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Generating groups of nilpotent varieties

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If $\underline{\underline{V}}$ is a variety of groups which are nilpotent of class c then $\underline{\underline{V}}$ is generated by its free group of rank c. It is proved that under certain general conditions $\underline{\underline{V}}$ cannot be generated by its free group of rank $c - 2$, and that under certain other conditions $\underline{\underline{V}}$ is generated by its free group of rank $c - 1$. It follows from these results that if $\underline{\underline{V}}$ is the variety of all groups which are nilpotent of class c, then the least value of k such that the free group of $\underline{\underline{V}}$ of rank k generates $\underline{\underline{V}}$ is $c - 1$. This extends known results of L.G. Kovács, M.F. Newman, P.F. Pentony (1969) and F. Levin (1970).

1. Introduction

If $\underline{\underline{V}}$ is a variety of groups let $d(\underline{\underline{V}})$ be the least value of k such that the free group of $\underline{\underline{V}}$ of rank k generates $\underline{\underline{V}}$. Of course $d(\underline{\underline{V}})$ may be infinite but in many cases it is finite. For instance if $\underline{\underline{V}}$ is nilpotent of class c then $d(\underline{\underline{V}}) \le c$ ([1], 35.12), and if $\underline{\underline{V}}$ is the variety of all metabelian groups which are nilpotent of class c $(c > 1)$ then $d(\underline{\underline{V}}) = 2$ [2].

Let $\underline{\underline{N}}_c$ denote the variety of all groups which are nilpotent of class c, and let $\underline{\underline{AN}}_2$ denote the variety of all groups which are abelian-by-(nilpotent of class 2). In this paper I shall prove the following theorems.

THEOREM 1. *If* $\left(\underline{\underline{N}}_c \wedge \underline{\underline{AN}}_2\right) \le \underline{\underline{V}} \le \underline{\underline{N}}_c$, $c > 2$, *then* $d(\underline{\underline{V}}) \ge c - 1$.

THEOREM 2. *If* $\underline{\underline{V}} \le \underline{\underline{N}}_c$, $c > 2$, *and if the free group of* $\underline{\underline{V}}$ *of*

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rank c *is torsion free then* $d(\underline{\underline{V}}) \leq c - 1$.

Since the free groups of $\underline{\underline{N}}_c$ are torsion free these theorems have the following corollary, which has been independently proved in [3] and [4].

COROLLARY. $d(\underline{\underline{N}}_c) = c - 1$ *for* $c > 2$.

Theorem 2 is not true in general without the condition that the free group of $\underline{\underline{V}}$ of rank c be torsion free, as the following example shows. Let $\underline{\underline{V}} \leq \underline{\underline{N}}_3$ be determined by the laws $[x_1, x_2, x_2]$, $[x_1, x_2, x_3, x_4]$. Then the free group of $\underline{\underline{V}}$ of rank two is nilpotent of class two but $\underline{\underline{V}}$ is not, and so $d(\underline{\underline{V}}) = 3$. However $\underline{\underline{V}}$ does satisfy the law $[x_1, x_2, x_3]^3$ and so the free group of $\underline{\underline{V}}$ of rank three is not torsion free.

The notation is generally consistent with [1]. If $\underline{\underline{V}}$ is a variety of groups then $F_k(\underline{\underline{V}})$ denotes the free group of $\underline{\underline{V}}$ generated by $x_1, x_2, \ldots, x_k$. If G is a group then $\gamma_n(G)$ denotes the n-th term of the lower central series of G .

2. Proof of Theorem 1

I shall show that for each $c > 2$ there is a word $w_c \in F_c(\underline{\underline{N}}_c)$ such that

(1) w_c is a law in $F_{c-2}(\underline{\underline{N}}_c)$,

(2) $w_c \notin \gamma_2\left(\gamma_3(F_c(\underline{\underline{N}}_c))\right)$.

Since $\underline{\underline{V}} \leq \underline{\underline{N}}_c$, $F_{c-2}(\underline{\underline{V}})$ is a homomorphic image of $F_{c-2}(\underline{\underline{N}}_c)$ and so (1) implies that w_c is a law in $F_{c-2}(\underline{\underline{V}})$. $F_c(\underline{\underline{V}}) \cong F_c(\underline{\underline{N}}_c)/V$ for some fully invariant subgroup V of $F_c(\underline{\underline{N}}_c)$. Since $(\underline{\underline{N}}_c \wedge \underline{\underline{AN}}_2) \leq \underline{\underline{V}}$, $V \leq \gamma_2\left(\gamma_3(F_c(\underline{\underline{N}}_c))\right)$, and so (2) implies that w_c is not a law in $\underline{\underline{V}}$. This shows that $F_{c-2}(\underline{\underline{V}})$ does not generate $\underline{\underline{V}}$, and so $d(\underline{\underline{V}}) \geq c - 1$.

The word w_c is defined as follows.

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Let P be the group of permutations of $(2, 3, \ldots, c)$. If $\sigma \in P$ let

$$\operatorname{sgn}\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

For $\sigma \in P$ let

$$w_c(\sigma) = \begin{cases} \left[\left[x_{\sigma(2)}, x_{\sigma(3)}\right], \left[x_{\sigma(4)}, x_1\right], \left[x_{\sigma(5)}, x_{\sigma(6)}\right], \ldots, \left[x_{\sigma(c-1)}, x_{\sigma(c)}\right]\right] & \text{if } c \text{ is even ,} \\ \left[\left[x_{\sigma(2)}, x_1, x_{\sigma(3)}\right], \left[x_{\sigma(4)}, x_{\sigma(5)}\right], \left[x_{\sigma(6)}, x_{\sigma(7)}\right], \ldots, \left[x_{\sigma(c-1)}, x_{\sigma(c)}\right]\right] & \text{if } c \text{ is odd .} \end{cases}$$

Let $w_c = \prod_{\sigma \in P} w_c(\sigma)^{\operatorname{sgn}\sigma}$. (The order of the product is immaterial since each term of the product is contained in $\gamma_c\left(F_c\left(\underline{\underline{N}}_c\right)\right)$ which is the centre of $F_c\left(\underline{\underline{N}}_c\right)$.)

First I shall show that w_c is a law in $F_{c-2}\left(\underline{\underline{N}}_c\right)$.

Since $w_c = w_c\left(x_1, x_2, \ldots, x_c\right)$ is contained in $\gamma_c\left(F_c\left(\underline{\underline{N}}_c\right)\right)$ it follows from repeated use of the identities

$$[xy, z] = [x, z][x, z, y][y, z] ,$$

$$[x, yz] = [x, z][x, y][x, y, z] ,$$

that

$$w_c\left(a_1, \ldots, a_{i-1}, ab, a_{i+1}, \ldots, a_c\right) = w_c\left(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_c\right) w_c\left(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_c\right)$$

and that

$$w_c\left(a_1, \ldots, a_{i-1}, a^{-1}, a_{i+1}, \ldots, a_c\right) = w_c\left(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_c\right)^{-1} ,$$

for any $a_1, a_2, \ldots, a_c, a, b \in F_c(\underline{\underline{N}}_c)$ and for each $i = 1, 2, \ldots, c$.

Hence to show that w_c is a law in $F_{c-2}(\underline{\underline{N}}_c)$ it is sufficient to show that whenever θ is an endomorphism of $F_c(\underline{\underline{N}}_c)$ which maps the set $\{x_1, x_2, \ldots, x_c\}$ into the set $\{x_1, x_2, \ldots, x_{c-2}\}$ then $w_c\theta = 1$. Now if θ is such a map then $x_i\theta = x_j\theta$ for some $i, j \geq 2$, $i \neq j$. Let (i, j) be the permutation of $(2, 3, \ldots, c)$ which interchanges i and j and fixes everything else, and let T be a transversal of (i, j) in P such that $P = T \cup (i, j)T$. If $\tau \in T$ let $\tau' = (i, j)\tau$. Then

$$w_c = \prod_{\sigma \in P} w_c(\sigma)^{\text{sgn}\sigma}$$

$$= \prod_{\tau \in T} w_c(\tau)^{\text{sgn}\tau} \cdot \prod_{\tau \in T} w_c(\tau')^{\text{sgn}\tau'} .$$

But $\text{sgn}\tau' = -\text{sgn}\tau$ and, since $x_i\theta = x_j\theta$, $w_c(\tau)\theta = w_c(\tau')\theta$. Therefore $w_c\theta = 1$.

To show that $w_c \notin \gamma_2\left(\gamma_3\left(F_c(\underline{\underline{N}}_c)\right)\right)$ I shall express w_c as a product of basic commutators.

The basic commutators of $F_c(\underline{\underline{N}}_c)$ are defined as follows.

(1) The basic commutators of weight one are $x_1, x_2, \ldots, x_c$.

(2) Having defined the basic commutators of weight less than n , and ordered them by $<$, the basic commutators of weight n are $[c, d]$ where

(a) c, d are basic commutators and $\text{wt}(c) + \text{wt}(d) = n$, and

(b) $c > d$, and if $c = [c_1, c_2]$ then $c_2 \leq d$.

(3) The basic commutators of weight n follow those of weight less than n under $<$, and are ordered arbitrarily with respect to each other.

The basic commutators of weight c form a free basis for the free abelian group $\gamma_c\left(F_c(\underline{\underline{N}}_c)\right)$ [5]. By Theorem 9.1 of [6] the basic

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commutators of weight c of the form $[c_1, c_2]$, $c_1, c_2 \in \gamma_3\left(F_c(\underline{\underline{N}}_c)\right)$, form a free basis for $\gamma_c\left(F_c(\underline{\underline{N}}_c)\right) \cap \gamma_2\left(\gamma_3(F_c(\underline{\underline{N}}_c))\right)$; and so to prove that $w_c \notin \gamma_2\left(\gamma_3(F_c(\underline{\underline{N}}_c))\right)$ it is sufficient to show that, modulo $\gamma_2\left(\gamma_3(F_c(\underline{\underline{N}}_c))\right)$, w_c can be expressed as a non-trivial product of basic commutators of weight c which are not of the form $[c_1, c_2]$, $c_1, c_2 \in \gamma_3\left(F_c(\underline{\underline{N}}_c)\right)$.

I shall need a specific ordering on the basic commutators of weights one to two.

Let $x_1 < x_2 < \ldots < x_c$.

Then the basic commutators of weight two are the commutators $[x_i, x_j]$, $i > j$. If $[x_i, x_j]$, $[x_k, x_l]$ are basic commutators let

$$[x_i, x_j] < [x_k, x_l]$$

if $j < l$ or $j = l$ and $i < k$.

First suppose that c is even. Then

$$w_c(\sigma) = \Big[[x_{\sigma(2)}, x_{\sigma(3)}], [x_{\sigma(4)}, x_1], [x_{\sigma(5)}, x_{\sigma(6)}], \ldots, [x_{\sigma(c-1)}, x_{\sigma(c)}]\Big] .$$

If τ is one of the permutations $(2, 3), (5, 6), (7, 8), \ldots, (c-1, c)$ then $w_c(\sigma\tau) = w_c(\tau)^{-1}$ for any $\sigma \in P$ since $[x_i, x_j] = [x_j, x_i]^{-1}$. But τ is then an odd permutation and so $\operatorname{sgn}\sigma\tau = -\operatorname{sgn}\sigma$. Hence

$$\begin{aligned} w_c &= \prod_{\sigma \in P} w_c(\sigma)^{\operatorname{sgn}\sigma} \\ &= \left(\prod_{\sigma \in Q} w_c(\sigma)^{\operatorname{sgn}\sigma}\right)^{2^{\frac{c-2}{2}}} \end{aligned}$$

where Q consists of those permutations in P such that

$$\sigma(2) > \sigma(3),\ \sigma(5) > \sigma(6),\ \sigma(7) > \sigma(8),\ \ldots,\ \sigma(c-1) > \sigma(c) .$$

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Now if $a_1, a_2, \ldots, a_7 \in F_c(\underline{\underline{N}}_c)$

$$\big[[a_1, a_2, a_3], [[a_4, a_5], [a_6, a_7]]\big] \in \gamma_2\big(\gamma_3(F_c(\underline{\underline{N}}_c))\big)$$

and so

$$[[a_1, a_2, a_3], [a_4, a_5], [a_6, a_7]] = [[a_1, a_2, a_3], [a_6, a_7], [a_4, a_5]] \bmod \gamma_2\big(\gamma_3(F_c(\underline{\underline{N}}_c))\big) .$$

Hence if τ is any of the permutations $(5, 7)(6, 8), (7, 9)(6, 10), \ldots, (c-3, c-1)(c-2, c)$, $w_c(\sigma\tau) = w_c(\sigma) \bmod \gamma_2\big(\gamma_3(F_c(\underline{\underline{N}}_c))\big)$ for any $\sigma \in P$. But τ is then an even permutation and so $\operatorname{sgn}\sigma\tau = \operatorname{sgn}\sigma$. Hence

$$w_c = \Big(\prod_{\sigma\in Q} w_c(\sigma)^{\operatorname{sgn}\sigma}\Big)^{2^{\frac{c-2}{2}}}$$

$$= \Big(\prod_{\sigma\in R} w_c(\sigma)^{\operatorname{sgn}\sigma}\Big)^{2^{\frac{c-2}{2}}\left(\frac{c-4}{2}\right)!} \bmod \gamma_2\big(\gamma_3(F_c(\underline{\underline{N}}_c))\big)$$

where R consists of those permutations in P such that

$$\sigma(2) > \sigma(3), \sigma(5) > \sigma(6), \sigma(7) > \sigma(8), \ldots, \sigma(c-1) > \sigma(c) ,$$

and

$$\sigma(6) < \sigma(8) < \ldots < \sigma(c) .$$

Now if $\sigma \in R$ then $w_c(\sigma)$ is a basic commutator of weight c which is not of the form $[c_1, c_2]$, $c_1, c_2 \in \gamma_3\big(F_c(\underline{\underline{N}}_c)\big)$, and so this proves that $w_c \notin \gamma_2\big(\gamma_3(F_c(\underline{\underline{N}}_c))\big)$.

Now suppose that c is odd. Then

$$w_c(\sigma) = \Big[[x_{\sigma(2)}, x_1, x_{\sigma(3)}], [x_{\sigma(4)}, x_{\sigma(5)}], [x_{\sigma(6)}, x_{\sigma(7)}], \ldots, [x_{\sigma(c-1)}, x_{\sigma(c)}]\Big] ,$$

and by an argument similar to that used above it can be shown that

$$w_c = \left(\prod_{\sigma\in S} w_c(\sigma)^{\text{sgn}\sigma}\right)^{2^{\frac{c-3}{2}}\left(\frac{c-3}{2}\right)!} \bmod \gamma_2\left(\gamma_3\left(F_c(\underline{\underline{N}}_c)\right)\right)$$

where S consists of those permutations in P such that

$$\sigma(4) > \sigma(5),\ \sigma(6) > \sigma(7),\ \ldots,\ \sigma(c-1) > \sigma(c) ,$$

and

$$\sigma(5) < \sigma(7) < \ldots < \sigma(c) .$$

But if $\sigma \in S$ then $w_c(\sigma)$ is a basic commutator of weight c which is not of the form $[c_1, c_2]$, $c_1, c_2 \in \gamma_3\left(F_c(\underline{\underline{N}}_c)\right)$, and so this proves that $w_c \notin \gamma_2\left(\gamma_3\left(F_c(\underline{\underline{N}}_c)\right)\right)$ if c is odd, which completes the proof of Theorem 1.

3. Proof of Theorem 2

Since $\underline{\underline{V}} \leq \underline{\underline{N}}_c$, $F_c(\underline{\underline{V}})$ generates $\underline{\underline{V}}$ ([1], 35.12), and so to prove Theorem 2 it is sufficient to show that there is no non-trivial word in $F_c(\underline{\underline{V}})$ which is a law in $F_{c-1}(\underline{\underline{V}})$.

Let $w \in F_c(\underline{\underline{V}})$ and suppose that w is a law in $F_{c-1}(\underline{\underline{V}})$. Let δ_i be the endormorphism of $F_c(\underline{\underline{V}})$ which maps $x_i \to 1$ and maps $x_j \to x_j$ for $j \neq i$. Then $w\delta_i = 1$ for each $i = 1, 2, \ldots, c$ and so w can be written as a product of commutators each involving all of $x_1, x_2, \ldots, x_c$ ([1], 33.37). Since $F_c(\underline{\underline{V}})$ is nilpotent of class c, w must be of weight one in each of the variables $x_1, x_2, \ldots, x_c$, and so, if $w = w(x_1, x_2, \ldots, x_c)$,

$$w(a_1, \ldots, a_{i-1}, ab, a_{i+1}, \ldots, a_c) = w(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_c)w(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_c)$$

for any $a_1, a_2, \ldots, a_c$, $a, b \in F_c(\underline{\underline{V}})$ and for each $i = 1, 2, \ldots, c$.

Let $1 \leq i < j \leq c$ and, for convenience, write

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$w(x_1, x_2, \ldots, x_c) = w(x_i, x_j)$, indicating only the variables in the i-th and j-th places. Then $w(x_i, x_i)$ is a word in $c-1$ variables and so $w(x_i, x_i) = 1$. Hence

$$\begin{aligned} 1 &= w(x_ix_j, x_ix_j) \\ &= w(x_i, x_i)w(x_j, x_j)w(x_i, x_j)w(x_j, x_i) \\ &= w(x_i, x_j)w(x_j, x_i) . \end{aligned}$$

Let P be the group of permutations of $(1, 2, \ldots, c)$ and for $\sigma \in P$ let $w(\sigma) = w(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(c)})$. Then the above remarks show that $w(\sigma)w(\sigma(i, j)) = 1$ for all $\sigma \in P$ and all (i, j) . Hence $w(\sigma)^{\mathrm{sgn}\sigma} = w$ and so $\prod_{\sigma\in P} w(\sigma)^{\mathrm{sgn}\sigma} = w^{c!}$. I shall show that $\prod_{\sigma\in P} w(\sigma)^{\mathrm{sgn}\sigma} = 1$. Since $F_c(\underline{\underline{V}})$ is torsion free this shows that $w = 1$, which completes the proof of Theorem 2.

Now $w(x_1, x_2, \ldots, x_c)$ can be written as a product of left normed commutators where each is of weight one in each of $x_1, x_2, \ldots, x_c$, that is w can be written as a product of elements of the form $[x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(c)}]$ where $\tau \in P$. Hence it is sufficient to show that

$$\prod_{\sigma\in P} [x_{\sigma(\tau(1))}, x_{\sigma(\tau(2))}, \ldots, x_{\sigma(\tau(c))}]^{\mathrm{sgn}\sigma} = 1 .$$

Let α be the permutation of $(1, 2, \ldots, c)$ which maps $1 \to 3$, $2 \to 1$, $3 \to 2$ and fixes everything else $(c > 2)$. Let T be a transversal of the subgroup $\{1, \alpha, \alpha^2\}$ in P such that $P = T \cup T\alpha \cup T\alpha^2$. Now α is an even permutation and so

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$$\prod_{\sigma\in P}\left[x_{\sigma(\tau(1))}, x_{\sigma(\tau(2))}, \ldots, x_{\sigma(\tau(c))}\right]^{\operatorname{sgn}\sigma}$$

$$= \prod_{\sigma\in P}\left[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(c)}\right]^{\operatorname{sgn}\tau\sigma}$$

$$= \prod_{\sigma\in T}\Big(\left[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, \ldots, x_{\sigma(c)}\right]\left[x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(1)}, x_{\sigma(4)}, \ldots, x_{\sigma(c)}\right]\left[x_{\sigma(3)}, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(4)}, \ldots, x_{\sigma(c)}\right]\Big)^{\operatorname{sgn}\tau\sigma}$$

$$= \prod_{\sigma\in T}\Big[\left[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right]\left[x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(1)}\right]\left[x_{\sigma(3)}, x_{\sigma(1)}, x_{\sigma(2)}\right], x_{\sigma(4)}, \ldots, x_{\sigma(c)}\Big]^{\operatorname{sgn}\tau\sigma}$$

$$= 1$$

since

$$\left[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right]\left[x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(1)}\right]\left[x_{\sigma(3)}, x_{\sigma(1)}, x_{\sigma(2)}\right] \in \gamma_4\left(F_c(\underline{\underline{V}})\right) .$$

This completes the proof of Theorem 2.

The methods used in this paper are similar to those used in [3] and [4], where it is proved that $d\left(\underline{\underline{N}}_c\right) = c - 1$ for $c > 2$; in fact the law w_c used here seems very close to the one introduced in [3]; but Theorems 1 and 2 apply to a wide range of varieties. For instance they show that $d\left(\underline{\underline{N}}_c \wedge \underline{\underline{A}}^l\right) = c - 1$ for $c > 2$, $l > 2$ $\left(\underline{\underline{A}}^l\right.$ is the variety of all groups which are soluble of derived length $\left.l\right)$, which should be compared with the result mentioned in the introduction that $d\left(\underline{\underline{N}}_c \wedge \underline{\underline{A}}_2\right) = 2$ [2].

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