists of piecewise constant functions. Thus,

$$\tilde{E}[u|\Lambda_n] = \tilde{E}[P_{T_{m-1}}u|\Lambda_n] = 0, \quad \forall n < m.$$

(b). We first look at $m = 0$ case. By definition of $\mathscr{C}$, we have $r^{-n\sigma d_W/2} I_p(u, r^n) \lesssim \|u\|_{L^p(K)}$, as $u \in \tilde{T}_m = \mathcal{H}_0$. The claim then follows by applying Lemma 4.3(a).

For general case, for each $w \in \Lambda_m$, we can see that

$$r^{-(n-m)\sigma d_W/2}\left\|(E[u|\Lambda_n]) \circ F_w\right\|_{L^p(K)} \lesssim \|u \circ F_w\|_{L^p(K)}, \quad \forall n \geq m.$$

(b) then follows by scaling and summing the estimates over Λ_m. ∎

Proposition 6.8 *For $1 < p < \infty$, $1 \le q \le \infty$ and $(\frac{1}{p}, \sigma) \in \mathscr{A}_2$, we have $B^{p,q}_\sigma(K) \subset \Lambda^{p,q}_\sigma(K)$ with $\|\cdot\|_{\Lambda^{p,q}_\sigma(K)} \lesssim \|\cdot\|_{B^{p,q}_\sigma(K)}$.*

Proof Let $f \in B^{p,q}_\sigma(K)$, it suffices to show that $\left\| r^{-m\sigma d_W/2} \|\tilde{E}[f|\Lambda_m]\|_{L^p(K)} \right\|_{l^q} \lesssim \|f\|_{B^{p,q}_\sigma(K)}$ by applying Proposition 4.4.

For convenience, we write $\mathscr{P} : L^p(K) \to \prod_{m=0}^\infty \tilde{T}_m$, defined as $\mathscr{P}(f)_m = P_{\tilde{T}_m} f$. Also, we equip each $\tilde{T}_m$ with the L^p norm. Then, P_{T_m} and $P_{\tilde{T}_m}$ are bounded linear maps on $L^p(K)$, which immediately provides Claim 1.

Claim 1 *$\mathscr{P}$ is bounded from $L^p(K)$ to $l^\infty(\tilde{T}_\cdot)$.* ■

Recall the definition of l^p_α from Lemma 6.1.

Claim 2 *$\mathscr{P}$ is bounded from $H^p_2(K)$ to $l^\infty_{r^{d_W}}(\tilde{T}_\cdot)$.*

Proof Let G be the Dirichlet Green's operator on K [26, 36]. For any $f \in H^p_2(K) = H^p_{2,D}(K) \oplus \mathcal{H}_0$, we have

$$\|f - P_{T_0} f\|_{L^p(K)} \lesssim \|G\Delta f\|_{L^p(K)} \lesssim \|\Delta f\|_{L^p(K)},$$

where the first inequality is due to the fact that $f - G(-\Delta)f \in T_0 = \mathcal{H}_0$, and the second inequality is due to the fact that G is bounded from $L^p(K)$ to $L^p(K)$. We apply the above estimate locally on each $F_w K$ with $w \in \Lambda_m$ to get

$$\|f - P_{T_m} f\|_{L^p(K)} \lesssim r^{m d_W} \|\Delta f\|_{L^p(K)},$$

by using the scaling property of Δf. Thus, we have

$$\|P_{\tilde{T}_m} f\|_{L^p(K)} \lesssim \|f - P_{T_m} f\|_{L^p(K)} + \|f - P_{T_{m-1}} f\|_{L^p(K)} \lesssim r^{m d_W} \|\Delta f\|_{L^p(K)}.$$

This finishes the proof of Claim 2. ■

Combining Claim 1 and Claim 2, and using Lemma 6.1, we see the following claim.

Claim 3 *For $1 < p < \infty$, $1 \le q \le \infty$ and $0 < \sigma < 2$, $\mathscr{P}$ is bounded from $B^{p,q}_\sigma(K)$ to $l^q_{r^{\sigma d_W/2}}(\tilde{T}_\cdot)$.*

Now, we turn to the proof of the proposition. We fix a parameter point $(\frac{1}{p}, \sigma)$ in $\mathscr{A}_2$. By Claim 3, we can see that, for each $f \in B^{p,q}_\sigma(K)$, we clearly have $f = \sum_{m=0}^\infty P_{\tilde{T}_m} f$, with the series absolute convergent in $L^p(K)$. Thus, we have

$$\tilde{E}[f|\Lambda_m] = \sum_{n=0}^\infty \tilde{E}[P_{\tilde{T}_n} f|\Lambda_m] = \sum_{n=0}^m \tilde{E}[P_{\tilde{T}_n} f|\Lambda_m],$$

where the second equality is due to Lemma 6.7(a). In addition, by applying Lemma 6.7(b), we have the estimate

$$\big\|\tilde{E}[f|\Lambda_m]\big\|_{L^p(K)} \lesssim \sum_{n=0}^{m} r^{(m-n)\eta d_W/2}\|P_{\tilde{T}_n}f\|_{L^p(K)},$$

where η is a fixed number such that $\sigma < \eta < \mathscr{C}(p)$. As a consequence, we then have

$$\begin{aligned}\Big\|r^{-m\sigma d_W/2}\|\tilde{E}[f|\Lambda_m]\|_{L^p(K)}\Big\|_{l^q} &\lesssim \Big\|r^{-m\sigma d_W/2}\sum_{n=0}^{m} r^{(m-n)\eta d_W/2}\|P_{\tilde{T}_n}f\|_{L^p(K)}\Big\|_{l^q}\\ &= \Big\|r^{-m\sigma d_W/2}\sum_{n=0}^{m} r^{n\eta d_W/2}\|P_{\tilde{T}_{m-n}}f\|_{L^p(K)}\Big\|_{l^q}\\ &\le \sum_{n=0}^{\infty} r^{n(\eta-\sigma)d_W/2}\cdot\Big\|r^{-m\sigma d_W/2}\|P_{\tilde{T}_m}f\|_{L^p(K)}\Big\|_{l^q} \lesssim \|f\|_{B^{p,q}_\sigma(K)},\end{aligned}$$

where we use Claim 3 again in the last inequality.

Remark We can apply a similar argument as Lemma 6.7 and Proposition 6.8 for $(\frac{1}{p},\sigma)\in\mathscr{B}$ to show that $B^{p,q}_\sigma(K)\subset\Lambda^{p,q}_{\sigma,(1)}(K)$, as stated in Proposition 6.5. The difference is that $f_m\in J_m$ in the tent function expansion of $f=\sum_{m=0}^{\infty} f_m$ depends on $P_{\tilde{T}_n}f$ for $n\ge m$. This gives a second proof of Proposition 6.5.

We finish this section with a conclusion that Theorem 1.1 holds.

Proof of Theorem 1.1 On $\mathscr{A}_1$, the theorem follows from Proposition 6.5; on $\mathscr{A}_2$, the theorem follows from Propositions 5.3 and 6.8; lastly, on the border between $\mathscr{A}_1$ and $\mathscr{A}_2$, we have $B^{p,q}_\sigma(K)\subset\Lambda^{p,q}_\sigma(K)$ by interpolation using Lemma 6.2(a), as well as the other direction is covered by Proposition 5.3. Finally, the region is sharp by Proposition 3.2. ■

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